LECTURE 5: SURFACES IN PROJECTIVE SPACE

1. Projective space

Definition: The n-dimensional *projective space* $\mathbb{P}^n$ is the set of lines through the origin in the vector space $\mathbb{R}^{n+1}$.

$\mathbb{P}^n$ may be thought of as the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$ where $\sim$ represents the equivalence relation

$$(x^0, \dots, x^n) \sim (\lambda x^0, \dots, \lambda x^n), \qquad \lambda \in \mathbb{R}^*.$$

The equivalence class of the point $(x^0, \dots, x^n)$ is denoted $[x^0, \dots, x^n]$.

In order to describe $\mathbb{P}^n$ as a homogenous space, we need to find its group of symmetries. Since the only structure on $\mathbb{P}^n$ is that of lines through the origin in $\mathbb{R}^{n+1}$, we should begin by finding those symmetries of $\mathbb{R}^{n+1}$ that preserve the set of lines through the origin. This is simply the matrix group $GL(n+1)$, so we might suppose that the group of symmetries of $\mathbb{P}^n$ is also $GL(n+1)$.

However, there is a subtle point to consider here. While it is true that all elements of $GL(n+1)$ are symmetries of $\mathbb{P}^n$, some of them act *trivially* on $\mathbb{P}^n$. A matrix $g \in GL(n+1)$ fixes every line in $\mathbb{R}^{n+1}$ if and only if $g = \lambda I$ for some $\lambda \neq 0$. Thus the most natural choice for the symmetry group of $\mathbb{P}^n$ is $GL(n+1)/\mathbb{R}^* I$. This group is isomorphic to $SL(n+1)$ if n is even and $SL(n+1)/\{\pm I\}$ if n is odd. In order to avoid the difficulties associated with working with a quotient group, we will take the symmetry group of $\mathbb{P}^n$ to be $SL(n+1)$ in either case.

Now given a point $[x] = [x^0, \dots, x^n] \in \mathbb{P}^n$, we need to find its isotropy group $H_{[x]}$. First take $[x_0] = [1, 0, \dots, 0]$. It is straightforward to show that for $g \in SL(n+1)$, $g \cdot [x_0] = [x_0]$ if and only if

$$g = \begin{bmatrix} (\det A)^{-1} & r_1 & \dots & r_n \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{bmatrix}$$

where $A \in GL(n)$. Thus

$$H_{[x_0]} = \{ \begin{bmatrix} e_0 & \dots & e_n \end{bmatrix} : e_0 = (\lambda, 0, \dots, 0) \text{ for some } \lambda \in \mathbb{R}^* \}.$$

Denote this group by H. For any other point $[x] \in \mathbb{P}^n$, $H_{[x]}$ is conjugate to H, and $\mathbb{P}^n$ is isomorphic to the set of left cosets of H in $SL(n+1)$. Thus $\mathbb{P}^n$ may be thought of as the homogenous space $\mathbb{P}^n \cong SL(n+1)/H$.

A *frame* on $\mathbb{P}^n$ is a set of vectors $(e_0, \dots, e_n)$, $e_i \in \mathbb{R}^{n+1}$, with $\det[e_0 \dots e_n] = 1$. We can regard $SL(n+1)$ as the frame bundle of $\mathbb{P}^n$; it is a principal bundle with fibers isomorphic to H. We can define a projection map $\pi : SL(n+1) \to \mathbb{P}^n$ by

$$\pi([e_0 \dots e_n]) = [e_0].$$

The Maurer-Cartan forms $\{\omega^\alpha_\beta,\ 0 \le \alpha, \beta \le n\}$ on $SL(n+1)$ are defined by the equations

$$de_\alpha = \sum_{\beta=0}^{n} e_\beta\, \omega^\beta_\alpha.$$

These forms satisfy the structure equations

$$d\omega^\alpha_\beta = -\sum_{\gamma=0}^{n} \omega^\alpha_\gamma \wedge \omega^\gamma_\beta$$

and the single relation

$$\sum_{\alpha=0}^{n} \omega^\alpha_\alpha = 0.$$

The forms $\omega^1_0, \dots, \omega^n_0$ are semi-basic for the projection $\pi : SL(n+1) \to \mathbb{P}^n$, while the remaining ω^α_β's form a basis for the 1-forms on each fiber of π and so may be thought of as connection forms on the frame bundle.

2. Surfaces in $\mathbb{P}^3$

Consider a smooth, embedded surface $[x] : \Sigma \to \mathbb{P}^3$, where Σ is an open set in $\mathbb{R}^2$. Because $\mathbb{P}^3 = \mathbb{R}^4/\sim$ is a quotient space, it is generally easier to work with the 3-dimensional submanifold $\tilde{\Sigma} \subset \mathbb{R}^4 \setminus \{0\}$ defined by the property that $x \in \tilde{\Sigma}$ if and only if $[x] \in \Sigma$. Clearly $\tilde{\Sigma}$ consists of a 2-parameter family of lines through the origin of $\mathbb{R}^4$ and so may be thought of as a cone over a 2-dimensional submanifold of $\mathbb{R}^4 \setminus \{0\}$. We will use the geometry of the surface to construct an adapted frame $\{e_0(x), e_1(x), e_2(x), e_3(x)\} \in SL(4)$ at each point $x \in \tilde{\Sigma}$.

For our first frame adaptation we will choose a frame at each point $x \in \tilde{\Sigma}$ such that $e_0 = x$ and $T_x\tilde{\Sigma}$ is spanned by the vectors e_0, e_1, e_2. These conditions are clearly invariant under the action of $SL(4)$ on $\mathbb{R}^4$, and any other frame $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ has the form

$$\begin{bmatrix}\tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3\end{bmatrix} = \begin{bmatrix}e_0 & e_1 & e_2 & e_3\end{bmatrix}\begin{bmatrix}1 & s_1 & s_2 & s_3 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} & s_4 \\ 0 & & & s_5 \\ 0 & 0 & 0 & (\det B)^{-1}\end{bmatrix}$$

where $B \in GL(2)$. For such a frame, dx must be a linear combination of e_0, e_1, e_2. Therefore the structure equation

$$dx = de_0 = \sum_{\beta=0}^{3} e_\beta\, \omega_0^\beta$$

implies that $\omega_0^3 = 0$, while the 1-forms $\omega_0^0, \omega_0^1, \omega_0^2$ form a basis for the 1-forms on $\tilde{\Sigma}$. Thus we have $d\omega_0^3 = 0$, and so

$$0 = d\omega_0^3 = -\omega_1^3 \wedge \omega_0^1 - \omega_2^3 \wedge \omega_0^2.$$

By Cartan's Lemma, there exist functions h_{11}, h_{12}, h_{22} such that

$$\begin{bmatrix}\omega_1^3 \\ \omega_2^3\end{bmatrix} = \begin{bmatrix}h_{11} & h_{12} \\ h_{12} & h_{22}\end{bmatrix}\begin{bmatrix}\omega_0^1 \\ \omega_0^2\end{bmatrix}.$$

In order to make our next frame adaptation we will compute how the matrix $[h_{ij}]$ varies if we choose a different frame. Suppose that $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is defined as above. Computing the Maurer-Cartan form of the new frame shows that

$$\begin{bmatrix}\tilde{\omega}_0^1 \\ \tilde{\omega}_0^2\end{bmatrix} = B^{-1}\begin{bmatrix}\omega_0^1 \\ \omega_0^2\end{bmatrix}, \qquad \begin{bmatrix}\tilde{\omega}_1^3 \\ \tilde{\omega}_2^3\end{bmatrix} = (\det B)B^t\begin{bmatrix}\omega_1^3 \\ \omega_2^3\end{bmatrix}$$

and therefore

$$\begin{bmatrix}\tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_{22}\end{bmatrix} = (\det B)B^t\begin{bmatrix}h_{11} & h_{12} \\ h_{12} & h_{22}\end{bmatrix}B.$$

This transformation has the property that $\det[\tilde{h}_{ij}] = (\det B)^4 \det[h_{ij}]$, so the sign of the determinant is fixed. We will assume that $\det[h_{ij}] > 0$; in this case the surface is said to be *elliptic*. Then we can choose the matrix B so that $[h_{ij}]$ is the identity matrix. This determines the frame up to a transformation of the form

$$\begin{bmatrix}\tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3\end{bmatrix} = \begin{bmatrix}e_0 & e_1 & e_2 & e_3\end{bmatrix}\begin{bmatrix}1 & s_1 & s_2 & s_3 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} & s_4 \\ 0 & & & s_5 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

with $B \in SO(2)$.

The quadratic form

$$I = \omega_1^3\, \omega_0^1 + \omega_2^3\, \omega_0^2 = (\omega_0^1)^2 + (\omega_0^2)^2$$

is now well-defined on $\tilde{\Sigma}$, but it is not well-defined on Σ; it varies by a constant multiple as we move along the fibers of the projection $\tilde{\Sigma} \to \Sigma$. Thus I determines a *conformal structure* on Σ which is invariant under the action of $SL(4)$.

The restricted Maurer-Cartan forms on our frame now have the property that $\omega_1^3 = \omega_0^1$, $\omega_2^3 = \omega_0^2$. Differentiating these equations yields

$$\begin{aligned}(2\omega_1^1 - \omega_0^0 - \omega_3^3) \wedge \omega_0^1 + (\omega_2^1 + \omega_1^2) \wedge \omega_0^2 = 0\\ (\omega_2^1 + \omega_1^2) \wedge \omega_0^1 + (2\omega_2^2 - \omega_0^0 - \omega_3^3) \wedge \omega_0^2 = 0.\end{aligned}$$

By Cartan's Lemma, there exist functions h_{111}, h_{112}, h_{122}, h_{222} such that

$$\begin{bmatrix} 2\omega_1^1 - \omega_0^0 - \omega_3^3 \\ \omega_2^1 + \omega_1^2 \\ 2\omega_2^2 - \omega_0^0 - \omega_3^3 \end{bmatrix} = \begin{bmatrix} h_{111} & h_{112} \\ h_{112} & h_{122} \\ h_{122} & h_{222} \end{bmatrix} \begin{bmatrix} \omega_0^1 \\ \omega_0^2 \end{bmatrix}.$$

In order to make further adaptations we need to compute how the h_{ijk}'s vary under a change of frame. This computation gets rather complicated, but we can make it simpler by breaking it down into two steps. Any two adapted frames at this stage vary by a composition of transformations of the form

$$\begin{bmatrix} \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1}$$

with $B \in SO(2)$ and

$$\begin{bmatrix} \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} 1 & s_1 & s_2 & s_3 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{I}} & s_4 \\ 0 & & & s_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.2}$$

First consider a change of frame of the form (2.2). It is left as an exercise that under such a change of frame,

$$\begin{aligned}\tilde{h}_{111} &= h_{111} + 3(s_1 - s_4)\\ \tilde{h}_{112} &= h_{112} + (s_2 - s_5)\\ \tilde{h}_{122} &= h_{122} + (s_1 - s_4)\\ \tilde{h}_{222} &= h_{222} + 3(s_2 - s_5).\end{aligned}$$

Thus we can choose the s_i so that $h_{122} = -h_{111}$, $h_{112} = -h_{222}$. For such a frame we have $\omega_0^0 + \omega_3^3 = \omega_1^1 + \omega_2^2 = 0$. (Exercise: why?) This condition is preserved under transformations of the form (2.1) and transformations of the form (2.2) with $s_4 = s_1,\ s_5 = s_2$. Transformations of the latter form fix all the h_{ijk}'s, while under a transformation of the form (2.1) we have

$$\begin{bmatrix} \tilde{h}_{111} \\ \tilde{h}_{222} \end{bmatrix} = B^3 \begin{bmatrix} h_{111} \\ h_{222} \end{bmatrix}$$

so the quantity $h_{111}^2 + h_{222}^2$ is invariant.

3. The case $h_{ijk} = 0$

Now suppose that $h_{111}^2 + h_{222}^2 \equiv 0$. Then we have

$$\omega_1^1 = \omega_2^2 = \omega_2^1 + \omega_1^2 = \omega_0^0 + \omega_3^3 = 0.$$

Differentiating these equations yields

$$\begin{aligned}
(\omega_1^0 - \omega_3^1) \wedge \omega_0^1 &= 0 \\
(\omega_2^0 - \omega_3^2) \wedge \omega_0^2 &= 0 \\
(\omega_2^0 - \omega_3^2) \wedge \omega_0^1 + (\omega_1^0 - \omega_3^1) \wedge \omega_0^2 &= 0 \\
-(\omega_1^0 - \omega_3^1) \wedge \omega_0^1 - (\omega_2^0 - \omega_3^2) \wedge \omega_0^2 &= 0.
\end{aligned}$$

The fourth equation is obviously a consequence of the first two. Applying Cartan's lemma to the first three of these equations shows that there exists a function λ such that

$$\begin{aligned}
\omega_1^0 - \omega_3^1 &= \lambda\, \omega_0^1 \\
\omega_2^0 - \omega_3^2 &= \lambda\, \omega_0^2.
\end{aligned}$$

Now consider a change of frame of the form

$$\begin{bmatrix} \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & s_3 \\ 0 & & & 0 \\ 0 & & I & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is left as an exercise that under this change of frame,

$$\tilde{\lambda} = \lambda - 2s_3.$$

Thus we can choose a frame with $\lambda = 0$, and for such a frame we have

$$\omega_3^1 = \omega_1^0, \qquad \omega_3^2 = \omega_2^0.$$

Differentiating these equations yields

$$\begin{aligned}
2\omega_3^0 \wedge \omega_0^1 &= 0 \\
2\omega_3^0 \wedge \omega_0^2 &= 0.
\end{aligned}$$

By Cartan's lemma we have $\omega_3^0 = 0$. Finally, differentiating this equation yields an identity.

At this point the Maurer-Cartan form for the reduced frame bundle is

$$\omega = \begin{bmatrix} \omega_0^0 & \omega_1^0 & \omega_2^0 & 0 \\ \omega_0^1 & 0 & \omega_2^1 & \omega_1^0 \\ \omega_0^2 & -\omega_2^1 & 0 & \omega_2^0 \\ 0 & \omega_0^1 & \omega_0^2 & -\omega_0^0 \end{bmatrix}.$$

We have not found a unique frame over each point of $\tilde{\Sigma}$, but since differentiating the structure equations yields no further relations, this is as far as the frame bundle can be reduced. What this means is that $\tilde{\Sigma}$ is itself a homogenous space G/H where G is the Lie group whose Lie algebra $\mathfrak{g}$ is the set of matrices with the symmetries of the Maurer-Cartan form above. All that remains is to identify this group G and to describe $\tilde{\Sigma}$ as a homogenous space G/H. Because $\tilde{\Sigma}$ is a homogenous space, perhaps it will not come as a surprise that $\tilde{\Sigma}$ is, up to a projective transformation, the cone over the sphere S^2. The details will be left to the exercises.

Exercises

1. Suppose that instead of being elliptic, $\tilde{\Sigma}$ has $h_{ij} = 0$. Prove that Σ is a plane in $\mathbb{P}^3$. (Hint: Σ is a plane if and only if $\tilde{\Sigma}$ is a hyperplane in $\mathbb{R}^4$. Show that the plane spanned by the vectors e_0, e_1, e_2 is constant, and that therefore $\tilde{\Sigma}$ must be contained in this plane.)

2. Suppose that $\Sigma \subset \mathbb{P}^3$ is elliptic and that we have adapted our frames so that $\omega_1^3 = \omega_0^1$, $\omega_2^3 = \omega_0^2$. Show that

a) The quadratic form I is well-defined on $\tilde{\Sigma}$.

b) $I_{\lambda x} = \lambda^2 I_x$, where I_x denotes the quadratic form I based at the point $x \in \tilde{\Sigma}$. (Hint: moving from x to λx corresponds to making a change of frame with $\tilde{e}_0 = \lambda e_0$. Since I is well-defined, the change of frame can otherwise be made as simple as possible for ease of computation. Set

$$\begin{bmatrix} \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \dfrac{1}{\lambda\mu^2} \end{bmatrix}$$

and show that in order to preserve the condition that $\omega_i^3 = \omega_0^i$, $i = 1, 2$, we must have $\mu = \pm 1$. Now show that under this change of frame, $\tilde{\omega}_0^i = \lambda \omega_0^i$, $i = 1, 2$ so that $I_{\lambda x} = \lambda^2 I_x$.)

3. Fill in the details of the frame computations:

a) Suppose that the surface $\Sigma \subset \mathbb{P}^2$ is elliptic and that we have restricted to frames for which $\omega_1^3 = \omega_0^1$, $\omega_2^3 = \omega_0^2$. Show that differentiating these equations yields

$$(2\omega_1^1 - \omega_0^0 - \omega_3^3) \wedge \omega_0^1 + (\omega_2^1 + \omega_1^2) \wedge \omega_0^2 = 0$$
$$(\omega_2^1 + \omega_1^2) \wedge \omega_0^1 + (2\omega_2^2 - \omega_0^0 - \omega_3^3) \wedge \omega_0^2 = 0$$

and that Cartan's Lemma implies that there exist functions $h_{111}, h_{112}, h_{122}, h_{222}$ such that

$$\begin{bmatrix} 2\omega_1^1 - \omega_0^0 - \omega_3^3 \\ \omega_2^1 + \omega_1^2 \\ 2\omega_2^2 - \omega_0^0 - \omega_3^3 \end{bmatrix} = \begin{bmatrix} h_{111} & h_{112} \\ h_{112} & h_{122} \\ h_{122} & h_{222} \end{bmatrix} \begin{bmatrix} \omega_0^1 \\ \omega_0^2 \end{bmatrix}.$$

b) Show that under a change of frame of the form (2.2),

$$\begin{aligned} \tilde{h}_{111} &= h_{111} + 3(s_1 - s_4) \\ \tilde{h}_{112} &= h_{112} + (s_2 - s_5) \\ \tilde{h}_{122} &= h_{122} + (s_1 - s_4) \\ \tilde{h}_{222} &= h_{222} + 3(s_2 - s_5). \end{aligned}$$

When we restrict to those frames for which $h_{122} = -h_{111}$, $h_{112} = -h_{222}$, why do we have $\omega_0^0 + \omega_3^3 = \omega_1^1 + \omega_2^2 = 0$?

c) Suppose that the invariant $h_{111}^2 + h_{222}^2$ vanishes identically. Show that differentiating the equations

$$\omega_1^1 = \omega_2^2 = \omega_2^1 + \omega_1^2 = \omega_0^0 + \omega_3^3 = 0$$

yields

$$\begin{aligned} (\omega_1^0 - \omega_3^1) \wedge \omega_0^1 &= 0 \\ (\omega_2^0 - \omega_3^2) \wedge \omega_0^2 &= 0 \\ (\omega_2^0 - \omega_3^2) \wedge \omega_0^1 + (\omega_1^0 - \omega_3^1) \wedge \omega_0^2 &= 0 \\ -(\omega_1^0 - \omega_3^1) \wedge \omega_0^1 - (\omega_2^0 - \omega_3^2) \wedge \omega_0^2 &= 0. \end{aligned}$$

Use Cartan's lemma to conclude that there exists a function λ such that

$$\begin{aligned} \omega_1^0 - \omega_3^1 &= \lambda\, \omega_0^1 \\ \omega_2^0 - \omega_3^2 &= \lambda\, \omega_0^2. \end{aligned}$$

d) Show that under a change of frame of the form

$$\begin{bmatrix} \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & s_3 \\ 0 & & & 0 \\ 0 & & I & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we have

$$\tilde{\lambda} = \lambda - 2s_3$$

so we can choose a frame for which $\lambda = 0$.

e) Show that differentiating the equations $\omega_3^1 = \omega_1^0$, $\omega_3^2 = \omega_1^0$ yields

$$2\omega_3^0 \wedge \omega_0^1 = 0$$
$$2\omega_3^0 \wedge \omega_0^2 = 0.$$

Use Cartan's lemma to conclude that $\omega_3^0 = 0$. Show that differentiating this equation yields no further relations among the ω_β^α's.

4. In this exercise we will show that when $h_{111}^2 + h_{222}^2 = 0$, Σ is a sphere in $\mathbb{P}^2$.

Let Q be the matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Q represents the quadratic form

$$q = (x^1)^2 + (x^2)^2 - 2x^0x^3$$

which is a Lorentzian metric on $\mathbb{R}^4$. The Lie Group $O(Q)$ is the group of matrices which preserves this metric; it is defined by

$$O(Q) = \{A \in GL(4) : {}^tAQA = Q\}$$

and is isomorphic to the Lie group $O(3,1)$.

a) Differentiate the equation

$${}^tAQA = Q$$

to show that the Lie algebra $\mathfrak{o}(Q)$ is defined by

$$\mathfrak{o}(Q) = \{a \in \mathfrak{gl}(4) : {}^taQ + Qa = 0.$$

b) Show that $\mathfrak{o}(Q)$ consists of the matrices of the form

$$\begin{bmatrix} a_0^0 & a_1^0 & a_2^0 & 0 \\ a_0^1 & 0 & a_2^1 & a_1^0 \\ a_0^2 & -a_2^1 & 0 & a_2^0 \\ 0 & a_0^1 & a_0^2 & -a_0^0 \end{bmatrix}.$$

Conclude that the reduced Maurer-Cartan form at the end of the lecture takes values in $\mathfrak{o}(Q)$.

c) Recall that for a given reduced frame

$$g(x) = \begin{bmatrix} e_0(x) & e_1(x) & e_2(x) & e_3(x) \end{bmatrix}$$

the Maurer-Cartan form is $\omega = g^{-1}dg$. Show that any reduced frame has the form

$$g(x) = CA(x)$$

where $C \in SL(4)$ is a constant matrix and $A(x) \in O(Q)$. C may be thought of as a symmetry of $\mathbb{P}^3$, so the surface $\tilde{\Sigma}$ is equivalent to the surface $C^{-1} \cdot \tilde{\Sigma}$. Thus we can assume that $g(x) \in O(Q)$.

d) Define a projection map $\pi : O(Q) \to \mathbb{R}^4 \setminus \{0\}$ by

$$\pi(\begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix}) = e_0.$$

Show that in the Lorentzian metric defined by Q, e_0 is a *null vector*, i.e., $\langle e_0, e_0 \rangle = 0$. Since the set of null vectors in $\mathbb{R}^4 \setminus \{0\}$ is 3-dimensional, $\tilde{\Sigma}$ must be an open subset of the hypersurface defined by the equation $\langle x, x \rangle = 0$. This hypersurface is the cone over the unit sphere S^2 of the hyperplane $\{x^0 = 1\} \subset \mathbb{R}^4$; therefore Σ is an open subset of the sphere in $\mathbb{P}^3$.