LECTURE 2: EUCLIDEAN SPACES, AFFINE SPACES, AND HOMOGENOUS SPACES IN GENERAL

1. Euclidean space

If the vector space $\mathbb{R}^n$ is endowed with a positive definite inner product $\langle , \rangle$ we say that it is a *Euclidean space* and denote it $\mathbb{E}^n$. The inner product gives a way of measuring distances and angles between points in $\mathbb{E}^n$, and this is the fundamental property of Euclidean spaces.

Question: What are the *symmetries* of Euclidean space? i.e., what kinds of transformations $\varphi : \mathbb{E}^n \to \mathbb{E}^n$ preserve the fundamental properties of lengths and angles between vectors?

Answer: Translations, rotations, and reflections, collectively known as "rigid motions." Any such transformation has the form

$$\varphi(x) = Ax + b$$

where $A \in O(n)$ and $b \in \mathbb{E}^n$. The set of such transformations forms a *Lie group*, called the *Euclidean group* $E(n)$. This group can be represented as a group of $(n+1) \times (n+1)$ matrices as follows: let

$$E(n) = \left\{ \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} : A \in O(n),\ b \in \mathbb{E}^n \right\}.$$

Here A is an $n \times n$ matrix, b is an $n \times 1$ column, and 0 represents a $1 \times n$ row of 0's. If we represent a vector $x \in \mathbb{E}^n$ by the $(n+1)$-dimensional vector $\begin{bmatrix} x \\ 1 \end{bmatrix}$, then elements of $E(n)$ acts on x by matrix multiplication:

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Ax + b \\ 1 \end{bmatrix}.$$

Question: Given a point $x \in \mathbb{E}^n$, which elements of $E(n)$ leave x fixed? This question is easiest to answer when $x = \vec{0}$. It is clear that an element $\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$ fixes $\vec{0}$ if and only if $b = 0$. The set of such elements $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ forms a subgroup $H_{\vec{0}}$, called the *isotropy group* of $\vec{0} \in \mathbb{E}^n$, which is clearly isomorphic to $O(n)$.

What about other points $x \in \mathbb{E}^n$? It seems reasonable to expect that there shouldn't be anything special about $\vec{0}$ since we can move any point to any other point via a translation. And in fact, the isotropy group H_x of any point $x \in \mathbb{E}^n$ is also isomorphic to $O(n)$. We can define an explicit isomorphism ϕ :

$H_{\vec{0}} \to H_x$ as follows. Let $t_x = \begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix}$, so that t_x represents the translation $t_x(y) = y + x$. Then define

$$\phi(h) = t_x h t_x^{-1}.$$

$\phi(h)$ clearly fixes x, and it is not difficult to see that ϕ is an isomorphism. Thus $H_x = t_x H_{\vec{0}} t_x^{-1}$, and all the isotropy groups H_x are conjugate in $E(n)$ and isomorphic to $O(n)$.

Note that the left cosets $t_x H_{\vec{0}}$ have the form

$$t_x H_{\vec{0}} = \left\{ \begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A & x \\ 0 & 1 \end{bmatrix} : A \in O(n) \right\}.$$

Thus the space $\mathbb{E}^n$ is isomorphic to the set of left cosets of the subgroup $O(n) \subset E(n)$. In fact, if we give $E(n)/O(n)$ the quotient topology, then we have a diffeomorphism $\mathbb{E}^n \cong E(n)/O(n)$.

2. Moving frames on Euclidean space

Another way to look at all this is in terms of frames on $\mathbb{E}^n$. We define a *frame* on $\mathbb{E}^n$ to be a set of vectors $(x; e_1, \ldots, e_n)$ where $x \in \mathbb{E}^n$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis for the tangent space to $\mathbb{E}^n$ at x. If we regard the vectors $e_1, \ldots, e_n$ as the columns of a matrix $A \in O(n)$, we see that the set of frames on $\mathbb{E}^n$ is isomorphic to the Euclidean group $E(n)$: the vector x represents the translation component, and the matrix A represents the rotation component. Regarded in this way, we can define a projection map $\pi : E(n) \to \mathbb{E}^n$ by

$$\pi(x; e_1, \ldots, e_n) = x.$$

The fiber of this map is the set of all orthonormal frames at x and so is isomorphic to $O(n)$. This map describes $E(n)$ as a principal bundle over $\mathbb{E}^n$ with fiber $O(n)$, called the *frame bundle* of $\mathbb{E}^n$ and sometimes denoted $\mathcal{F}(\mathbb{E}^n)$.

The vector components $x, e_1, \ldots, e_n$ may all be thought of as $\mathbb{E}^n$-valued functions on $E(n)$. Thus their exterior derivatives dx, de_i should be $T\mathbb{E}^n$-valued 1-forms on $E(n)$. Since $\{e_1, \ldots, e_n\}$ is a basis for the tangent space to $\mathbb{E}^n$ at each point, these exterior derivatives can be expressed as linear combinations of $e_1, \ldots, e_n$ whose coefficients are ordinary scalar-valued 1-forms. It will become clear over the next several lectures that *the power of the method of moving frames lies in expressing the derivatives of a frame in terms of the frame itself.* We define 1-forms ω^i, ω_i^j, $1 \le i, j \le n$ by the equations

$$dx = \sum_{i=1}^{n} e_i\,\omega^i \tag{2.1}$$
$$de_i = \sum_{j=1}^{n} e_j\,\omega^j_i.$$

Since $dx : T_x\mathbb{E}^n \to T_x\mathbb{E}^n$ is the identity map, the 1-forms $\omega^1, \ldots, \omega^n$ form a basis for the 1-forms on $\mathbb{E}^n$. They are often called the *dual forms* of the frame $\{e_1, \ldots, e_n\}$. (For instance, if we take e_i to be the ith standard basis vector ε_i for $i = 1, \ldots, n$, then $\omega^i = dx^i$.) The dual forms have the property that $\omega^i(v) = 0$ for any vector v which is tangent to the fibers of the projection $\pi : E(n) \to \mathbb{E}^n$; we say that forms with this property are *semi-basic* for the projection π. The ω^i_j, on the other hand, form a basis for the 1-forms on each fiber of π. They are often called the *connection forms* of the frame $\{e_1, \ldots, e_n\}$.

Differentiating equations (2.1) shows that the forms ω^i, ω^j_i satisfy the *structure equations*

$$d\omega^i = -\sum_{j=1}^{n} \omega^i_j \wedge \omega^j \tag{2.2}$$
$$d\omega^i_j = -\sum_{k=1}^{n} \omega^i_k \wedge \omega^k_j.$$

We have not yet used the fact that we have a Euclidean strcture on $\mathbb{E}^n$. Since the e_i are orthonormal vectors, we have

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Differentiating these equations shows that the ω^i_j are skew-symmetric; i.e., $\omega^i_j = -\omega^j_i$.

This can all be described in terms of the Lie group $E(n)$. Let $\mathfrak{e}(n)$ denote the tangent space to $E(n)$ at the identity element; $\mathfrak{e}(n)$ is called the *Lie algebra* of $E(n)$, and it is not difficult to show that

$$\mathfrak{e}(n) = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a \text{ is a skew-symmetric } n \times n \text{ matrix and } b \in \mathbb{E}^n \right\}.$$

Let $g : E(n) \to E(n)$ represent the identity map, and consider the 1-form

$$\omega = g^{-1}\,dg.$$

ω is an $\mathfrak{e}(n)$-valued 1-form, called the *Maurer-Cartan form* of $E(n)$, and it is *left-invariant*. This means that if $h \in E(n)$ and $L_h : E(n) \to E(n)$ denotes left multiplication by h, then $L_h^*\omega = \omega$. As a consequence, ω is completely determined by its values at the identity. Moreover, the matrix entries of ω form a basis for the left-invariant scalar 1-forms on $E(n)$.

It is straightforward to check that ω satisfies the *Maurer-Cartan* equation

$$d\omega = -\omega \wedge \omega.$$

The entries of ω are exactly the scalar 1-forms ω^i, ω^i_j. For instance, when $n = 3$, we have

$$\omega = \begin{bmatrix} 0 & \omega^1_2 & -\omega^3_1 & \omega^1 \\ -\omega^1_2 & 0 & \omega^2_3 & \omega^2 \\ \omega^3_1 & -\omega^2_3 & 0 & \omega^3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is not difficult to show that the Maurer-Cartan equation is equivalent to the structure equations given above for the ω^i, ω^i_j.

3. Affine spaces

Now we consider a different structure on $\mathbb{R}^n$. If $\mathbb{R}^n$ is endowed with a *volume form* dV (i.e., a measure of the volume of the parallelepiped spanned by any n vectors $v_1, \dots, v_n$), we say that it is an *affine space* (or *special affine space*) and denote it $\mathbb{A}^n$. Affine space has less structure than Euclidean space; there is no inner product, and thus no way of measuring distances or angles between vectors.

Question: What are the symmetries of affine space? Since there is less structure that must be preserved, we might expect that the symmetry group would be larger. Indeed, the transformations $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ which preserve the affine structure are those of the form

$$\varphi(x) = Ax + b$$

where $A \in SL(n)$ and $b \in \mathbb{A}^n$. Just as in the Euclidean case, these transformations form a Lie group, called the *affine group* $A(n)$, which can be represented as

$$A(n) = \left\{ \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} : A \in SL(n),\ b \in \mathbb{A}^n \right\}.$$

Now the isotropy group $H_{\vec{0}}$ is

$$H_{\vec{0}} = \left\{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : A \in SL(n) \right\}$$

and the isotropy groups H_x are all conjugate and isomorphic to $SL(n)$. By the same reasoning as in the Euclidean case, the affine space $\mathbb{A}^n$ is isomorphic to $A(n)/SL(n)$, the set of left cosets of $SL(n)$ in $A(n)$.

A *frame* on affine space $\mathbb{A}^n$ is a set of vectors $(x; e_1, \dots, e_n)$ where $x \in \mathbb{A}^n$ and $\{e_1, \dots, e_n\}$ is any basis for the tangent space to $\mathbb{A}^n$ at x which spans a parallelepiped of volume 1. (Such a frame is called *unimodular*.) By the same reasoning as in the Euclidean case, the set of frames is isomorphic to the affine group $A(n)$. We can define a projection map $\pi : A(n) \to \mathbb{A}^n$ by

$$\pi(x; e_1, \dots, e_n) = x.$$

The fiber of this map is the set of all affine frames at x and so is isomorphic to $SL(n)$. This map describes $A(n)$ as a principal bundle over $\mathbb{A}^n$ with fiber $SL(n)$.

The Maurer-Cartan forms are defined as in the Euclidean case by the equations

$$dx = \sum_{i=1}^{n} e_i\, \omega^i$$
$$de_i = \sum_{j=1}^{n} e_j\, \omega_i^j.$$

The structure equations are the same as in Euclidean case; the only difference is that without the Euclidean structure it is no longer true that the ω_j^i are skew-symmetric. However, differentiating the equation

$$e_1 \wedge \cdots \wedge e_n = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$$

shows that

$$\sum_{i=1}^{n} \omega_i^i = 0.$$

4. Homogenous spaces

For both Euclidean and affine spaces, we began with the vector space $\mathbb{R}^n$ endowed with a certain structure, and by considering the symmetries of the structure we arrived at a description of the space as the set of left cosets of a closed subgroup H of some Lie group G. In both cases, the group G ($E(n)$ or $A(n)$) was the group of symmetries of the structure, and the subgroup H ($O(n)$ or $SL(n)$) was the isotropy group of a particular point in the space. In general, we define a *homogenous space* to be the set of left cosets of a closed subgroup H of a Lie group G. (We will restrict our attention to the case where G is a subgroup of some matrix group $GL(m)$.) G acts on G/H by left multiplication in the obvious way, and we are generally interested in those properties (such as the inner product in the Euclidean case and the volume form in the affine case) which are preserved under this action. For any point $x = gH \in G/H$, the isotropy group H_x is the group $H_x = gHg^{-1}$, which is clearly conjugate to H in G. Moreover, the projection map $\pi : G \to G/H$ defined by

$$\pi(g) = gH$$

describes G as a principal bundle over G/H with fiber H.

Just as we identified the groups $E(n)$ and $A(n)$ with the set of frames on Euclidean and affine space, respectively, in general we can regard the group G as the set of "frames" on the space G/H. The Maurer-Cartan form $\omega = g^{-1}\, dg$ is well-defined on G; it takes values in the Lie algebra $\mathfrak{g}$ of G, and it satisfies the same structure equation as before:

$$d\omega = -\omega \wedge \omega.$$

This equation will turn out to play a crucial role in the geometry of submanifolds of the space G/H.

Exercises

1. a) Prove that the Lie algebra of the Lie group $E(n)$ is

$$\mathfrak{e}(n) = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a \text{ is a skew-symmetric } n \times n \text{ matrix and } b \in \mathbb{E}^n \right\}.$$

b) Prove that the Lie algebra of the Lie group $A(n)$ is

$$\mathfrak{a}(n) = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a \text{ is an } n \times n \text{ matrix with trace}(a) = 0 \text{ and } b \in \mathbb{A}^n \right\}.$$

2. Show that the Maurer-Cartan equation $d\omega = -\omega \wedge \omega$ is equivalent to the structure equations (2.2).

3. Suppose that we choose a particular frame $\{e_i(x)\}$ for $T_x\mathbb{E}^n$ at each point $x \in \mathbb{E}^n$. This amounts to choosing a section $\sigma : \mathbb{E}^n \to E(n)$ of the frame bundle $\pi : E(n) \to \mathbb{E}^n$. The pullbacks $\sigma^*(\omega^i), \sigma^*(\omega^i_j)$ are then 1-forms on $\mathbb{E}^n$. (We will generally omit the σ^* and denote these pullbacks by ω^i, ω^i_j if the context is clear.) Show that if

$$\begin{bmatrix} e_1 & \dots & e_n \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \dots & \varepsilon_n \end{bmatrix} A(x)$$

(here ε_i represents the ith standard basis vector), the dual forms and connection forms of the frame are

$$\begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix} = A(x)^{-1} \begin{bmatrix} dx^1 \\ \vdots \\ dx^n \end{bmatrix} \qquad \begin{bmatrix} \omega^1_1 & \dots & \omega^1_n \\ \vdots & & \vdots \\ \omega^n_1 & \dots & \omega^n_n \end{bmatrix} = A(x)^{-1}\, dA(x).$$

4. Consider $\mathbb{E}^3$ with cylindrical coordinates (r, θ, z). Apply the result of Exercise 3 to the cylindrical frame field

$$\begin{aligned} e_1 &= (\cos\theta)\varepsilon_1 + (\sin\theta)\varepsilon_2 \\ e_2 &= (-\sin\theta)\varepsilon_1 + (\cos\theta)\varepsilon_2 \\ e_3 &= \varepsilon_3 \end{aligned}$$

to compute the dual forms ω^i and the connection forms ω^i_j for this frame field. Show by direct computation that these forms satisfy the structure equations (2.2).

5. Repeat Exercise 4 for the spherical frame field

$$\begin{aligned} e_1 &= (\cos\varphi\cos\theta)\varepsilon_1 + (\cos\varphi\sin\theta)\varepsilon_2 + (\sin\varphi)\varepsilon_3 \\ e_2 &= (-\sin\theta)\varepsilon_1 + (\cos\theta)\varepsilon_2 \\ e_3 &= (-\sin\varphi\cos\theta)\varepsilon_1 - (\sin\varphi\sin\theta)\varepsilon_2 + (\cos\varphi)\varepsilon_3 \end{aligned}$$

where (ρ, φ, θ) are spherical coordinates on $\mathbb{E}^3$.

6. *Elliptic space* S^n. Let S^n represent the unit sphere in $\mathbb{E}^{n+1}$. The symmetry group of S^n is defined to be the subgroup of $E(n+1)$ which preserves S^n.

a) Show that the symmetry group of S^n is naturally isomorphic to $O(n+1)$.

b) Given an element $g \in O(n+1)$, let $\{e_0, \ldots, e_n\}$ represent the columns of the matrix g. Define a map $\pi : O(n+1) \to S^n$ by

$$\pi([e_0 \ \ldots \ e_n]) = e_0.$$

Show that this map describes $O(n+1)$ as a principal bundle over S^n whose fibers are isomorphic to $O(n)$. Thus $S^n \cong O(n+1)/O(n)$.

c) Let $\omega^\alpha_\beta = -\omega^\beta_\alpha,\ 0 \le \alpha, \beta \le n$ be the Maurer-Cartan forms on $O(n+1)$. Show that the forms $\omega^i_0,\ 1 \le i \le n$ are semi-basic for the projection $\pi : O(n+1) \to S^n$ and so may be regarded as the dual forms of any frame field $\{e_1, \ldots, e_n\}$ on S^n. Set $\omega^i = \omega^i_0$. The forms $\{\omega^i_j = -\omega^j_i,\ 1 \le i, j \le n\}$ may then be regarded as the connection forms. Show that these forms satisfy the structure equations

$$\begin{aligned} d\omega^i &= -\sum_{j=1}^n \omega^i_j \wedge \omega^j \\ d\omega^i_j &= -\sum_{k=1}^n \omega^i_k \wedge \omega^k_j + \omega^i \wedge \omega^j. \end{aligned}$$

These look *almost* like the structure equations for $\mathbb{E}^n$. The extra term in the equation for $d\omega^i_j$ reflects the fact that S^n has sectional curvature identically equal to 1, whereas $\mathbb{E}^n$ is flat.