LECTURE 6: PSEUDOSPHERICAL SURFACES AND BÄCKLUND'S THEOREM

1. Line congruences

Let $G_1(\mathbb{E}^3)$ denote the Grassmanian of lines in $\mathbb{E}^3$. A *line congruence* in $\mathbb{E}^3$ is an immersed surface $L : U \to G_1(\mathbb{E}^3)$, where $U \subset \mathbb{R}^2$ is open. The points of the line $L(u,v)$ are given by

$$L(u,v) = \{y(u,v) + \lambda w(u,v) : \lambda \in \mathbb{R}\}$$

for some $y(u,v),\ w(u,v) \in \mathbb{E}^3$ with $|w(u,v)| = 1$.

A parametric curve $u = u(t),\ v = v(t)$ in U defines a ruled surface

$$X(t,\lambda) = y(u(t),v(t)) + \lambda w(u(t),v(t)) = y(t) + \lambda w(t)$$

belonging to the congruence. The surface is called *developable* if

$$\det \begin{bmatrix} w(t) & w'(t) & y'(t) \end{bmatrix} = 0.$$

This is a quadratic equation for $u'(t), v'(t)$. If it has distinct real roots, then the solutions of this equation define two distinct families of developable surfaces X. In the generic case each family consists of the tangent lines to a surface, and these two surfaces $\Sigma, \bar{\Sigma}$ are called the *focal surfaces* of the congruence. The congruence gives a mapping $f : \Sigma \to \bar{\Sigma}$ with the property that the congruence consists of lines which are tangent to both Σ and $\bar{\Sigma}$ and join $x \in \Sigma$ to $\bar{x} = f(x) \in \bar{\Sigma}$.

2. Bäcklund's Theorem

Definition: let L be a line congruence in $\mathbb{E}^3$ with focal surfaces $\Sigma, \bar{\Sigma}$, and let $f : \Sigma \to \bar{\Sigma}$ be the function defined above. The congruence is called *pseudospherical* if

1. The distance $r = |\bar{x} - x|$ is a constant independent of x.
2. The angle α between the surface normals $N(x), N(\bar{x})$ is a constant independent of x.

Bäcklund's Theorem: Suppose that L is a pseudospherical line congruence in $\mathbb{E}^3$ with focal surfaces $\Sigma, \bar{\Sigma}$. Then both Σ and $\bar{\Sigma}$ have constant negative Gauss curvature $K = -\dfrac{\sin^2 \alpha}{r^2}$. (Such surfaces are called *pseudospherical surfaces.*)

This theorem can be proved using local coordinates on the surfaces $\Sigma, \bar{\Sigma}$, but it is a computational mess. The proof can be greatly simplified by using the method of moving frames, because the frames can be adapted to the geometry of the problem in a way that local coordinates cannot. Whereas in previous lectures we have adapted our frames according to the geometry of a single surface, here we have to consider two surfaces and geometrical conditions relating them. We will use these considerations to choose frames on the surfaces $\Sigma, \bar{\Sigma}$.

Proof: Let $\{e_1, e_2, e_3\}$ be an orthonormal frame of $T_x\mathbb{E}^3$ at $x \in \Sigma$ such that e_3 is the unit normal to Σ at x (and hence e_1, e_2 span $T_x\Sigma$) and e_1 is the unit vector in the direction of $\bar{x} - x$. We can then define an orthonormal frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ of $T_{\bar{x}}\mathbb{E}^3$ by

$$\begin{aligned}
\bar{e}_1 &= e_1 \\
\bar{e}_2 &= (\cos\alpha)e_2 + (\sin\alpha)e_3 \\
\bar{e}_3 &= (-\sin\alpha)e_2 + (\cos\alpha)e_3
\end{aligned}$$

Note that $\bar{e}_3$ is the unit normal to $\bar{\Sigma}$ at $\bar{x}$.

A comment about the domain of definition of these frames may be in order. Since the line congruence gives a map $f : \Sigma \to \bar{\Sigma}$, we can think of the immersions $x : \Sigma \to \mathbb{E}^3$ and $\bar{x} = f \circ x : \Sigma \to \mathbb{E}^3$ as being defined on the same abstract surface Σ. Thus the pullback bundles $x^{-1}(T\mathbb{E}^3)$ and $\bar{x}^{-1}(T\mathbb{E}^3)$ are naturally isomorphic as vector bundles over Σ, and this is the setting where it makes sense to say $\bar{e}_1 = e_1$, etc. We will shortly have similar relations between the Maurer-Cartan forms restricted to Σ and $\bar{\Sigma}$. These will make sense because all the forms in question are really pullbacks via x or $\bar{x}$ and so are forms on the abstract surface Σ.

With frames chosen as above, the immersions $x : \Sigma \to \mathbb{E}^3$ and $\bar{x} : \bar{\Sigma} \to \mathbb{E}^3$ are related by the equation

$$\bar{x} = x + re_1.$$

Taking the exterior derivative of this equation yields

$$\begin{aligned}
d\bar{x} &= dx + r\,de_1 \\
&= e_1\,\omega^1 + e_2\,\omega^2 + r(e_2\,\omega_1^2 + e_3\,\omega_1^3) \\
&= e_1\,\omega^1 + e_2(\omega^2 + r\,\omega_1^2) + e_3(r\,\omega_1^3).
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
d\bar{x} &= \bar{e}_1\,\bar{\omega}^1 + \bar{e}_2\,\bar{\omega}^2 \\
&= e_1\,\bar{\omega}^1 + e_2(\cos\alpha\;\bar{\omega}^2) + e_3(\sin\alpha\;\bar{\omega}^2).
\end{aligned}$$

Comparing these equations yields

$$
\begin{aligned}
\bar{\omega}^1 &= \omega^1 \\
\cos\alpha\ \bar{\omega}^2 &= \omega^2 + r\,\omega_1^2 \qquad (2.1)\\
\sin\alpha\ \bar{\omega}^2 &= r\,\omega_1^3.
\end{aligned}
$$

The last two of these equations imply that

$$\omega^2 + r\omega_1^2 = r\cot\alpha\ \omega_1^3. \tag{2.2}$$

We will use the fact that the Gauss curvature $\bar{K}$ of $\bar{\Sigma}$ satisfies the equation

$$\bar{\omega}_1^3 \wedge \bar{\omega}_2^3 = \bar{K}\,\bar{\omega}^1 \wedge \bar{\omega}^2$$

to compute $\bar{K}$. Recall that for an adapted frame with $e_3 = 0$ on a surface $\Sigma \subset \mathbb{E}^3$ we have

$$
\begin{aligned}
\omega_1^3 &= h_{11}\,\omega^1 + h_{12}\,\omega^2 \\
\omega_2^3 &= h_{12}\,\omega^1 + h_{22}\,\omega^2
\end{aligned}
$$

for some functions h_{11}, h_{12}, h_{22}. (We will denote the corresponding functions for $\bar{\Sigma}$ by $\bar{h}_{11}, \bar{h}_{12}, \bar{h}_{22}$.) Since $\bar{\omega}^1, \bar{\omega}^2$ are linearly independent forms, the first and third equations of (2.1) imply that $h_{12} \neq 0$. Using equation (2.2) we compute that

$$
\begin{aligned}
\bar{\omega}_1^3 &= \langle d\bar{e}_1, \bar{e}_3\rangle \\
&= \langle de_1, (-\sin\alpha)e_2 + (\cos\alpha)e_3\rangle \\
&= (\cos\alpha)\,\omega_1^3 - (\sin\alpha)\,\omega_1^2 \\
&= \frac{\sin\alpha}{r}\omega^2 \\
\bar{\omega}_2^3 &= \langle d\bar{e}_2, \bar{e}_3\rangle \\
&= \langle (\cos\alpha)\ de_2 + (\sin\alpha)\ de_3,\ (-\sin\alpha)\ e_2 + (\cos\alpha)\ e_3\rangle \\
&= (\cos^2\alpha)\,\omega_2^3 - (\sin^2\alpha)\,\omega_3^2 \\
&= \omega_2^3
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\bar{\omega}_1^3 \wedge \bar{\omega}_2^3 &= \frac{\sin\alpha}{r}\omega^2 \wedge \omega_2^3 \\
&= -\frac{\sin\alpha}{r}h_{12}\,\omega^1 \wedge \omega^2.
\end{aligned}
$$

But by the last equation in (2.1) we also have

$$
\begin{aligned}
\bar{\omega}_1^3 \wedge \bar{\omega}_2^3 &= \bar{K}\,\bar{\omega}^1 \wedge \bar{\omega}^2 \\
&= \bar{K}\,\omega^1 \wedge \left(\frac{r}{\sin\alpha}\omega_1^3\right) \\
&= \bar{K}\frac{r}{\sin\alpha}h_{12}\,\omega^1 \wedge \omega^2.
\end{aligned}
$$

Since $h_{12} \neq 0$, comparing coefficients yields

$$\bar{K} = -\frac{\sin^2 \alpha}{r^2}.$$

An analogous argument shows that $K = -\dfrac{\sin^2 \alpha}{r^2}$ as well. $\square$

The map $f : \Sigma \to \bar{\Sigma}$ given by the line congruence is called a *Bäcklund transformation* of the surface Σ.

3. Pseudospherical surfaces and the sine-Gordon equation

Let Σ be a pseudospherical surface, and for simplicity assume that its Gauss curvature is $K = -1$. Since the Gauss curvature of Σ is negative, Σ must have no umbilic points. Therefore every point $x \in \Sigma$ has a neighborhood on which there exists a local coordinate chart whose coordinate curves are principal curves in Σ. In such a coordinate system (u^1, u^2) we can choose an orthonormal frame $\{\underline{e}_1, \underline{e}_2\}$ with

$$\underline{e}_1 = \frac{1}{a_1}\frac{\partial}{\partial u^1} \quad \underline{e}_2 = \frac{1}{a_2}\frac{\partial}{\partial u^2}$$

for some nonvanishing functions a_1, a_2 on Σ. Then we have

$$\underline{\omega}^1 = a_1\, du^1, \qquad \underline{\omega}^2 = a_2\, du^2, \qquad \underline{\omega}_1^3 = \kappa_1 a_1\, du^1, \qquad \underline{\omega}_2^3 = \kappa_2 a_2\, du^2$$

where κ_1, κ_2 are the principal curvatures of Σ. The first and second fundamental forms of Σ are

$$I = (\underline{\omega}^1)^2 + (\underline{\omega}^2)^2 = (a_1)^2\, (du^1)^2 + (a_2)^2\, (du^2)^2$$
$$II = \underline{\omega}_1^3\, \underline{\omega}^1 + \underline{\omega}_2^3\, \underline{\omega}^2 = \kappa_1 (a_1)^2\, (du^1)^2 + \kappa_2 (a_2)^2\, (du^2)^2.$$

The structure equations of the Maurer-Cartan forms imply that

$$\underline{\omega}_2^1 = \frac{1}{a_2}\frac{\partial a_1}{\partial u^2}\, du^1 - \frac{1}{a_1}\frac{\partial a_2}{\partial u^1}\, du^2$$

and the Codazzi equations take the form

$$\frac{1}{\kappa_1 - \kappa_2}\frac{\partial \kappa_1}{\partial u^2} = -\frac{\partial(\ln a_1)}{\partial u^2}$$
$$\frac{1}{\kappa_2 - \kappa_1}\frac{\partial \kappa_2}{\partial u^1} = -\frac{\partial(\ln a_2)}{\partial u^1}.$$

Now since $K = -1$, we have $\kappa_1 \kappa_2 = -1$. Thus the Codazzi equations can be written as

$$\frac{\kappa_1}{\kappa_1(\kappa_1 - \kappa_2)}\frac{\partial \kappa_1}{\partial u^2} = -\frac{\partial(\ln a_1)}{\partial u^2}$$
$$\frac{\kappa_2}{\kappa_2(\kappa_2 - \kappa_1)}\frac{\partial \kappa_2}{\partial u^1} = -\frac{\partial(\ln a_2)}{\partial u^1},$$

or

$$\frac{1}{2}\frac{\partial(\ln(\kappa_1^2+1))}{\partial u^2} = -\frac{\partial(\ln a_1)}{\partial u^2}$$
$$\frac{1}{2}\frac{\partial(\ln(\kappa_2^2+1))}{\partial u^1} = -\frac{\partial(\ln a_2)}{\partial u^1}.$$

Therefore there must exist functions $c_1(u^1), c_2(u^2)$ such that

$$\kappa_i^2 + 1 = \frac{c_i(u^i)}{a_i^2}, \quad i = 1, 2.$$

Making a change of coordinates of the form $\tilde{u}^1 = \tilde{u}^1(u^1)$, $\tilde{u}^2 = \tilde{u}^2(u^2)$, we can assume that $c_i \equiv 1$. Then there exists a function ψ such that

$$\kappa_1 = \tan\psi, \qquad \kappa_2 = -\cot\psi, \qquad a_1 = \cos\psi, \qquad a_2 = \sin\psi,$$

so the first and second fundamental forms of Σ are

$$I = \cos^2\psi\,(du^1)^2 + \sin^2\psi\,(du^2)^2$$
$$II = \sin\psi\cos\psi\,((du^1)^2 - (du^2)^2).$$

From this we can compute that the angle between the asymptotic directions at any point is 2ψ. The connection form is

$$\underline{\omega}_2^1 = -\frac{\partial\psi}{\partial u^2}\,du^1 - \frac{\partial\psi}{\partial u^1}\,du^2$$

and the Gauss equation is equivalent to

$$\frac{\partial^2\psi}{\partial(u^1)^2} - \frac{\partial^2\psi}{\partial(u^2)^2} = \sin\psi\cos\psi.$$

In other words, the angle $\phi = 2\psi$ between the asymptotic directions satisfies the sine-Gordon equation

$$\frac{\partial^2\phi}{\partial(u^1)^2} - \frac{\partial^2\phi}{\partial(u^2)^2} = \sin\phi. \tag{3.1}$$

In fact, there is a one-to-one correspondence between local solutions ϕ of the sine-Gordon equation with $0 < \phi < \pi$ and local surfaces of constant Gauss curvature $K = -1$ in $\mathbb{E}^3$ up to rigid motion.

Exercises

1. Consider the change of coordinates

$$x = \frac{1}{2}(u^1 + u^2), \qquad y = \frac{1}{2}(u^1 - u^2)$$

where u^1, u^2 are the coordinates for which

$$I = \cos^2\psi\,(du^1)^2 + \sin^2\psi\,(du^2)^2$$
$$II = \sin\psi\cos\psi\,((du^1)^2 - (du^2)^2)$$

on the surface Σ with $K = -1$.

a) Show that x, y are asymptotic coordinates on Σ (this is equivalent to the statement that $II = f\, dx\, dy$ for some function f on Σ) and that the first and second fundamental forms on Σ are

$$\begin{aligned} I &= dx^2 + 2\cos(2\psi)\, dx\, dy + dy^2 \\ II &= 2\sin(2\psi)\, dx\, dy \end{aligned}$$

b) Show that the Maurer-Cartan forms corresponding to the principal frame $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ are

$$\begin{aligned} \underline{\omega}^1 &= \cos\psi(dx + dy) \\ \underline{\omega}^2 &= \sin\psi(dx - dy) \\ \underline{\omega}_1^3 &= \sin\psi(dx + dy) \\ \underline{\omega}_2^3 &= -\cos\psi(dx - dy) \\ \underline{\omega}_2^1 &= \frac{\partial\psi}{\partial y}\, dy - \frac{\partial\psi}{\partial x}\, dx. \end{aligned}$$

c) Show that in these coordinates, the sine-Gordon equation takes the form

$$\frac{\partial^2\phi}{\partial x \partial y} = \sin\phi.$$

2. Suppose that we have a Bäcklund transformation between two pseudo-spherical surfaces $\Sigma, \bar{\Sigma}$ with $K = -1$. Let $\{e_1, e_2, e_3\}$ be the frame adapted to the transformation as in the lecture, and let η denote the angle between e_1 and $\underline{e}_1$. Then we have

$$\begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 \end{bmatrix} \begin{bmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{bmatrix}.$$

a) Show that

$$\begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix} = \begin{bmatrix} \cos\eta & \sin\eta \\ -\sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} \underline{\omega}^1 \\ \underline{\omega}^2 \end{bmatrix}$$

$$\begin{bmatrix} \omega_1^3 \\ \omega_2^3 \end{bmatrix} = \begin{bmatrix} \cos\eta & \sin\eta \\ -\sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} \underline{\omega}_1^3 \\ \underline{\omega}_2^3 \end{bmatrix}$$

$$\omega_2^1 = \underline{\omega}_2^1 - d\eta.$$

b) Show that the Bäcklund equation (2.2) is equivalent to the first-order system of partial differential equations

$$\psi_x + \eta_x = \lambda \sin(\psi - \eta)$$
$$\psi_y - \eta_y = \frac{1}{\lambda} \sin(\psi + \eta)$$

where $\lambda = \cot\alpha - \csc\alpha$ is constant.

3. Suppose that $\psi(x, y)$, $\eta(x, y)$ are any two solutions of the PDE system

$$\psi_x + \eta_x = \lambda \sin(\psi - \eta) \tag{3.2}$$
$$\psi_y - \eta_y = \frac{1}{\lambda} \sin(\psi + \eta)$$

where $\lambda \neq 0$ is constant.

a) Show that the functions $2\psi, 2\eta$ must both be solutions of the sine-Gordon equation

$$\frac{\partial^2 \phi}{\partial x \partial y} = \sin\phi.$$

b) If 2ψ is any known solution of the sine-Gordon equation, then the system (3.2) is a compatible, overdetermined system for the unknown function η. Therefore it can be solved using only techniques of ordinary differential equations. The system (3.2) is called a *Bäcklund transformation* for the sine-Gordon solution. Suppose that ψ is the trivial solution $\psi(x, y) \equiv 0$. Show that the corresponding solutions η are

$$\eta(x, y) = 2\tan^{-1}(Ce^{-(\lambda x + \frac{1}{\lambda} y)})$$

where $C \neq 0$ is constant. (Hint: you may find the trig identity $\csc\eta + \cot\eta = \cot(\frac{1}{2}\eta)$ useful.) The functions

$$2\eta = 4\tan^{-1}(Ce^{-(\lambda x + \frac{1}{\lambda} y)})$$

are called the *1-soliton solutions* of the sine-Gordon equation. Iterating this procedure gives the 2-solitons, etc. The trivial solution $\psi = 0$ corresponds to the degenerate "surface" consisting of a straight line in $\mathbb{E}^3$, while the family of surfaces corresponding to the 1-solitons includes the classical pseudosphere.