LECTURE 8: WEIERSTRASS FORMULAE

1. The Weierstrass formula for minimal surfaces in $\mathbb{E}^3$

Weierstrass showed that any minimal surface in $\mathbb{E}^3$ could be described in terms of holomorphic functions of a complex variable. We will use adapted frames on the surface to derive this result.

Let $\{e_1, e_2, e_3\}$ be an orthonormal frame on a surface $\Sigma \subset \mathbb{E}^3$ with e_3 normal to the surface at each point. Recall that for such a frame we have

$$\begin{bmatrix} \omega_1^3 \\ \omega_2^3 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$$

for some functions h_{11}, h_{12}, h_{22} and that Σ is minimal if and only if $h_{11} + h_{22} = 0$.

Consider the complex vector-valued 1-form

$$\begin{aligned} \xi &= (e_1 - ie_2)(\omega^1 + i\omega^2) \\ &= (e_1\,\omega^1 + e_2\,\omega^2) + i(e_1\,\omega^2 - e_2\,\omega^1). \end{aligned}$$

ξ is well-defined independent of the choice of adapted frame, and its exterior derivative is

$$\begin{aligned} d\xi &= i(de_1 \wedge \omega^2 + e_1\,d\omega^2 - de_2 \wedge \omega^1 - e_2\,d\omega^1) \\ &= i(h_{11} + h_{22})e_3\,\omega^1 \wedge \omega^2. \end{aligned}$$

Therefore $d\xi = 0$ if and only if Σ is minimal.

Suppose that Σ is minimal and let z, f be local complex-valued functions on Σ such that

$$\omega^1 + i\omega^2 = f\,dz.$$

(Such functions always exist for any 1-form on a surface, although this is not true on higher-dimensional manifolds.) The function f must be everywhere nonzero, and since

$$dz \wedge d\bar{z} = -\frac{2i}{|f|^2}\,\omega^1 \wedge \omega^2 \neq 0,$$

we can regard z as a local complex coordinate on Σ; this defines a complex structure on Σ. If we define $F(z, \bar{z})$ to be the vector valued function

$$F = (e_1 - ie_2)f$$

then

$$\xi = F\,dz.$$

The fact that Σ is minimal, and hence that $d\xi = 0$, implies that F is a function of z alone and so is a holomorphic function on Σ. Moreover,

$$\langle F, F\rangle = f^2\langle e_1 - ie_2,\ e_1 - ie_2\rangle = 0.$$

Since ξ is a closed (1,0)-form, locally there exists a holomorphic function $X(z)$ such that $\xi = dX$ (and so $X'(z) = F(z)$), and

$$\mathrm{Re}(dX) = \mathrm{Re}(\xi) = e_1\,\omega^1 + e_2\,\omega^2 = dx$$

where x is the position vector on the surface Σ. Therefore, up to a translation in $\mathbb{E}^3$ we have

$$X(z) = x(z) + iy(z)$$

for some real vector-valued function y on Σ. Conversely, if $X(z)$ is any holomorphic $\mathbb{C}^3$-valued function with $\langle X', X'\rangle = 0$, then the surface $x = \mathrm{Re}(X)$ is a minimal surface in $\mathbb{E}^3$.

This gives rise to the *Weierstrass representation* for minimal surfaces. Let $U \subset \mathbb{C}$ be open, $g : U \to \mathbb{C}$ a meromorphic function, and $f : U \to \mathbb{C}$ a holomorphic function with the property that if g has a pole of order k at $z_0 \in U$ then f has a zero of order $2k$ at z_0. Choose $z_0 \in U$ and define $X : U \to \mathbb{C}^3$ by

$$X(z) = \int_{z_0}^{z} \begin{bmatrix} \frac{1}{2} f(\zeta)(1 - g(\zeta)^2) \\ \frac{i}{2} f(\zeta)(1 + g(\zeta)^2) \\ f(\zeta)g(\zeta) \end{bmatrix} d\zeta.$$

Then $\langle X', X'\rangle = 0$ and so $x = \mathrm{Re}(X)$ is the position vector of a minimal surface $\Sigma \subset \mathbb{E}^3$. Conversely, any minimal surface has a local representation of this form (up to translation) in a neighborhood of any point.

The first fundamental form of Σ may be written as

$$\begin{aligned} I &= \tfrac{1}{2}\langle \xi, \bar{\xi}\rangle \\ &= \tfrac{1}{2}\langle dX, d\bar{X}\rangle \\ &= (\omega^1 + i\omega^2)(\omega^1 - i\omega^2) \\ &= (\omega^1)^2 + (\omega^2)^2. \end{aligned}$$

This leads to the following observation. Let $t \in \mathbb{R}$, and set

$$X_t = e^{it}X.$$

The family of minimal surfaces Σ_t with position vector $x_t = \mathrm{Re}(X_t)$ is called the *associated family* of Σ. All the surfaces in this family clearly have the same first fundamental form and so are *isometric*. In particular, the surface $\Sigma_{3\pi/2}$ with position vector $y = \mathrm{Im}(X)$ is isometric to Σ; this surface is called the *conjugate surface* of Σ.

2. A Weierstrass-type formula for minimal surfaces in $\mathbb{A}^3$

Now let $\Sigma \subset \mathbb{A}^3$ be an elliptic surface, and let $\{e_1, e_2, e_3\}$ be an orthonormal frame on Σ for which the Maurer-Cartan forms satisfy the conditions

$$\omega_1^3 = \omega^1, \qquad \omega_2^3 = \omega^2, \qquad \omega_3^3 = 0.$$

Recall that for such a frame we have

$$\begin{bmatrix} \omega_3^1 \\ \omega_3^2 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{12} & \ell_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$$

$$\begin{bmatrix} 2\omega_1^1 \\ \omega_2^1 + \omega_1^2 \\ 2\omega_2^2 \end{bmatrix} = \begin{bmatrix} h_1 & -h_2 \\ -h_2 & -h_1 \\ -h_1 & h_2 \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$$

where $h_1 = h_{111} = -h_{122},\ h_2 = h_{222} = -h_{112}$, and that Σ is affine minimal if and only if $\ell_{11} + \ell_{22} = 0$.

Let $\mathbb{A}^3_{\mathbb{C}}$ denote the complexified affine space $\mathbb{A}^3 \otimes \mathbb{C}$, and consider the $\Lambda^2\mathbb{A}^3_{\mathbb{C}}$-valued 1-form

$$\begin{aligned} \xi &= \tfrac{1}{2} e_3 \wedge (e_1 - ie_2)(\omega^1 + i\omega^2) \\ &= \tfrac{1}{2} e_3 \wedge [(e_1\,\omega^1 + e_2\,\omega^2) + i(e_1\,\omega^2 - e_2\,\omega^1)]. \end{aligned}$$

ξ is well-defined independent of the choice of adapted frame, and a straightforward computation shows that its exterior derivative is

$$d\xi = \tfrac{1}{2}(\ell_{11} + \ell_{22})(e_1 \wedge e_2)\,\omega^1 \wedge \omega^2.$$

Therefore $d\xi = 0$ if and only if Σ is affine minimal.

Suppose that Σ is affine minimal and let z, f be complex-valued functions on Σ such that

$$\omega^1 + i\omega^2 = f\,dz.$$

By the same reasoning as in the Euclidean case, z can be thought of as a local complex coordinate on Σ, and locally there exists a holomorphic $\Lambda^2\mathbb{A}^3_{\mathbb{C}}$-valued function $X(z)$ on Σ such that $\xi = dX$.

For ease of notation, let

$$\begin{aligned} e &= \tfrac{1}{2}(e_1 - ie_2) \\ \omega &= \omega^1 + i\omega^2. \end{aligned}$$

Then $dX = \xi = e_3 \wedge e\,\omega$, and by conjugation $d\bar{X} = \bar{\xi} = e_3 \wedge \bar{e}\,\bar{\omega}$. A computation shows that

$$d(e \wedge \bar{e}) = \tfrac{1}{2}(e_3 \wedge \bar{e}\,\bar{\omega} - e_3 \wedge e\,\omega) = \tfrac{1}{2}(d\bar{X} - dX).$$

It follows that

$$\bar{X} - X = 2e \wedge \bar{e} + 2ic$$

for some real-valued constant $c \in \Lambda^2\mathbb{A}^3$. By adding an imaginary constant to X, we can assume that $c = 0$.

At this point we need the "special linear cross product". This is the unique skew-symmetric bilinear map

$$\times : \Lambda^2\mathbb{A}^3 \times \Lambda^2\mathbb{A}^3 \to \mathbb{A}^3$$

that is $SL(3)$-equivariant and satisfies

$$(e_1 \wedge e_2) \times (e_1 \wedge e_3) = e_1$$

for any unimodular basis $\{e_1, e_2, e_3\}$ of $\mathbb{A}^3$. Geometrically, we can think of $v_1 \wedge v_2 \in \Lambda^2\mathbb{A}^3$ as the plane spanned by v_1 and v_2. The cross product of $v_1 \wedge v_2$ and $w_1 \wedge w_2$ is a vector which spans the line of intersection of the two planes. It can be computed using the ordinary cross product formula in $\mathbb{R}^3$ by

$$(v_1 \wedge v_2) \times (w_1 \wedge w_2) = (v_1 \times v_2) \times (w_1 \times w_2).$$

This cross product can be extended in the obvious way to $\Lambda^2\mathbb{A}^3_{\mathbb{C}}$.

Now, using this formula for the cross product we compute that

$$(\bar{X} - X) \times d(\bar{X} + X) = -i(e\,\omega + \bar{e}\,\bar{\omega}) = -i\,dx$$

where x is the position vector of Σ. Therefore

$$\begin{aligned} dx &= i[(\bar{X} - X) \times d(\bar{X} + X)] \\ &= i[\bar{X} \times d\bar{X} - X \times dX + d(\bar{X} \times X)] \end{aligned}$$

and so the position vector x of the surface Σ is given by

$$x(z) = x(z_0) + i[\overline{X(z)} \times X(z) - \overline{X(z_0)} \times X(z_0) + \int_{z_0}^{z} (\bar{X} \times d\bar{X} - X \times dX)].$$

for some $z_0 \in \Sigma$.

Conversely, let $U \subset \mathbb{C}$ be open, and let $X : U \to \Lambda^2\mathbb{A}^3_{\mathbb{C}}$ be a holomorphic function that satisfies the open conditions $dX \neq 0$ and $\bar{X} \neq X$. Then the formula above gives the position vector x of an affine minimal surface Σ. This Weierstrass-type representation for affine minimal surfaces is due to Blaschke.

Exercises

1. In the Weierstrass representation for surfaces in $\mathbb{E}^3$, let $f(z) = 2$ and $g(z) = z$. Show that the resulting minimal surface is parametrized by

$$x(u, v) = \begin{bmatrix} u - \frac{1}{3}u^3 + uv^2 \\ -v + \frac{1}{3}v^3 - vu^2 \\ u^2 - v^2 \end{bmatrix}$$

where $z = u + iv$. This is called *Enneper's surface.* If you have access to a software package such as Maple, try sketching the surface over various intervals in u and v.

2. Recall from Lecture 7, Exercise 1 that the catenoid is parametrized by

$$x(u,v) = \begin{bmatrix} \cos u \cosh v \\ \sin u \cosh v \\ v \end{bmatrix}.$$

Show that the Weierstrass representation of the catenoid is obtained by taking $f(z) = -ie^{-iz}$, $g(z) = e^{iz}$, and that its conjugate surface is the helicoid. (Hint: the formula you'll find for the conjugate surface will require a change of parameters before it looks like the parametrization for the helicoid from Lecture 7, Exercise 2.)

3. Consider the affine elliptic paraboloid $z = \frac{1}{2}(x^2 + y^2)$ with its adapted frame

$$\begin{aligned} e_1 &= (1, 0, x) \\ e_2 &= (0, 1, y) \\ e_3 &= (0, 0, 1). \end{aligned}$$

Show that its Weierstrass-type representation is obtained by taking

$$X(z) = -\tfrac{i}{2}\varepsilon_1 \wedge \varepsilon_2 + \tfrac{i}{2} z\, \varepsilon_2 \wedge \varepsilon_3 + \tfrac{1}{2} z\, \varepsilon_3 \wedge \varepsilon_1$$

where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ represents the standard basis of $\mathbb{A}^3$. (Hint: Write the e_i as

$$\begin{aligned} e_1 &= \varepsilon_1 + x\varepsilon_3 \\ e_2 &= \varepsilon_2 + y\varepsilon_3 \\ e_3 &= \varepsilon_3 \end{aligned}$$

and note that, since $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a unimodular basis,

$$\begin{aligned} (\varepsilon_1 \wedge \varepsilon_2) \times (\varepsilon_1 \wedge \varepsilon_3) &= \varepsilon_1 \\ (\varepsilon_2 \wedge \varepsilon_3) \times (\varepsilon_2 \wedge \varepsilon_1) &= \varepsilon_2 \\ (\varepsilon_3 \wedge \varepsilon_1) \times (\varepsilon_3 \wedge \varepsilon_2) &= \varepsilon_3.) \end{aligned}$$