LECTURE 7: MINIMALITY AND VARIATIONAL CALCULATIONS

1. Minimal surfaces in $\mathbb{E}^3$

We will say that a regular surface in $\mathbb{E}^3$ is *minimal* if it is locally area minimizing. More precisely, $\Sigma \subset \mathbb{E}^3$ is minimal if for any sufficiently small open set $U \subset \Sigma$, U has the minimum area of all surfaces in $\mathbb{E}^3$ with the same boundary as U. Classical examples are the plane, catenoid, and helicoid.

How would we go about finding minimal surfaces? If we define the *area functional* of a surface Σ to be

$$\mathcal{A}(\Sigma) = \int_\Sigma dA$$

then minimal surfaces should be critical points of this functional. But the space of surfaces in $\mathbb{E}^3$ is infinite-dimensional, so finding critical points of the functional $\mathcal{A}$ is somewhat complicated. The idea goes something like this: if Σ is a critical point of $\mathcal{A}$, then for any smooth curve $t \to \Sigma_t$ in the space of surfaces in $\mathbb{E}^3$ with $\Sigma_0 = \Sigma$ we should have

$$\frac{d}{dt}\,|_{t=0}\; \mathcal{A}(\Sigma_t) = 0.$$

Conversely, if Σ is not a critical point of $\mathcal{A}$ then there must exist a smooth curve $t \to \Sigma_t$ with $\Sigma_0 = \Sigma$ and $\frac{d}{dt}\,|_{t=0}\; \mathcal{A}(\Sigma_t) \neq 0$.

In order to make use of this idea we have to define what we mean by a smooth curve in the space of surfaces in $\mathbb{E}^3$. This leads us to the notion of a variation of a surface Σ. Given a regular surface $x : \Sigma \to \mathbb{E}^3$, consider a smooth map

$$X : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{E}^3$$

where $\Sigma_t = X(\Sigma, t) \subset \mathbb{E}^3$ is a regular surface for all $t \in (-\varepsilon, \varepsilon)$ and $\Sigma_0 = \Sigma$. Such a map is called a *variation* of Σ. The variation is said to be *compactly supported* if there is a compact set U in the interior of Σ such that for every $t \in (-\varepsilon, \varepsilon)$,

$$X(x, t) = X(x, 0)$$

for all $x \in \Sigma \setminus U$. If X is a compactly supported variation of Σ and Σ is a critical point of $\mathcal{A}$, then

$$\frac{d}{dt}\,|_{t=0}\; \mathcal{A}(\Sigma_t) = 0.$$

Conversely, if $\frac{d}{dt}\,|_{t=0}\ \mathcal{A}(\Sigma_t) = 0$ for *every* compactly supported variation of Σ, then Σ is a critical point of the functional $\mathcal{A}$. We will use this fact to investigate minimal surfaces.

Let $\{e_1, e_2, e_3\}$ be an orthonormal frame on Σ with e_3 normal to the tangent plane $T_x\Sigma$ at each point $x \in \Sigma$. Recall that when the Maurer-Cartan forms of $\mathbb{E}^3$ are restricted to this frame, we have $\omega^3 = 0$ and

$$\begin{bmatrix} \omega_1^3 \\ \omega_2^3 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}.$$

Rotating the frame $\{e_1, e_2\}$ changes the matrix $[h_{ij}]$, but its determinant $K = h_{11}h_{22} - h_{12}^2$ and its trace $2H = h_{11} + h_{22}$ are invariant under such changes of frame. K is the Gauss curvature of Σ at x, and H is called the *mean curvature* of Σ at x.

Now consider a compactly supported variation

$$X : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{E}^3.$$

Since reparametrizing the surface does not affect the area functional, we can assume that X is a *normal variation* of Σ. This means that the vector $\dfrac{\partial X}{\partial t}$ is parallel to the unit normal of the surface Σ_t at each point. In order to compute $\frac{d}{dt}\,|_{t=0}\ \mathcal{A}(\Sigma_t)$, we will define a frame on the variation X (i.e., a lifting $\tilde{X} : \Sigma \times (-\varepsilon, \varepsilon) \to E(3)$) and consider the restriction of the Maurer-Cartan forms on $\mathbb{E}^3$ to this frame. For each $(x, t) \in \Sigma \times (-\varepsilon, \varepsilon)$, let $\{e_1(x,t), e_2(x,t), e_3(x,t)\}$ be an orthonormal frame for the surface Σ_t at x with e_3 normal to the tangent plane $T_x\Sigma_t$. The restrictions of the forms $\omega^1, \omega^2, \omega^3$ to this frame are defined by the equation

$$dX = \sum_{i=1}^{3} e_i\, \omega^i.$$

Because $\{e_1, e_2, e_3\}$ is adapted to the surface Σ_t, the forms

$$\omega^1 = \langle dX, e_1 \rangle$$
$$\omega^2 = \langle dX, e_2 \rangle$$

are the usual dual forms on the surface Σ_t. But instead of having $\omega^3 = 0$, we have

$$\omega^3 = \langle dX, e_3 \rangle = |\frac{\partial X}{\partial t}|\, dt.$$

Set $f(x,t) = |\dfrac{\partial X}{\partial t}|$. Then $\omega^3 = f\, dt$. Differentiating this equation yields

$$-\omega_1^3 \wedge \omega^1 - \omega_2^3 \wedge \omega^2 = df \wedge dt,$$

and therefore

$$\omega_1^3 \wedge \omega^1 + \omega_2^3 \wedge \omega^2 + df \wedge dt = 0.$$

By Cartan's Lemma,

$$\begin{bmatrix}\omega_1^3\\ \omega_2^3\\ df\end{bmatrix} = \begin{bmatrix}h_{11} & h_{12} & f_1\\ h_{12} & h_{22} & f_2\\ f_1 & f_2 & f_3\end{bmatrix}\begin{bmatrix}\omega^1\\ \omega^2\\ dt\end{bmatrix}$$

for some functions $h_{11}, h_{12}, h_{22}, f_1, f_2, f_3$ on $\Sigma \times (-\varepsilon, \varepsilon)$. The h_{ij} are the coefficients of the second fundamental form of the surface Σ_t, while the f_i are the directional derivatives of f in the directions of the e_i.

Now the area form on the surface Σ_t is $dA = \omega^1 \wedge \omega^2$, so the area functional is

$$\mathcal{A}(\Sigma_t) = \int_{\Sigma_t} \omega^1 \wedge \omega^2.$$

Its derivative at $t = 0$ is

$$\begin{aligned}
\frac{d}{dt}\,|_{t=0}\ \mathcal{A}(\Sigma_t) &= \frac{d}{dt}\,|_{t=0} \int_{\Sigma_t} \omega^1 \wedge \omega^2\\
&= \int_\Sigma \mathcal{L}_{\partial/\partial t}(\omega^1 \wedge \omega^2)\\
&= \int_\Sigma \frac{\partial}{\partial t} \lrcorner\, d(\omega^1 \wedge \omega^2)\\
&= \int_\Sigma \frac{\partial}{\partial t} \lrcorner\, (-\omega_3^1 \wedge \omega^3 \wedge \omega^2 + \omega^1 \wedge \omega_3^2 \wedge \omega^3)\\
&= \int_\Sigma \frac{\partial}{\partial t} \lrcorner\, (-h_{11} - h_{22}) f\, \omega^1 \wedge \omega^2 \wedge dt\\
&= \int_\Sigma -2Hf\, \omega^1 \wedge \omega^2.
\end{aligned}$$

This computation shows that the derivative of the area functional at $t = 0$ is the integral over Σ of the function $H|\frac{\partial X}{\partial t}|$. This integral vanishes for all compactly supported normal variations of Σ if and only if the mean curvature H of Σ is identically zero.

We have proved the following theorem, which is often taken as a definition of minimal surfaces:

Theorem: A regular surface in $\mathbb{E}^3$ is minimal if and only if its mean curvature H is identically zero.

2. Minimal surfaces in $\mathbb{A}^3$

Recall that for elliptic surfaces $\Sigma \subset \mathbb{A}^3$ we found adapted frames $\{e_1, e_2, e_3\}$ for which the Maurer-Cartan forms satisfy the conditions

$$\omega_1^3 = \omega^1, \qquad \omega_2^3 = \omega^2, \qquad \omega_3^3 = 0.$$

Such a frame is determined up to a transformation of the form

$$\begin{bmatrix}\tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3\end{bmatrix} = \begin{bmatrix}e_1 & e_2 & e_3\end{bmatrix}\begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{B}} & 0 \\ & & 0 \\ 0 & 0 & 1\end{bmatrix}$$

with $B \in SO(2)$, and we also have

$$\begin{bmatrix}\omega_3^1 \\ \omega_3^2\end{bmatrix} = \begin{bmatrix}\ell_{11} & \ell_{12} \\ \ell_{12} & \ell_{22}\end{bmatrix}\begin{bmatrix}\omega^1 \\ \omega^2\end{bmatrix}$$

for some functions $\ell_{11}, \ell_{12}, \ell_{22}$. The quadratic forms

$$I = \omega_1^3\,\omega^1 + \omega_2^3\,\omega^2 = (\omega^1)^2 + (\omega^2)^2$$

$$II = \omega_3^1\,\omega^1 + \omega_3^2\,\omega^1 = \ell_{11}(\omega^1)^2 + 2\ell_{12}\,\omega^1\,\omega^2 + \ell_{22}(\omega^2)^2$$

are well-defined; the affine first fundamental form I defines a metric on Σ, and the affine second fundamental form II is the analog of the second fundamental form for surfaces in $\mathbb{E}^3$. Its trace

$$2L = \ell_{11} + \ell_{22}$$

is well-defined, and the quantity L is called the *affine mean curvature* of Σ.

The affine first fundamental form gives rise to a well-defined *affine area form* $dA = \omega^1 \wedge \omega^2$ on Σ, and we define the *affine area* of Σ to be

$$\mathcal{A}(\Sigma) = \int_\Sigma \omega^1 \wedge \omega^2.$$

By analogy with the Euclidean case, we can ask what properties a surface $\Sigma \subset \mathbb{A}^3$ must satisfy in order to be a critical point of this area functional. Such surfaces are called *affine minimal surfaces.*

We proceed as in the Euclidean case by considering a compactly supported normal variation

$$X : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{A}^3.$$

Here "normal" means that the vector $\dfrac{\partial X}{\partial t}$ is parallel to the affine normal of the surface $\Sigma_t = X(\Sigma, t)$ at each point. For each $(x,t) \in \Sigma \times (-\varepsilon, \varepsilon)$ let $\{e_1(x,t), e_2(x,t), e_3(x,t)\}$ be a frame for the surface Σ_t at x which is adapted as described above. As in the Euclidean case, the Maurer-Cartan forms ω^1, ω^2 are the usual dual forms on the surface Σ_t, while $\omega^3 = f\,dt$ where $f(x,t) = |\dfrac{\partial X}{\partial t}|$. The derivative of the affine area functional at $t = 0$

is

$$
\begin{aligned}
\frac{d}{dt}\,|_{t=0}\,\mathcal{A}(\Sigma_t) &= \frac{d}{dt}\,|_{t=0}\int_{\Sigma_t}\omega^1\wedge\omega^2\\
&= \int_\Sigma \mathcal{L}_{\partial/\partial t}(\omega^1\wedge\omega^2)\\
&= \int_\Sigma \frac{\partial}{\partial t}\lrcorner d(\omega^1\wedge\omega^2)\\
&= \int_\Sigma \frac{\partial}{\partial t}\lrcorner(-\omega^1_1\wedge\omega^1\wedge\omega^2-\omega^1_3\wedge\omega^3\wedge\omega^2\\
&\qquad\qquad +\omega^1\wedge\omega^2_2\wedge\omega^2+\omega^1\wedge\omega^2_3\wedge\omega^3)\\
&= \int_\Sigma \frac{\partial}{\partial t}\lrcorner(\ell_{11}+\ell_{22})f\,\omega^1\wedge\omega^2\wedge dt\\
&= \int_\Sigma 2Lf\,\omega^1\wedge\omega^2.
\end{aligned}
$$

This integral vanishes for all compactly supported normal variations of Σ if and only if the affine mean curvature L of Σ is identically zero. Thus we have proved the following theorem:

Theorem: A regular surface in $\mathbb{A}^3$ is affine minimal if and only if its affine mean curvature L is identically zero.

Exercises

1. The *catenoid* is the surface $\Sigma\subset\mathbb{E}^3$ obtained by rotating the curve $x=\cosh z$ about the z axis. It can be parametrized by

$$x(u,v)=(\cos(u)\cosh(v),\ \sin(u)\cosh(v),\ v).$$

a) Show that the frame

$$
\begin{aligned}
e_1 &= \frac{x_u}{|x_u|} = (-\sin(u),\ \cos(u),\ 0)\\
e_2 &= \frac{x_v}{|x_v|} = \frac{1}{\cosh(v)}(\cos(u)\sinh(v),\ \sin(u)\sinh(v),\ 1)\\
e_3 &= e_1\times e_2 = \frac{1}{\cosh(v)}(\cos(u),\ \sin(u),\ -\sinh(v))
\end{aligned}
$$

is orthonormal and that e_1, e_2 span the tangent space to Σ at each point.

b) Show that the dual forms of this frame are

$$\omega^1=\cosh(v)\,du,\qquad \omega^2=\cosh(v)\,dv.$$

c) Compute de_3 and show that

$$\omega_1^3 = -\frac{1}{\cosh(v)}\, du, \qquad \omega_2^3 = \frac{1}{\cosh(v)}\, dv.$$

Use this to compute the matrix $[h_{ij}]$ and show that the mean curvature of Σ is $H \equiv 0$.

2. Repeat the computation of Exercise 1 for an arbitrary surface of revolution parametrized by

$$x(u,v) = (f(v)\cos(u),\ f(v)\sin(u),\ v)$$

(you may want to use a computer algebra package such as Maple to assist with part c) and show that the surface is minimal if and only if f satisfies the differential equation

$$ff'' = (f')^2 + 1.$$

Show that the only solutions of this equation are

$$f(v) = \frac{1}{a}\cosh(av + b)$$

where a, b are constants. Conclude that catenoids are the only non-planar minimal surfaces of revolution.

3. The *helicoid* is the ruled surface $\Sigma \subset \mathbb{E}^3$ parametrized by

$$x(u,v) = (v\cos(u),\ v\sin(u),\ u).$$

a) Show that the frame

$$\begin{aligned}
e_1 &= \frac{x_u}{|x_u|} = \frac{1}{\sqrt{v^2+1}}(-v\sin(u),\ v\cos(u),\ 1)\\
e_2 &= \frac{x_v}{|x_v|} = (\cos(u),\ \sin(u),\ 0)\\
e_3 &= e_1 \times e_2 = \frac{1}{\sqrt{v^2+1}}(-\sin(u),\ \cos(u),\ -v)
\end{aligned}$$

is orthonormal and that e_1, e_2 span the tangent space to Σ at each point.

b) Show that the dual forms of this frame are

$$\omega^1 = \sqrt{v^2+1}\, du, \qquad \omega^2 = dv.$$

c) Compute de_3 and show that

$$\omega_1^3 = \frac{1}{v^2+1}\, dv, \qquad \omega_2^3 = \frac{1}{\sqrt{v^2+1}}\, du.$$

Use this to compute the matrix $[h_{ij}]$ and show that the mean curvature of Σ is $H \equiv 0$.

4. a) Show that for any values of a, b, c with $ac - b^2 > 0$ the elliptic paraboloid $z = a\,x^2 + b\,xy + c\,y^2$ is affinely equivalent to the paraboloid $\Sigma \subset \mathbb{A}^3$ given by $z = \frac{1}{2}(x^2 + y^2)$.

b) Show that the affine frame

$$\begin{aligned} e_1 &= (1, 0, x) \\ e_2 &= (0, 1, y) \\ e_3 &= (0, 0, 1) \end{aligned}$$

is adapted to Σ. Compute its dual and connection forms and show that they satisfy the conditions

$$\begin{aligned} \omega_1^3 &= \omega^1 \\ \omega_2^3 &= \omega^2 \\ \omega_3^3 &= 0. \end{aligned}$$

c) Show that the affine mean curvature of Σ is identically zero. Conclude that any elliptic paraboloid is affine minimal.

5. (This exercise should be done with the aid of a computer algebra package such as Maple.) Suppose that a surface $\Sigma \subset \mathbb{A}^3$ is described by a graph $z = f(x, y)$. Consider the affine frame

$$\begin{aligned} \underline{e}_1 &= (1, 0, f_x) \\ \underline{e}_2 &= (0, 1, f_y) \\ \underline{e}_3 &= (0, 0, 1). \end{aligned}$$

a) Show that the dual forms of this frame are

$$\underline{\omega}^1 = dx, \qquad \underline{\omega}^2 = dy, \qquad \underline{\omega}^3 = 0$$

and that the only nonzero connection forms are

$$\begin{aligned} \underline{\omega}_1^3 &= f_{xx}\, dx + f_{xy}\, dy \\ \underline{\omega}_2^3 &= f_{xy}\, dx + f_{yy}\, dy. \end{aligned}$$

Thus we have

$$\begin{bmatrix} \underline{h}_{11} & \underline{h}_{12} \\ \underline{h}_{12} & \underline{h}_{22} \end{bmatrix} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}.$$

Assume that $f_{xx} f_{yy} - f_{xy}^2 > 0$, so that Σ is elliptic, and for simplicity assume that $f_{xx} > 0$. In order to decide whether Σ is affine minimal, we must adapt frames so that

$$\begin{aligned} \omega_1^3 &= \omega^1 \\ \omega_2^3 &= \omega^2 \\ \omega_3^3 &= 0. \end{aligned}$$

Consider an affine change of frame

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \end{bmatrix} \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{B}} & r_1 \\ & & r_2 \\ 0 & 0 & (\det B)^{-1} \end{bmatrix}$$

with $B \in GL(2)$.

b) Show that if we take

$$B = \begin{bmatrix} \dfrac{(f_{xx}f_{yy} - f_{xy}^2)^{1/8}}{\sqrt{f_{xx}}} & -\dfrac{f_{xy}}{\sqrt{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2)^{3/8}} \\ 0 & \dfrac{\sqrt{f_{xx}}}{(f_{xx}f_{yy} - f_{xy}^2)^{3/8}} \end{bmatrix}$$

then

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\omega_1^3 = \omega^1, \ \omega_2^3 = \omega^2$.

c) Show that under this change of basis (with r_1, r_2 still arbitrary),

$$\begin{aligned} \omega_3^3 = &\left[\frac{f_{xx}r_1}{(f_{xx}f_{yy} - f_{xy}^2)^{1/4}} + \frac{f_{xy}r_2}{(f_{xx}f_{yy} - f_{xy}^2)^{1/4}} + \frac{f_{xx}f_{xyy} - 2f_{xy}f_{xxy} + f_{yy}f_{xxx}}{4(f_{xx}f_{yy} - f_{xy}^2)} \right] dx \\ &+ \left[\frac{f_{xy}r_1}{(f_{xx}f_{yy} - f_{xy}^2)^{1/4}} + \frac{f_{yy}r_2}{(f_{xx}f_{yy} - f_{xy}^2)^{1/4}} + \frac{f_{xx}f_{yyy} - 2f_{xy}f_{xyy} + f_{yy}f_{xxy}}{4(f_{xx}f_{yy} - f_{xy}^2)} \right] dy. \end{aligned}$$

Conclude that by choosing

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = - \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{f_{xx}f_{xyy} - 2f_{xy}f_{xxy} + f_{yy}f_{xxx}}{4(f_{xx}f_{yy} - f_{xy}^2)^{3/4}} \\ \dfrac{f_{xx}f_{yyy} - 2f_{xy}f_{xyy} + f_{yy}f_{xxy}}{4(f_{xx}f_{yy} - f_{xy}^2)^{3/4}} \end{bmatrix}$$

we can arrange that $\omega_3^3 = 0$.

d) Show that under this change of frame we have

$$\omega^1 = \frac{\sqrt{f_{xx}}}{(f_{xx}f_{yy} - f_{xy}^2)^{1/8}}\, dx + \frac{f_{xy}}{\sqrt{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2)^{1/8}}\, dy$$

$$\omega^2 = \frac{(f_{xx}f_{yy} - f_{xy}^2)^{3/8}}{\sqrt{f_{xx}}}\, dy$$

Also compute ω_3^1, ω_3^2 and find the functions ℓ_{ij} such that

$$\begin{bmatrix} \omega_3^1 \\ \omega_3^2 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{12} & \ell_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}.$$

Show that the affine mean curvature equation

$$L = \ell_{11} + \ell_{22} = 0$$

is a fourth-order differential equation for f. Pretty messy, huh?