LECTURE 9: MOVING FRAMES IN THE NONHOMOGENOUS CASE: FRAME BUNDLES

1. Introduction

Until now we have been considering homogenous spaces G/H where G is a Lie group and H is a closed subgroup. The natural projection map

$$\pi : G \to G/H$$

leads to a description of G as the frame bundle of G/H, and the set of frames over a given point $x \in G/H$ is isomorphic to H. The fundamental property of homogenous spaces is that for any two points $x, y \in G/H$ and any frames F_x, F_y based at x and y respectively, there is a symmetry of G/H that takes x to y and F_x to F_y.

However, there are many interesting spaces that are not homogenous. In this lecture we will consider the case of a Riemannian manifold M. In general M is not a homogenous space, but the method of moving frames can still be applied to the study of submanifolds of M.

2. Frames and connections on Riemannian manifolds

Let M be an n-dimensional Riemannian manifold. Then M is a differentiable manifold with a smoothly varying inner product $\langle , \rangle_x$ on each tangent space T_xM. An *orthonormal frame at the point* $x \in M$ is an orthonormal basis $\{e_1, \dots , e_n\}$ of the tangent space T_xM. The set of orthonormal frames at each point is isomorphic to the Lie group $O(n)$, and the set of orthonormal frames on M forms a principal bundle over M with fiber $O(n)$, called the *frame bundle* of M and denoted $\mathcal{F}(M)$. An *orthonormal frame on the open set* $U \subset M$ is a choice of an orthonormal frame $\{e_1(x), \dots , e_n(x)\}$ at each point $x \in U$ such that each e_i is a smooth local section of the tangent bundle TM; any such frame is a smooth section of the frame bundle $\mathcal{F}(M)$.

What changes when M is not a homogenous space? First of all, while each fiber of the frame bundle is isomorphic to $O(n)$ and so has a group structure, there is no group structure on the entire frame bundle $\mathcal{F}(M)$. Moreover, given an orthonormal frame $\{e_1(x), \dots , e_n(x)\}$ on an open set $U \subset M$, there is no natural way of thinking of the frame vectors $e_i(x)$ as functions from M to a fixed vector space. Each $e_i(x)$ takes values in the vector space T_xM, and while this space is isomorphic to $\mathbb{E}^n$, there is no *canonical* isomorphism $T_xM \cong \mathbb{E}^n$ and hence no natural way of regarding the $e_i(x)$ as $\mathbb{E}^n$-valued functions. Instead we must regard them as sections of the vector bundle TM.

This raises the question of how to differentiate the $e_i(x)$. The exterior derivative d is defined for functions and differential forms that take values in a fixed vector space, but not for sections of vector bundles. In order to differentiate sections of vector bundles we need the notion of a connection on the vector bundle TM.

Definition: Let $\Gamma(TM)$ denote the space of smooth local sections of TM. An *affine connection* ∇ on TM is a map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM),$$

with $\nabla(w, v)$ denoted $\nabla_w v$, that satisfies

1. $\nabla_{fw_1+gw_2} v = f\nabla_{w_1} v + g\nabla_{w_2} v$
2. $\nabla_w(v_1 + v_2) = \nabla_w v_1 + \nabla_w v_2$
3. $\nabla_w(fv) = w(f)v + f\nabla_w v$

for $v, v_i, w, w_i \in \Gamma(TM)$ and f, g smooth real-valued functions on M.

By the linearity properties, a connection ∇ is completely determined by its action on any given frame $\{e_1, \ldots, e_n\}$ on M. Associated to ∇ and to the frame are scalar-valued 1-forms ω^i_j, $1 \leq i, j \leq n$ on M, called the *connection forms*, which are uniquely determined by the condition that for any $w \in T_xM$,

$$\nabla_w e_i = \sum_{j=1}^n e_j \, \omega^j_i(w).$$

We can think of the connection ∇ as defining a map from vector fields $v \in \Gamma(TM)$ to TM-valued 1-forms by letting ∇v be the 1-form defined by

$$(\nabla v)(w) = \nabla_w v.$$

Then we have

$$\nabla e_i = \sum_{j=1}^n e_j \, \omega^j_i$$

and by the linearity properties, if $v = \sum v^i e_i$, then

$$\begin{aligned}\nabla v &= \sum_{i=1}^n (e_i \, dv^i + v^i \sum_{j=1}^n e_j \, \omega^j_i) \\ &= \sum_{i=1}^n e_i (dv^i + \sum_{j=1}^n v^j \omega^i_j).\end{aligned}$$

∇ is the analog of the exterior derivative d.

A connection ∇ is called *symmetric* if for any vector fields $v, w \in \Gamma(TM)$,

$$\nabla_v w - \nabla_w v = [v, w]$$

where $[v, w]$ denotes the usual Lie bracket of vector fields. A connection ∇ is said to be *compatible with the metric* on M if for any vector fields $v, w \in \Gamma(TM)$,

$$d\langle v, w\rangle = \langle \nabla v, w\rangle + \langle v, \nabla w\rangle.$$

Theorem: (Levi-Civita) Given a Riemannian manifold M, there exists a unique connection ∇ on TM which is both symmetric and compatible with the metric. This connection is called the *Levi-Civita connection* on TM.

From now on we will assume that ∇ is the Levi-Civita connection on TM.

Now suppose that $\{e_1, \dots, e_n\}$ is an orthonormal frame on an open set in M, and let $\{\omega^i_j,\ 1 \le i, j \le n\}$ be the connection forms associated to the frame. Because the connection is compatible with the metric, we have

$$\begin{aligned} 0 &= d\langle e_i,\, e_j\rangle \\ &= \langle \nabla e_i,\, e_j\rangle + \langle e_i,\, \nabla e_j\rangle \\ &= \langle \sum_{k=1}^n e_k\, \omega^k_i,\, e_j\rangle + \langle e_i,\, \sum_{k=1}^n e_k\, \omega^k_j\rangle \\ &= \omega^j_i + \omega^i_j. \end{aligned}$$

Therefore we have $\omega^j_i = -\omega^i_j$, just as in the homogenous case.

In addition to the connection forms ω^i_j, we also have the *dual forms* ω^i defined by the equation

$$dx = \sum_{i=1}^n e_i\, \omega^i.$$

The forms $\{\omega^i,\, \omega^i_j\}$ form a basis for the space of 1-forms on the frame bundle $\mathcal{F}(M)$. In order to compute the structure equations for these forms, we will first need to differentiate the equation

$$dx = \sum_{i=1}^n e_i\, \omega^i. \tag{2.1}$$

This requires some care. Since all the terms in these equations are 1-forms that take values in TM, we must use the connection ∇ to differentiate them. First consider the left-hand side of equation (2.1). In terms of any local coordinate system $(x^1, \dots, x^n)$ on M we have

$$dx = \sum_{i=1}^n \frac{\partial}{\partial x^i}\, dx^i.$$

Because ∇ is symmetric and $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ we have

$$\begin{aligned}
\nabla(dx) &= \sum_{i=1}^{n} \nabla(\tfrac{\partial}{\partial x^i}) \wedge dx^i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \nabla_{\partial/\partial x^j}(\tfrac{\partial}{\partial x^i})\, dx^j \wedge dx^i \\
&= \sum_{i<j} [\nabla_{\partial/\partial x^i}(\tfrac{\partial}{\partial x^j}) - \nabla_{\partial/\partial x^j}(\tfrac{\partial}{\partial x^i})]\, dx^i \wedge dx^j \\
&= 0.
\end{aligned}$$

Thus differentiating equation (2.1) yields

$$\begin{aligned}
0 &= \sum_{i=1}^{n} [\nabla e_i \wedge \omega^i + e_i\, d\omega^i] \\
&= \sum_{i=1}^{n} [\sum_{j=1}^{n} e_j\, \omega^j_i \wedge \omega^i + e_i\, d\omega^i] \\
&= \sum_{i=1}^{n} e_i \left(d\omega^i + \sum_{j=1}^{n} \omega^i_j \wedge \omega^j\right).
\end{aligned}$$

Therefore,

$$d\omega^i = -\sum_{j=1}^{n} \omega^i_j \wedge \omega^j. \tag{2.2}$$

Next we want to compute the exterior derivatives of the ω^i_j. Differentiating equation (2.2) yields

$$\begin{aligned}
0 &= -\sum_{j=1}^{n} (d\omega^i_j \wedge \omega^j + \sum_{k=1}^{n} \omega^i_j \wedge \omega^j_k \wedge \omega^k) \\
&= -\sum_{j=1}^{n} (d\omega^i_j + \sum_{k=1}^{n} \omega^i_k \wedge \omega^k_j) \wedge \omega^j. \qquad (2.3)
\end{aligned}$$

By a higher-degree version of Cartan's lemma, there must exist 1-forms α^i_{jk} such that

$$d\omega^i_j + \sum_{k=1}^{n} \omega^i_k \wedge \omega^k_j = \sum_{k=1}^{n} \alpha^i_{jk} \wedge \omega^k.$$

Furthermore, I claim that we can choose the α^i_{jk} so that $\alpha^i_{jk} = -\alpha^j_{ik}$. It follows from the skew-symmetry of the ω^i_j that

$$\sum_{k=1}^{n} (\alpha^i_{jk} + \alpha^j_{ik}) \wedge \omega^k = 0.$$

By Cartan's lemma, this implies that

$$\alpha^i_{jk} + \alpha^j_{ik} = \sum_{\ell=1}^{n} T^i_{jk\ell}\omega^\ell$$

where $T^i_{jk\ell} = T^i_{j\ell k}$. Moreover, $T^i_{jk\ell} = T^j_{ik\ell}$. Now the α^i_{jk}'s are only determined up to transformations of the form

$$\tilde{\alpha}^i_{jk} = \alpha^i_{jk} + \sum_{\ell=1}^{n} U^i_{jk\ell}\omega^\ell$$

with $U^i_{jk\ell} = U^i_{j\ell k}$. So by changing the α^i_{jk} if necessary, we can assume that $T^i_{jk\ell} \equiv 0$, and so $\alpha^i_{jk} = -\alpha^j_{ik}$.

Now since the forms $\{\omega^i, \omega^i_j\}$ form a basis for the 1-forms on the frame bundle $\mathcal{F}(M)$, there must exist functions R^i_{jkl}, $S^{i\ell}_{jkm} = -S^{im}_{jk\ell}$ such that

$$\alpha^i_{jk} = -\sum_{\ell=1}^{n} R^i_{jkl}\,\omega^\ell + \sum_{\ell,m=1}^{n} S^{i\ell}_{jkm}\,\omega^m_\ell.$$

Substituting these equations into (2.3) yields

$$\sum_{j,k,\ell=1}^{n} R^i_{jk\ell}\,\omega^k \wedge \omega^\ell \wedge \omega^j + \sum_{j,k,\ell,m=1}^{n} S^{i\ell}_{jkm}\,\omega^m_\ell \wedge \omega^k \wedge \omega^j = 0.$$

The vanishing of the terms involving ω^m_ℓ implies that $S^{i\ell}_{jkm} = S^{i\ell}_{kjm}$. But because $S^{i\ell}_{jkm}$ is skew-symmetric in i and j, this implies that the $S^{i\ell}_{jkm}$ must all be zero. Thus we have

$$d\omega^i_j = -\sum_{k=1}^{n} \omega^i_k \wedge \omega^k_j + \sum_{k,\ell=1}^{n} R^i_{jkl}\omega^k \wedge \omega^\ell. \tag{2.4}$$

Without loss of generality we can assume that $R^i_{jk\ell} = -R^i_{j\ell k}$; then the functions $R^i_{jk\ell}$ are uniquely determined. They form the components of a tensor, called the *Riemannian curvature tensor* of the metric on M. In addition to the symmetries

$$R^i_{jk\ell} = -R^j_{ik\ell}, \qquad R^i_{jk\ell} = -R^i_{j\ell k}$$

equation (2.3) implies that

$$R^i_{jk\ell} + R^i_{k\ell j} + R^i_{\ell ji} = 0.$$

This is called the *Bianchi identity.* Together with the other symmetries, it implies that

$$R^i_{jk\ell} = R^k_{\ell ij}.$$

3. Surfaces in 3-dimensional Riemannian manifolds

Suppose that M is a 3-dimensional Riemannian manifold, and let $x : \Sigma \to M$ be an embedded surface in M. How much of the analysis from the case when $M = \mathbb{E}^3$ will remain valid in this more general setting?

We can still choose an orthonormal frame $\{e_1(x), e_2(x), e_3(x)\}$ for T_xM at each point $x \in \Sigma$ such that e_3 is orthogonal to the tangent plane $T_x\Sigma$. For such a frame, e_1 and e_2 form a basis for the tangent plane $T_x\Sigma$, and the same reasoning as before implies that $\omega^3 = 0$ and that ω^1, ω^2 form a basis for the 1-forms on Σ. The metric of M naturally induces a metric on Σ, and the first fundamental form of this metric is

$$I = (\omega^1)^2 + (\omega^2)^2.$$

As in the Euclidean case, differentiating the equation $\omega^3 = 0$ and applying Cartan's Lemma implies that there exist functions h_{11}, h_{12}, h_{22} on Σ such that

$$\begin{bmatrix} \omega_1^3 \\ \omega_2^3 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}.$$

The second fundamental form of Σ is given by

$$II = \omega_1^3\,\omega^1 + \omega_2^3\,\omega^2 = h_{11}(\omega^1)^2 + 2h_{12}\omega^1\omega^2 + h_{22}(\omega^2)^2.$$

As in the Euclidean case, the first and second fundamental forms are invariants of the surface, so if two surfaces have different first and second fundamental forms then they cannot be equivalent via a symmetry of M. However, in general there is no analog of Important Lemmas 1 and 2 that allows us to decide when we have a complete set of invariants for a surface or when two surfaces are equivalent via a symmetry of M. In particular, given arbitrary quadratic forms I and II on an abstract surface Σ, checking that the structure equations are satisfied is not sufficient to guarantee that there exists an immersion $x : \Sigma \to M$ whose first and second fundamental forms are I and II. The best we can do is to arbitrarily specify one quadratic form; for instance, we have the following isometric embedding theorem which is due independently to Cartan and Janet.

Theorem: Let M be a 3-dimensional real analytic Riemannian manifold, and let Σ be a surface with a prescribed real analytic metric I. Then locally there exists an embedding $x : \Sigma \to M$ for which the metric on Σ induced from M agrees with I.

Exercises

1. Let M be a Riemannian manifold, and let $\{e_1, \dots, e_n\}$ be a frame on an open set in M with associated dual forms $\omega^1, \dots, \omega^n$. Show directly that there exist unique 1-forms $\omega^i_j = -\omega^j_i$ such that

$$d\omega^i = -\sum_{j=1}^{n} \omega^i_j \wedge \omega^j.$$

(Hint: suppose that there are two such sets of 1-forms ω^i_j, $\bar{\omega}^i_j$ and use Cartan's lemma. Then use the skew-symmetry of the ω^i_j's to show that in fact $\omega^i_j = \bar{\omega}^i_j$.) These are the connection forms associated to the Levi-Civita connection on M, and this argument shows that the Levi-Civita connection is uniquely determined.

2. Let M be the standard 3-sphere S^3. From Lecture 2, Exercise 6 it follows that the structure equations of the dual and connection forms on S^3 are

$$\begin{aligned}
d\omega^1 &= -\omega^1_2 \wedge \omega^2 + \omega^3_1 \wedge \omega^3 \\
d\omega^2 &= \omega^1_2 \wedge \omega^2 + \omega^3_2 \wedge \omega^3 \\
d\omega^3 &= -\omega^3_1 \wedge \omega^1 - \omega^3_2 \wedge \omega^2 \\
d\omega^1_2 &= \omega^3_1 \wedge \omega^3_2 + \omega^1 \wedge \omega^2 \\
d\omega^3_1 &= \omega^3_2 \wedge \omega^1_2 + \omega^3 \wedge \omega^1 \\
d\omega^3_2 &= -\omega^3_1 \wedge \omega^3_2 + \omega^3 \wedge \omega^2.
\end{aligned}$$

Let $x : \Sigma \to S^3$ be an embedding. The induced metric on Σ has Gauss curvature K given by the equation

$$d\omega^1_2 = K\, \omega^1 \wedge \omega^2.$$

Show that

$$K = h_{11}h_{22} - h_{12}^2 + 1$$

where the h_{ij} are the coefficients of the second fundamental form. Why is this equation different than in the Euclidean case?