CHAPTER III
FUNCTIONS AND CONTINUITY
DEFINITION OF THE NUMBER π.

The concept of a function is perhaps the most basic one in mathematical analysis. The objects of interest in our subject can often be represented as functions, and the " unknowns" in our equations are frequently functions. Therefore, we will spend some time developing and understanding various kinds of functions, including functions defined by polynomials, by power series, and as limits of other functions. In particular, we introduce in this chapter the elementary transcendental functions. We begin with the familiar set theoretical notion of a function, and then move quickly to their analytical properties, specifically that of continuity.

The main theorems of this chapter include:

(1) The **Intermediate Value Theorem** (Theorem 3.6),
(2) the theorem that asserts that a **continuous real-valued function on a compact set attains a maximum and minimum value** (Theorem 3.8),
(3) A **continuous function on a compact set is uniformly continuous** (Theorem 3.9),
(4) The **Identity Theorem for Power Series Functions** (Theorem 3.14),
(5) The definition of the real number π,
(6) The theorem that asserts that the **uniform limit of a sequence of continuous functions is continuous** (Theorem 3.17), and
(7) the **Weierstrass M-Test** (Theorem 3.18).

FUNCTIONS

DEFINITION. Let S and T be sets. A *function* from S into T (notation $f : S \to T$) is a rule that assigns to each element x in S a unique element denoted by $f(x)$ in T.

It is useful to think of a function as a mechanism or black box. We use the elements of S as inputs to the function, and the outputs are elements of the set T.

If $f : S \to T$ is a function, then S is called the *domain* of f, and the set T is called the *codomain* of f. The *range* or *image* of f is the set of all elements y in the codomain T for which there exists an x in the domain S such that $y = f(x)$. We denote the range by $f(S)$. The codomain is the set of all potential outputs, while the range is the set of actual outputs.

Suppose f is a function from a set S into a set T. If $A \subseteq S$, we write $f(A)$ for the subset of T containing all the elements $t \in T$ for which there exists an $s \in A$ such that $t = f(s)$. We call $f(A)$ the *image* of A under f. Similarly, if $B \subseteq T$, we write $f^{-1}(B)$ for the subset of S containing all the elements $s \in S$ such that $f(s) \in B$, and we call the set $f^{-1}(B)$ the *inverse image* or *preimage* of B. The symbol $f^{-1}(B)$ is a little confusing, since it could be misinterpreted as the image of the set B under a function called f^{-1}. We will discuss inverse functions later on, but this notation is not meant to imply that the function f has an inverse.

If $f : S \to T$, then the *graph* of f is the subset G of the Cartesian product $S \times T$ consisting of all the pairs of the form $(x, f(x))$.

If $f : S \to \mathbb{R}$ is a function, then we call f a *real-valued* function, and if $f : S \to \mathbb{C}$, then we call f a *complex-valued* function. If $f : S \to \mathbb{C}$ is a complex-valued function, then for each $x \in S$ the complex number $f(x)$ can be written as $u(x) + iv(x)$, where

$u(x)$ and $v(x)$ are the real and imaginary parts of the complex number $f(x)$. The two real-valued functions $u : S \to \mathbb{R}$ and $v : S \to \mathbb{R}$ are called respectively the *real* and *imaginary* parts of the complex-valued function f.

If $f : S \to T$ and $S \subseteq \mathbb{R}$, then f is called a function of a *real variable*, and if $S \subseteq \mathbb{C}$, then f is called a function of a *complex variable*.

If the range of f equals the codomain, then f is called *onto*.

The function $f : S \to T$ is called one-to-one if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

The domain of f is the set of x's for which $f(x)$ is defined. If we are given a function $f : S \to T$, we are free to regard f as having a smaller domain, i.e., a subset S' of S. Although this restricted function is in reality a different function, we usually continue to call it by the same name f. Enlarging the domain of a function, in some consistent manner, is often impossible, but is nevertheless frequently of great importance. The codomain of f is distinguished from the range of f, which is frequently a proper subset of the codomain. For example, since every real number is a complex number, any real-valued function $f : S \to \mathbb{R}$ is also a (special kind of) complex-valued function.

We consider in this book functions either of a real variable or of complex variable. that is, the domains of functions here will be subsets either of $\mathbb{R}$ or of $\mathbb{C}$. Frequently, we will indicate what kind of variable we are thinking of by denoting real variables with the letter x and complex variables with the letter z. Be careful about this, for this distinction is **not always** made.

Many functions, though not all by any means, are defined by a single equation:

$$y = 3x - 7,$$

$$y = (x^2 + x + 1)^{2/3},$$

$$x^2 + y^2 = 4,$$

(How does this last equation define a function?)

$$(1 - x^7 y^{11})^{2/3} = (x/(1 - y))^{8/17}.$$

(How does this equation determine a function?)

There are various types of functions, and they can be combined in a variety of ways to produce other functions. It is necessary therefore to spend a fair amount of time at the beginning of this chapter to present these definitions.

DEFINITION. If f and g are two complex-valued functions with the same domain S, i.e., $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$, and if c is a complex number, we define $f + g,\ fg,\ f/g$ (if $g(x)$ is never 0), and cf by the familiar formulas:

$$(f + g)(x) = f(x) + g(x),$$

$$(fg)(x) = f(x)g(x),$$

$$(f/g)(x) = f(x)/g(x),$$

and

$$(cf)(x) = cf(x).$$

If f and g are real-valued functions, we define functions $\max(f,g)$ and $\min(f,g)$ by

$$[\max(f,g)](x) = \max(f(x), g(x))$$

(the maximum of the numbers $f(x)$ and $g(x)$), and

$$[\min(f,g)](x) = \min(f(x), g(x)),$$

(the minimum of the two numbers $f(x)$ and $g(x)$).

If f is either a real-valued or a complex-valued function on a domain S, then we say that f is *bounded* if there exists a positive number M such that $|f(x)| \leq M$ for all $x \in S$.

There are two special types of functions of a real or complex variable, the even functions and the odd functions. In fact, every function that is defined on all of $\mathbb{R}$ or $\mathbb{C}$ (or, more generally, any function whose domain S equals $-S$) can be written uniquely as a sum of an even part and an odd part. This decomposition of a general function into simpler parts is frequently helpful.

DEFINITION. A function f whose domain S equals $-S$, is called an *even* function if $f(-z) = f(z)$ for all z in its domain. It is called an *odd* function if $f(-z) = -f(z)$ for all z in its domain.

We next give the definition for perhaps the most familiar kinds of functions.

DEFINITION. A nonzero *polynomial* or *polynomial function* is a complex-valued function of a complex variable, $p : \mathbb{C} \to \mathbb{C}$, that is defined by a formula of the form

$$p(z) = \sum_{k=0}^{n} a_k z^k = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

where the a_k's are complex numbers and $a_n \neq 0$. The integer n is called the *degree* of the polynomial p and is denoted by $\deg(p)$. The numbers $a_0, a_1, \ldots, a_n$ are called the *coefficients* of the polynomial. The domain of a polynomial function is all of $\mathbb{C}$; i.e., $p(z)$ is defined for every complex number z.

For technical reasons of consistency, the identically 0 function is called the *zero polynomial*. All of its coefficients are 0 and its degree is defined to be $-\infty$.

A *rational function* is a function r that is given by an equation of the form $r(z) = p(z)/q(z)$, where q is a nonzero polynomial and p is a (possibly zero) polynomial. The domain of a rational function is the set S of all $z \in \mathbb{C}$ for which $q(z) \neq 0$, i.e., for which $r(z)$ is defined.

Two other kinds of functions that are simple and important are step functions and polygonal functions.

DEFINITION. Let $[a,b]$ be a closed bounded interval of real numbers. By a *partition* of $[a,b]$ we mean a finite set $P = \{x_0 < x_1 < \ldots < x_n\}$ of $n+1$ points, where $x_0 = a$ and $x_n = b$.

The n intervals $\{[x_{i-1}, x_i]\}$, for $1 \leq i \leq n$, are called the *closed subintervals* of the partition P, and the n intervals $\{(x_{i-1}, x_i)\}$ are called the *open subintervals* of P.

We write $\|P\|$ for the maximum of the numbers (lengths of the subintervals) $\{x_i - x_{i-1}\}$, and call the number $\|P\|$ the *mesh size* of the partition P.

A function $h : [a,b] \to \mathbb{C}$ is called a *step function* if there exists a partition $P = \{x_0 < x_1 < \ldots < x_n\}$ of $[a,b]$ and n numbers $\{a_1, a_2, \ldots, a_n\}$ such that $h(x) = a_i$ if $x_{i-1} < x < x_i$. That is, h is a step function if it is a constant function on each of the (open) subintervals (x_{i-1}, x_i) determined by a partition P. Note that the values of a step function at the points $\{x_i\}$ of the partition are not restricted in any way.

A function $l : [a,b] \to \mathbb{R}$ is called a *polygonal function*, or a *piecewise linear function*, if there exists a partition $P = \{x_0 < x_1 < \ldots < x_n\}$ of $[a,b]$ and $n+1$ numbers $\{y_0, y_1, \ldots, y_n\}$ such that for each $x \in [x_{i-1}, x_i]$, $l(x)$ is given by the linear equation

$$l(x) = y_{i-1} + m_i(x - x_{i-1}),$$

where $m_i = (y_i - y_{i_1})/(x_i - x_{i-1})$. That is, l is a polygonal function if it is a linear function on each of the closed subintervals $[x_{i-1}, x_i]$ determined by a partition P. Note that the values of a piecewise linear function at the points $\{x_i\}$ of the partition P are the same, whether we think of x_i in the interval $[x_{i-1}, x_i]$ or $[x_i, x_{i+1}]$. (Check the two formulas for $l(x_i)$.)

The graph of a piecewise linear function is the polygonal line joining the $n+1$ points $\{(x_i, y_i)\}$.

There is a natural generalization of the notion of a step function that works for any domain S, e.g., a rectangle in the plane $\mathbb{C}$. Thus, if S is a set, we define a *partition* of S to be a finite collection $\{E_1, E_2, \ldots, E_n\}$ of subsets of S for which

(1) $\cup_{i=1}^n E_i = S$, and
(2) $E_i \cap E_j = \emptyset$ if $i \neq j$.

Then, a *step function* on S would be a function h that is constant on each subset E_i. We will encounter an even more elaborate generalized notion of a step function in Chapter V, but for now we will restrict our attention to step functions defined on intervals $[a,b]$.

The set of polynomials and the set of step functions are both closed under addition and multiplication, and the set of rational functions is closed under addition, multiplication, and division.

Exercise 3.1. (a) Prove that the sum and product of two polynomials is again a polynomial. Show that $\deg(p+q) \leq \max(\deg(p), \deg(q))$ and $\deg(pq) = \deg(p) + \deg(q)$. Show that a constant function is a polynomial, and that the degree of a nonzero constant function is 0.

(b) Show that the set of step functions is closed under addition and multiplication. Show also that the maximum and minimum of two step functions is again a step function. (Be careful to note that different step functions may be determined by different partitions. For instance, a partition determining the sum of two step functions may be different from the partitions determining the two individual step functions.) Note, in fact, that a step function can be determined by infinitely many different partitions. Prove that the sum, the maximum, and the minimum of two piecewise linear functions is again a piecewise linear function. Show by example that the product of two piecewise linear functions need not be piecewise linear.

(c) Prove that the sum, product, and quotient of two rational functions is again a rational function.

(d) Prove the **Root Theorem:** If $p(z) = \sum_{k=0}^n a_k z^k$ is a nonzero polynomial of degree n, and if c is a complex number for which $p(c) = 0$, then there exists a

nonzero polynomial $q(z) = \sum_{j=0}^{n-1} b_j z^j$ of degree $n-1$ such that $p(z) = (z-c)q(z)$ for all z. That is, if c is a "root" of p, then $z-c$ is a factor of p. Show also that the leading coefficient b_{n-1} of q equals the leading coefficient a_n of p.
HINT: Write

$$p(z) = p(z) - p(c) = \sum_{k=0}^{n} a_k(z^k - c^k) = \dots .$$

(e) Let f be a function whose domain S equals $-S$. Define functions f_e and f_o by the formulas

$$f_e(z) = \frac{f(z) + f(-z)}{2} \text{ and } f_o(z) = \frac{f(z) - f(-z)}{2}.$$

Show that f_e is an even function, that f_o is an odd function, and that $f = f_e + f_o$. Show also that, if $f = g + h$, where g is an even function and h is an odd function, then $g = f_e$ and $h = f_o$. That is, there is only one way to write f as the sum of an even function and an odd function.

(f) Use part (e) to show that a polynomial p is an even function if and only if its only nonzero coefficients are even ones, i.e., the a_{2k}'s. Show also that a polynomial is an odd function if and only if its only nonzero coefficients are odd ones, i.e., the a_{2k+1}'s.

(g) Suppose $p(z) = \sum_{k=0}^{n} a_{2k} z^{2k}$ is a polynomial that is an even function. Show that

$$p(iz) = \sum_{k=0}^{n} (-1)^k a_{2k} z^{2k} = p^a(z),$$

where p^a is the polynomial obtained from p by alternating the signs of its nonzero coefficients.

(h) If $q(z) = \sum_{k=0}^{n} a_{2k+1} z^{2k+1}$ is a polynomial that is an odd function, show that

$$q(iz) = i\sum_{k=0}^{n} (-1)^k a_{2k+1} z^{2k+1} = iq^a(z),$$

where again q^a is the polynomial obtained from q by alternating the signs of its nonzero coefficients.

(i) If p is any polynomial, show that

$$p(iz) = p_e(iz) + p_o(iz) = p_e^a(z) + ip_o^a(z),$$

and hence that $p_e(iz) = p_e^a(z)$ and $p_o(iz) = ip_o^a(z)$.

POLYNOMIAL FUNCTIONS

If $p(z) = \sum_{k=0}^{n} a_k z^k$ and $q(z) = \sum_{j=0}^{m} b_j z^j$ are two polynomials, it certainly seems clear that they determine the same function only if they have identical coefficients. This is true, but by no means an obvious fact. Also, it seems clear that, as $|z|$ gets larger and larger, a polynomial function is more and more comparable to its leading term $a_n z^n$. We collect in the next theorem some elementary properties of polynomial functions, and in particular we verify the above "uniqueness of coefficients" result and the "behavior at infinity" result.

THEOREM 3.1.

(1) *Suppose $p(z) = \sum_{k=0}^n a_k z^k$ is a nonconstant polynomial of degree $n > 0$. Then $p(z) = 0$ for at most n distinct complex numbers.*

(2) *If r is a polynomial for which $r(z) = 0$ for an infinite number of distinct points, then r is the zero polynomial. That is, all of its coefficients are 0.*

(3) *Suppose p and q are nonzero polynomials, and assume that $p(z) = q(z)$ for an infinite number of distinct points. Then $p(z) = q(z)$ for all z, and p and q have the same coefficients. That is, they are the same polynomial.*

(4) *Let $p(z) = \sum_{j=0}^n c_j z^j$ be a polynomial of degree $n > 0$. Then there exist positive constants m and B such that*

$$\frac{|c_n|}{2}|z|^n \leq |p(z)| \leq M|z|^n$$

for all complex numbers z for which $|z| \geq B$. That is, For all complex numbers z with $|z| \geq B$, the numbers $|p(z)|$ and $|z|^n$ are "comparable."

(5) *If $f : [0, \infty) \to \mathbb{C}$ is defined by $f(x) = \sqrt{x}$, then there is no polynomial p for which $f(x) = p(x)$ for all $x \geq 0$. That is, the square root function does not agree with any polynomial function.*

PROOF. We prove part (1) using an argument by contradiction. Thus, suppose there does exist a counterexample to the claim, i.e., a nonzero polynomial p of degree n and $n + 1$ distinct points $\{c_1, c_2, \ldots, c_{n+1}\}$ for which $p(c_j) = 0$ for all $1 \leq j \leq n + 1$. From the set of all such counterexamples, let p_0 be one with minimum degree n_0. That is, the claim in part (1) is true for any polynomial whose degree is smaller than n_0. We write

$$p_0(z) = \sum_{k=0}^{n_0} a_k z^k,$$

and we suppose that $p_0(c_j) = 0$ for $j = 1$ to $n_0 + 1$, where these c_k's are distinct complex numbers. We use next the Root Theorem (part (d) of Exercise 3.1) to write $p_0(z) = (z - c_{n_0+1})q(z)$, where $q(z) = \sum_{k=0}^{n_0-1} b_k z^k$. We have that q is a polynomial of degree $n_0 - 1$ and the leading coefficient a_{n_0} of p_0 equals the leading coefficient b_{n_0-1} of q. Note that for $1 \leq j \leq n_0$ we have

$$0 = p_0(c_j) = (c_j - c_{n_0+1})q(c_j),$$

which implies that $q(c_j) = 0$ for $1 \leq j \leq n_0$, since $c_j - c_{n_0+1} \neq 0$. But, since $\deg(q) < n_0$, the nonzero polynomial q can not be a counterexample to part (1), implying that $q(z) = 0$ for at most $n_0 - 1$ distinct points. We have arrived at a contradiction, and part (1) is proved.

Next, let r be a polynomial for which $r(z) = 0$ for an infinite number of distinct points. It follows from part (1) that r cannot be a nonzero polynomial, for in that case it would have a degree $n \geq 0$ and could be 0 for at most n distinct points. Hence, r is the zero polynomial, and part (2) is proved.

Now, to see part (3), set $r = p - q$. Then r is a polynomial for which $r(z) = 0$ for infinitely many z's. By part (2), it follows then that $r(z) = 0$ for all z, whence $p(z) = q(z)$ for all z. Moreover, $p - q$ is the zero polynomial, all of whose coefficients are 0, and this implies that the coefficients for p and q are identical.

To prove the first inequality in part (4), suppose that $|z| > 1$, and from the backwards triangle inequality, note that

$$\begin{aligned}
|p(z)| &= |\sum_{k=0}^{n} c_k z^k| \\
&= |z|^n |\sum_{k=0}^{n} \frac{c_k}{z^{n-k}}| \\
&= |z|^n |(\sum_{k=0}^{n-1} \frac{c_k}{z^{n-k}}) + c_n| \\
&\geq |z|^n (|c_n| - |\sum_{k=0}^{n-1} \frac{c_k}{z^{n-k}}|) \\
&\geq |z|^n (|c_n| - \sum_{k=0}^{n-1} \frac{|c_k|}{|z|^{n-k}}) \\
&\geq |z|^n (|c_n| - \sum_{k=0}^{n-1} \frac{|c_k|}{|z|}) \\
&\geq |z|^n (|c_n| - \frac{1}{|z|} \sum_{k=0}^{n-1} |c_k|).
\end{aligned}$$

Set B equal to the constant $(2/|c_n|)\sum_{j=0}^{n-1} |c_j|$. Then, replacing the $1/|z|$ in the preceding calculation by $1/B$, we obtain

$$|p(z)| \geq m|z|^n$$

for every z for which $|z| \geq B$. This proves the first half of part (4).

To get the other half of part (4), suppose again that $|z| > 1$. We have

$$|p(z)| \leq \sum_{k=0}^{n} |c_k||z|^k \leq \sum_{k=0}^{n} |c_k||z|^n,$$

so that we get the other half of part (4) by setting $M = \sum_{k=0}^{n} |c_k|$.

Finally, to see part (5), suppose that there does exist a polynomial p of degree n such that $\sqrt{x} = p(x)$ for all $x \geq 0$. Then $x = (p(x))^2$ for all $x \geq 0$. Now p^2 is a polynomial of degree $2n$. By part (2), the two polynomials $q(x) = x$ and $(p(x))^2$ must be the same, implying that they have the same degree. However, the degree of q is 1, which is odd, and the degree of p^2 is $2n$, which is even. Hence, we have arrived at a contradiction.

Exercise 3.2. (a) Let $r(z) = p(z)/q(z)$ and $r'(z) = p'(z)/q'(z)$ be two rational functions. Suppose $r(z) = r'(z)$ for infinitely many z's. Prove that $r(z) = r'(z)$ for all z in the intersection of their domains. Is it true that $p = p'$ and $q = q'$?

(b) Let p and q be polynomials of degree n and m respectively, and define a rational function r by $r = p/q$. Prove that there exist positive constants C and B such that $|r(z)| < C|z|^{n-m}$ for all complex numbers z for which $|z| > B$.

(c) Define $f : [0,\infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Show that there is no rational function r such that $f(x) = r(x)$ for all $x \geq 0$. That is, the square root function does not agree with a rational function.

(d) Define the real-valued function r on $\mathbb{R}$ by $r(x) = 1/(1+x^2)$. Prove that there is no polynomial p such that $p(x) = r(x)$ for infinitely many real numbers x.

(e) If f is the real-valued function of a real variable given by $f(x) = |x|$, show that f is **not** a rational function.
HINT: Suppose $|x| = p(x)/q(x)$. Then $|x|q(x) = p(x)$ implying that $|x|q(x)$ is a polynomial $s(x)$. Now use Theorem 3.1 to conclude that $p(x) = xq(x)$ for all x and that $p(x) = -xq(x)$ for all x.

(f) Let f be any complex-valued function of a complex variable, and let $c_1, \ldots, c_n$ be n distinct complex numbers that belong to the domain of f. Show that there does exist a polynomial p of degree n such that $p(c_j) = f(c_j)$ for all $1 \leq j \leq n$.
HINT: Describe p in factored form.

(g) Give examples to show that the maximum and minimum of two polynomials need not be a polynomial or even a rational function.

Very important is the definition of the *composition* $g \circ f$ of two functions f and g.

DEFINITION. Let $f : S \to T$ and $g : T \to U$ be functions. We define a function $g \circ f$, with domain S and codomain U, by $(g \circ f)(x) = g(f(x))$.

If $f : S \to T$, $g : T \to S$, and $g \circ f(x) = x$ for all $x \in S$, then g is called a *left inverse* of f. If $f \circ g(y) = y$ for all $y \in T$, then g is called a *right inverse* for f. If g is both a left inverse and a right inverse, then g is called an *inverse* for f, f is called *invertible*, and we denote g by f^{-1}.

Exercise 3.3. (a) Suppose $f : S \to T$ has a left inverse. Prove that f is 1-1.

(b) Suppose $f : S \to T$ has a right inverse. Prove that f is onto.

(c) Show that the composition of two polynomials is a polynomial and that the composition of two rational functions is a rational function.
HINT: If p is a polynomial, show by induction that p^n is a polynomial. Now use Exercise 3.1.

(d) Find formulas for $g \circ f$ and $f \circ g$ for the following. What are the domains of these compositions?

(1) (i) $f(x) = 1 + x^2$ and $g(x) = 1/(1+x)^{1/2}$.
(2) (ii) $f(x) = x/(x+1)$ and $g(x) = x/(1-x)$.
(3) (iii) $f(x) = ax + b$ and $g(x) = cx + d$.

CONTINUITY

Next, we come to the definition of continuity. Unlike the preceding discussion, which can be viewed as being related primarily to the algebraic properties of functions, this one is an analytic notion.

DEFINITION. Let S and T be sets of complex numbers, and let $f : S \to T$. Then f is said to be *continuous at a point c* of S if for every positive ϵ, there exists a positive δ such that if $x \in S$ satisfies $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. The function f is called *continuous on S* if it is continuous at every point c of S.

If the domain S of f consists of real numbers, then the function f is called *right continuous at c* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$

whenever $x \in S$ and $0 \leq x - c < \delta$, and is called *left continuous* at c if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $x \in S$ and $0 \geq x - c > -\delta$.

REMARK. If f is continuous at a point c, then the positive number δ of the preceding definition is not unique (any smaller number would work as well), but it does depend both on the number ϵ and on the point c. Sometimes we will write $\delta(\epsilon, c)$ to make this dependence explicit. Later, we will introduce a notion of uniform continuity in which δ only depends on the number ϵ and not on the particular point c.

The next theorem indicates the interaction between the algebraic properties of functions and continuity.

THEOREM 3.2. *Let S and T be subsets of $\mathbb{C}$, let f and g be functions from S into T, and suppose that f and g are both continuous at a point c of S. Then*

(1) *There exists a $\delta > 0$ and a positive number M such that if $|y - c| < \delta$ and $y \in S$ then $|f(y)| \leq M$. That is, if f is continuous at c, then it is bounded near c.*
(2) *$f + g$ is continuous at c.*
(3) *fg is continuous at c.*
(4) *$|f|$ is continuous at c.*
(5) *If $g(c) \neq 0$, then f/g is continuous at c.*
(6) *If f is a complex-valued function, and u and v are the real and imaginary parts of f, then f is continuous at c if and only if u and v are continuous at c.*

PROOF. We prove parts (1) and (5), and leave the remaining parts to the exercise that follows.

To see part (1), let $\epsilon = 1$. Then, since f is continuous at c, there exists a $\delta > 0$ such that if $|y - c| < \delta$ and $y \in S$ then $|f(y) - f(c)| < 1$. Since $|z - w| \geq ||z| - |w||$ for any two complex numbers z and w (backwards Triangle Inequality), it then follows that $||f(y)| - |f(c)|| < 1$, from which it follows that if $|y - c| < \delta$ then $|f(y)| < |f(c)| + 1$. Hence, setting $M = |f(c)| + 1$, we have that if $|y - c| < \delta$ and $y \in S$, then $|f(y)| \leq M$ as desired.

To prove part (5), we first make use of part 1. Let δ_1, M_1 and δ_2, M_2 be chosen so that if $|y - c| < \delta_1$ and $y \in S$ then

$$|f(y)| < M_1 \tag{3.1}$$

and if $|y - c| < \delta_2$ and $y \in S$ then

$$|g(y)| < M_2 \tag{3.2.}$$

Next, let ϵ' be the positive number $|g(c)|/2$. Then, there exists a $\delta' > 0$ such that if $|y - c| < \delta'$ and $y \in S$ then $|g(y) - g(c)| < \epsilon' = |g(c)|/2$. It then follows from the backwards triangle inequality that

$$|g(y)| > \epsilon' = |g(c)|/2 \text{ so that } |1/g(y)| < 2/|g(c)| \tag{3.3.}$$

Now, to finish the proof of part (5), let $\epsilon > 0$ be given. If $|y - c| < \min(\delta_1, \delta_2, \delta')$

and $y \in S$, then from Inequalities (3.1), (3.2), and (3.3) we obtain

$$\begin{aligned} |\frac{f(y)}{g(y)} - \frac{f(c)}{g(c)}| &= \frac{|f(y)g(c) - f(c)g(y)|}{|g(y)g(c)|} \\ &= \frac{|f(y)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(y)|}{|g(y)||g(c)|} \\ &\leq \frac{|f(y) - f(c)||g(c)| + |f(c)||g(c) - g(y)|}{|g(y)||g(c)|} \\ &< (|f(y) - f(c)|M_2 + M_1|g(c) - g(y)|) \times \frac{2}{|g(c)|^2}. \end{aligned}$$

Finally, using the continuity of both f and g applied to the positive numbers $\epsilon_1 = \epsilon/(4M_2|g(c)|^2)$ and $\epsilon_2 = \epsilon/(4M_1|g(c)|^2)$, choose $\delta > 0$, with $\delta < \min(\delta_1, \delta_2, \delta')$, and such that if $|y - c| < \delta$ and $y \in S$ then $|f(y) - f(c)| < \frac{\epsilon}{4M_2/|g(c)|^2}$ and $|g(c) - g(y)| < \frac{\epsilon}{4M_1/|g(c)|^2}$. Then, if $|y - c| < \delta$ and $y \in S$ we have that

$$|\frac{f(y)}{g(y)} - \frac{f(c)}{g(c)}| < \epsilon$$

as desired.

Exercise 3.4. (a) Prove part (2) of the preceding theorem. (It's an $\epsilon/2$ argument.)

(b) Prove part (3) of the preceding theorem. (It's similar to the proof of part (5) only easier.)

(c) Prove part (4) of the preceding theorem.

(d) Prove part (6) of the preceding theorem.

(e) Suppose S is a subset of $\mathbb{R}$. Verify the above theorem replacing " continuity" with left continuity and right continuity.

(f) If S is a subset of $\mathbb{R}$, show that f is continuous at a point $c \in S$ if and only if it is both right continuous and left continuous at c.

THEOREM 3.3. (The composition of continuous functions is continuous.) Let S, T, and U be subsets of $\mathbb{C}$, and let $f : S \to T$ and $g : T \to U$ be functions. Suppose f is continuous at a point $c \in S$ and that g is continuous at the point $f(c) \in T$. Then the composition $g \circ f$ is continuous at c.

PROOF. Let $\epsilon > 0$ be given. Because g is continuous at the point $f(c)$, there exists an $\alpha > 0$ such that $|g(t) - g(f(c))| < \epsilon$ if $|t - f(c)| < \alpha$. Now, using this positive number α, and using the fact that f is continuous at the point c, there exists a $\delta > 0$ so that $|f(s) - f(c)| < \alpha$ if $|s - c| < \delta$. Therefore, if $|s - c| < \delta$, then $|f(s) - f(c)| < \alpha$, and hence $|g(f(s)) - g(f(c))| = |g \circ f(s) - g \circ f(c)| < \epsilon$, which completes the proof.

Exercise 3.5. (a) If $f : \mathbb{C} \to \mathbb{C}$ is the function defined by $f(z) = z$, prove that f is continuous at each point of $\mathbb{C}$.

(b) Use part (a) and Theorem 3.2 to conclude that every rational function is continuous on its domain.

(c) Prove that a step function $h : [a, b] \to \mathbb{C}$ is continuous everywhere on $[a, b]$ except possibly at the points of the partition P that determines h.

Exercise 3.6. (a) Let S be the set of nonnegative real numbers, and define $f : S \to S$ by $f(x) = \sqrt{x}$. Prove that f is continuous at each point of S.
HINT: For $c = 0$, use $\delta = \epsilon^2$. For $c \neq 0$, use the identity

$$\sqrt{y} - \sqrt{c} = (\sqrt{y} - \sqrt{c})\frac{\sqrt{y} + \sqrt{c}}{\sqrt{y} + \sqrt{c}} = \frac{y - c}{\sqrt{y} + \sqrt{c}} \leq \frac{y - c}{\sqrt{c}}.$$

(b) If $f : \mathbb{C} \to \mathbb{R}$ is the function defined by $f(z) = |z|$, show that f is continuous at every point of its domain.

Exercise 3.7. Using the previous theorems and exercises, explain why the following functions f are continuous on their domains. Describe the domains as well.

(a) $f(z) = (1 - z^2)/(1 + z^2)$.
(b) $f(z) = |1 + z + z^2 + z^3 - (1/z)|$.
(c) $f(z) = \sqrt{1 + \sqrt{1 - |z|^2}}$.

Exercise 3.8. (a) If c and d are real numbers, show that $\max(c, d) = (c + d)/2 + |c - d|/2$.

(b) If f and g are functions from S into $\mathbb{R}$, show that $\max(f, g) = (f + g)/2 + |f - g|/2$.

(c) If f and g are real-valued functions that are both continuous at a point c, show that $\max(f, g)$ and $\min(f, g)$ are both continuous at c.

Exercise 3.9. Let $\mathbb{N}$ be the set of natural numbers, let P be the set of positive real numbers, and define $f : \mathbb{N} \to P$ by $f(n) = \sqrt{1 + n}$. Prove that f is continuous at each point of $\mathbb{N}$. Show in fact that every function $f : \mathbb{N} \to \mathbb{C}$ is continuous on this domain $\mathbb{N}$.
HINT: Show that for any $\epsilon > 0$, the choice of $\delta = 1$ will work.

Exercise 3.10. (Negations)

(a) Negate the statement: "For every $\epsilon > 0$, $|x| < \epsilon$."
(b) Negate the statement: "For every $\epsilon > 0$, there exists an x for which $|x| < \epsilon$."
(c) Negate the statement that " f is continuous at c."

The next result establishes an equivalence between the basic ϵ, δ definition of continuity and a sequential formulation. In many cases, maybe most, this sequential version of continuity is easier to work with than the ϵ, δ version.

THEOREM 3.4. *Let $f : S \to \mathbb{C}$ be a complex-valued function on S, and let c be a point in S. Then f is continuous at c if and only if the following condition holds: For every sequence $\{x_n\}$ of elements of S that converges to c, the sequence $\{f(x_n)\}$ converges to $f(c)$. Or, said a different way, if $\{x_n\}$ converges to c, then $\{f(x_n)\}$ converges to $f(c)$. And, said yet a third (somewhat less precise) way, the function f converts convergent sequences to convergent sequences.*

PROOF. Suppose first that f is continuous at c, and let $\{x_n\}$ be a sequence of elements of S that converges to c. Let $\epsilon > 0$ be given. We must find a natural number N such that if $n \geq N$ then $|f(x_n) - f(c)| < \epsilon$. First, choose $\delta > 0$ so that $|f(y) - f(c)| < \epsilon$ whenever $y \in S$ and $|y - c| < \delta$. Now, choose N so that $|x_n - c| < \delta$ whenever $n \geq N$. Then if $n \geq N$, we have that $|x_n - c| < \delta$, whence $|f(x_n) - f(c)| < \epsilon$. This shows that the sequence $\{f(x_n)\}$ converges to $f(c)$, as desired.

We prove the converse by proving the contrapositive statement; i.e., we will show that if f is not continuous at c, then there does exist a sequence $\{x_n\}$ that converges to c but for which the sequence $\{f(x_n)\}$ does not converge to $f(c)$. Thus, suppose f is **not** continuous at c. Then there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ there is a $y \in S$ such that $|y - c| < \delta$ but $|f(y) - f(c)| \geq \epsilon_0$. To obtain a sequence, we apply this statement to δ's of the form $\delta = 1/n$. Hence, for every natural number n there exists a point $x_n \in S$ such that $|x_n - c| < 1/n$ but $|f(x_n) - f(c)| \geq \epsilon_0$. Clearly, the sequence $\{x_n\}$ converges to c since $|x_n - c| < 1/n$. On the other hand, the sequence $\{f(x_n)\}$ cannot be converging to $f(c)$, because $|f(x_n) - f(c)|$ is always $\geq \epsilon_0$.

This completes the proof of the theorem.

CONTINUITY AND TOPOLOGY

Let $f : S \to T$ be a function, and let A be a subset of the codomain T. Recall that $f^{-1}(A)$ denotes the subset of the domain S consisting of all those $x \in S$ for which $f(x) \in A$.

Our original definition of continuity was in terms of ϵ's and δ's. Theorem 3.4 established an equivalent form of continuity, often called "sequential continuity," that involves convergence of sequences. The next result shows a connection between continuity and topology, i.e., open and closed sets.

THEOREM 3.5. *(1) Suppose S is a closed subset of $\mathbb{C}$ and that $f : S \to \mathbb{C}$ is a complex-valued function on S. Then f is continuous on S if and only if $f^{-1}(A)$ is a closed set whenever A is a closed subset of $\mathbb{C}$. That is, f is continuous on a closed set S if and only if the inverse image of every closed set is closed.*

(2) Suppose U is an open subset of $\mathbb{C}$ and that $f : U \to \mathbb{C}$ is a complex-valued function on U. Then f is continuous on U if and only if $f^{-1}(A)$ is an open set whenever A is an open subset of $\mathbb{C}$. That is, f is continuous on an open set U if and only if the inverse image of every open set is open.

PROOF. Suppose first that f is continuous on a closed set S and that A is a closed subset of $\mathbb{C}$. We wish to show that $f^{-1}(A)$ is closed. Thus, let $\{x_n\}$ be a sequence of points in $f^{-1}(A)$ that converges to a point c. Because S is a closed set, we know that $c \in S$, but in order to see that $f^{-1}(A)$ is closed, we need to show that $c \in f^{-1}(A)$. That is, we need to show that $f(c) \in A$. Now, $f(x_n) \in A$ for every n, and, because f is continuous at c, we have by Theorem 3.4 that $f(c) = \lim f(x_n)$. Hence, $f(c)$ is a limit point of A, and so $f(c) \in A$ because A is a closed set. Therefore, $c \in f^{-1}(A)$, and $f^{-1}(A)$ is closed.

Conversely, still supposing that S is a closed set, suppose f is not continuous on S, and let c be a point of S at which f fails to be continuous. Then, there exists an $\epsilon > 0$ and a sequence $\{x_n\}$ of elements of S such that $c = \lim x_n$ but such that $|f(c) - f(x_n)| \geq \epsilon$ for all n. (Why? See the proof of Theorem 3.4.) Let A be the complement of the open disk $B_\epsilon(f(c))$. Then A is a closed subset of $\mathbb{C}$. We have that $f(x_n) \in A$ for all n, but $f(c)$ is not in A. So, $x_n \in f^{-1}(A)$ for all n, but $c = \lim x_n$ is not in $f^{-1}(A)$. Hence, $f^{-1}(A)$ does not contain all of its limit points, and so $f^{-1}(A)$ is not closed. Hence, if f is not continuous on S, then there exists a closed set A such that $f^{-1}(A)$ is not closed. This completes the proof of the second half of part (1).

Next, suppose U is an open set, and assume that f is continuous on U. Let A be an open set in $\mathbb{C}$, and let c be an element of $f^{-1}(A)$. In order to prove that$f^{-1}(A)$ is open, we need to show that c belongs to the interior of $f^{-1}(A)$. Now, $f(c) \in A$, A is open, and so there exists an $\epsilon > 0$ such that the entire disk $B_\epsilon(f(c)) \subseteq A$. Then, because f is continuous at the point c, there exists a $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$. In other words, if $x \in B_\delta(c)$, then $f(x) \in B_\epsilon(f(c)) \subseteq A$. This means that $B_\delta(c)$ is contained in $f^{-1}(A)$, and hence c belongs to the interior of $f^{-1}(A)$. Hence, if f is continuous on an open set U, then $f^{-1}(A)$ is open whenever A is open. This proves half of part (2).

Finally, still assuming that U is open, suppose $f^{-1}(A)$ is open whenever A is open, let c be a point of S, and let us prove that f is continuous at c. Thus, let $\epsilon > 0$ be given, and let A be the open set $A = B_\epsilon(f(c))$. Then, by our assumption, $f^{-1}(A)$ is an open set. Also, c belongs to this open set $f^{-1}(A)$, and hence c belongs to the interior of $f^{-1}(A)$. Therefore, there exists a $\delta > 0$ such that the entire disk $b_\delta(c) \subseteq f^{-1}(A)$. But this means that if $\in S$ satisfies $|x - c| < \delta$, then $x \in B_\delta(c) \subseteq f^{-1}(A)$, and so $f(x) \in A = B_\epsilon(f(c))$. Therefore, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$, which proves that f is continuous at c, and the theorem is completely proved.

DEEPER ANALYTIC PROPERTIES OF CONTINUOUS FUNCTIONS

We collect here some theorems that show some of the consequences of continuity. Some of the theorems apply to functions either of a real variable or of a complex variable, while others apply only to functions of a real variable. We begin with what may be the most famous such result, and this one is about functions of a real variable.

THEOREM 3.6. (Intermediate Value Theorem) If $f : [a, b] \to \mathbb{R}$ is a real-valued function that is continuous at each point of the closed interval $[a, b]$, and if v is a number (value) between the numbers $f(a)$ and $f(b)$, then there exists a point c between a and b such that $f(c) = v$.

PROOF. If $v = f(a)$ or $f(b)$, we are done. Suppose then, without loss of generality, that $f(a) < v < f(b)$. Let S be the set of all $x \in [a, b]$ such that $f(x) \leq v$, and note that S is nonempty and bounded above. ($a \in S$, and b is an upper bound for S.) Let $c = \sup S$. Then there exists a sequence $\{x_n\}$ of elements of S that converges to c. (See Exercise 2.20.) So, $f(c) = \lim f(x_n)$ by Theorem 3.4. Hence, $f(c) \leq v$. (Why?)

Now, arguing by contradiction, if $f(c) < v$, let ϵ be the positive number $v - f(c)$. Because f is continuous at c, there must exist a $\delta > 0$ such that $|f(y) - f(c)| < \epsilon$ whenever $|y - c| < \delta$ and $y \in [a, b]$. Since any smaller δ satisfies the same condition, we may also assume that $\delta < b - c$. Consider $y = c + \delta/2$. Then $y \in [a, b]$, $|y - c| < \delta$, and so $|f(y) - f(c)| < \epsilon$. Hence $f(y) < f(c) + \epsilon = v$, which implies that $y \in S$. But, since $c = \sup S$, c must satisfy $c \geq y = c + \delta/2$. This is a contradiction, so $f(c) = v$, and the theorem is proved.

The Intermediate Value Theorem tells us something qualitative about the range of a continuous function on an interval $[a, b]$. It tells us that the range is "connected;" i.e., if the range contains two points c and d, then the range contains all the points between c and d. It is difficult to think what the analogous assertion would be for functions of a complex variable, since "between" doesn't mean anything for complex numbers. We will eventually prove something called the Open Mapping Theorem in Chapter VII that could be regarded as the complex analog of the Intermediate Value Theorem.

The next theorem is about functions of either a real or a complex variable.

THEOREM 3.7. *Let $f : S \to \mathbb{C}$ be a continuous function, and let C be a compact (closed and bounded) subset of S. Then the image $f(C)$ of C is also compact. That is, the continuous image of a compact set is compact.*

PROOF. First, suppose $f(C)$ is not bounded. Thus, let $\{x_n\}$ be a sequence of elements of C such that, for each n, $|f(x_n)| > n$. By the Bolzano-Weierstrass Theorem, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $x = \lim x_{n_k}$. Then $x \in C$ because $C$is a closed subset of $\mathbb{C}$. Co, $f(x) = \lim f(x_{n_k})$ by Theorem 3.4. But since $|f(x_{n_k})| > n_k$, the sequence $\{f(x_{n_k})\}$ is not bounded, so cannot be convergent. Hence, we have arrived at a contradiction, and the set $f(C)$ must be bounded.

Now, we must show that the image $f(C)$ is closed. Thus, let y be a limit point of the image $f(C)$ of C, and let $y = \lim y_n$ where each $y_n \in f(C)$. For each n, let $x_n \in C$ satisfy $f(x_n) = y_n$. Again, using the Bolzano-Weierstrass Theorem, let $\{x_{n_k}\}$ be a convergent subsequence of the bounded sequence $\{x_n\}$, and write $x = \lim x_{n_k}$. Then $x \in C$, since C is closed, and from Theorem 3.4

$$y = \lim f(x_n) = \lim f(x_{n_k}) = f(x),$$

showing that $y \in f(C)$, implying that $f(C)$ is closed.

This theorem tells us something about the range of a continuous function of a real or complex variable. It says that if a subset of the domain is closed and bounded, so is the image of that subset.

The next theorem is about continuous real-valued functions of a complex variable, and it is one of the theorems to remember.

THEOREM 3.8. *Let f be a continuous real-valued function on a compact subset S of $\mathbb{C}$. Then f attains both a maximum and a minimum value on S. That is, there exist points z_1 and z_2 in S such that $f(z_1) \leq f(z) \leq f(z_2)$ for all $z \in S$.*

PROOF. We prove that f attains a maximum value, leaving the fact that f attains a minimum value to the exercise that follows. Let M_0 be the supremum of the set of all numbers $f(x)$ for $x \in S$. (How do we know that this supremum exists?) We will show that there exists an $z_2 \in S$ such that $f(z_2) = M_0$. This will finish the proof, since we would then have $f(z_2) = M_0 \geq f(z)$ for all $z \in S$. Thus, let $\{y_n\}$ be a sequence of elements in the range of f for which the sequence $\{y_n\}$ converges to M_0. (This is Exercise 2.20 again.) For each n, let x_n be an element of S such that $y_n = f(x_n)$. Then the sequence $\{f(x_n)\}$ converges to M_0. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$. (How?) Let $z_2 = \lim x_{n_k}$. Then $z_2 \in S$, because S is closed, and $f(z_2) = \lim f(x_{n_k})$, because f is continuous. Hence, $f(z_2) = M_0$, as desired.

Exercise 3.11. (a) Prove that the f of the preceding theorem attains a minimum value on S.

(b) Give an alternate proof of Theorem 3.8 by using Theorem 3.7, and then proving that a closed and bounded subset of $\mathbb{R}$ contains both its supremum and its infimum.

(c) Let S be a compact subset of $\mathbb{C}$, and let c be a point of $\mathbb{C}$ that is not in S. Prove that there is a closest point to c in S. That is, show that there exists a point $w \in S$ such that $|w-c| \leq |z-c|$ for all points $z \in S$.
HINT: The function $z \to |z-c|$ is continuous on the set S.

Exercise 3.12. Let $f : [a,b] \to \mathbb{R}$ be a real-valued function that is continuous at each point of $[a,b]$.

(a) Prove that the range of f is a closed interval $[a',b']$. Show by example that the four numbers $f(a), f(b)$, a' and b' can be distinct.

(b) Suppose f is 1-1. Show that, if c is in the open interval (a,b), then $f(c)$ is in the open interval (a',b').

We introduce next a different kind of continuity called uniform continuity. The difference between regular continuity and uniform continuity is a bit subtle, and well worth some thought.

DEFINITION. A function $f : S \to \mathbb{C}$ is called *uniformly continuous* on S if for each positive number ϵ, there exists a positive number δ such that $|f(x)-f(y)| < \epsilon$ for all $x, y \in S$ satisfying $|x-y| < \delta$.

Basically, the difference between regular continuity and uniform conintuity is that the same δ works for all points in S.

Here is another theorem worth remembering.

THEOREM 3.9. *A continuous complex-valued function on a compact subset S of $\mathbb{C}$ is uniformly continuous.*

PROOF. We argue by contradiction. Thus, suppose f is continuous on S but not uniformly continuous. Then, there exists an $\epsilon > 0$ for which no positive number δ satisfies the uniform continuity definition. Therefore, thinking of the δ's as ranging through the numbers $1/n$, we know that for each positive integer n, there exist two points x_n and y_n in S so that

(1) $|y_n - x_n| < 1/n$, and
(2) $|f(y_n) - f(x_n)| \geq \epsilon$.

Otherwise, some $1/n$ would suffice for a δ. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ with limit x. By (1) and the triangle inequality, we deduce that x is also the limit of the corresponding subsequence $\{y_{n_k}\}$ of $\{y_n\}$. But then $f(x) = \lim f(x_{n_k}) = \lim f(y_{n_k})$, implying that $0 = \lim |f(y_{n_k}) - f(x_{n_k})|$, which implies that $|f(y_{n_k}) - f(x_{n_k})| < \epsilon$ for all large enough k. But that contradicts (2), and this completes the proof.

Continuous functions whose domains are not compact sets may or may not be uniformly continuous, as the next exercise shows.

Exercise 3.13. (a) Let $f : (0,1) \to \mathbb{R}$ be defined by $f(x) = 1/x$. Prove that f is continuous at each x in its domain but that f is not uniformly continuous there.
HINT: Set $\epsilon = 1$, and consider the pairs of points $x_n = 1/n$ and $y_n = 1/(n+1)$.

(b) Let $f : [1,\infty) \to [1,\infty)$ be defined by $f(x) = \sqrt{x}$. Prove that f is not bounded, but is nevertheless uniformly continuous on its domain.
HINT: Take $\delta = \epsilon$.

THEOREM 3.10. *Let $f : S \to T$ be a continuous 1-1 function from a compact (closed and bounded) subset of $\mathbb{C}$ onto the (compact) set T. Let $g : T \to S$ denote the inverse function f^{-1} of f. Then g is continuous. The inverse of a continuous function, that has a compact domain, is also continuous.*

PROOF. We prove that g is continuous by using Theorem 3.5; i.e., we will show that $g^{-1}(A)$ is closed whenever A is a closed subset of $\mathbb{C}$. But this is easy, since $g^{-1}(A) = g^{-1}(A \cap S) = f(A \cap S)$, and this is a closed set by Theorem 3.7, because $A \cap S$ is compact. See part (e) of Exercise 2.24.

REMARK. Using the preceding theorem, and the exercise below, we will show that taking nth roots is a continuous function. that is, the function f defined by $f(x) = x^{1/n}$ is continuous.

Exercise 3.14. Use the preceding theorem to show the continuity of the following functions.

(a) Show that if n is an odd positive integer, then there exists a continuous function g defined on all of $\mathbb{R}$ such that $g(x)$ is an nth root of x for all real numbers x. That is, $(g(x))^n = x$ for all real x. (The function $f(x) = x^n$ is 1-1 and continuous.)

(b) Show that if n is any positive integer then there exists a unique continuous function g defined on $[0, \infty)$ such that $g(x)$ is an nth root of x for all nonnegative x.

(c) Let $r = p/q$ be a rational number. Prove that there exists a continuous function $g : [0, \infty) \to [0, \infty)$ such that $g(x)^q = x^p$ for all $x \geq 0$; i.e., $g(x) = x^r$ for all $x \geq 0$.

THEOREM 3.11. *Let f be a continuous 1-1 function from the interval $[a, b]$ onto the interval $[c, d]$. Then f must be strictly monotonic, i.e., strictly increasing everywhere or strictly decreasing everywhere.*

PROOF. Since f is 1-1, we clearly have that $f(a) \neq f(b)$, and, without loss of generality, let us assume that $c = f(a) < f(b) = d$. It will suffice to show that if α and β belong to the open interval (a, b), and $\alpha < \beta$, then $f(\alpha) \leq f(\beta)$. (Why will this suffice?) Suppose by way of contradiction that there exists $\alpha < \beta$ in (a, b) for which $f(\alpha) > f(\beta)$. We use the intermediate value theorem to derive a contradiction. Consider the four points $a < \alpha < \beta < b$. Either $f(a) < f(\alpha)$ or $f(\beta) < f(b)$. (Why?) In the first case ($f(a) < f(\alpha)$), $f([a, \alpha])$ contains every value between $f(a)$ and $f(\alpha)$. And, $f([\alpha, \beta])$ contains every value between $f(\alpha)$ and $f(\beta)$. So, let v be a number such that $f(a) < v$, $f(\beta) < v$, and $v < f(\alpha)$ (why does such a number v exist?). By the Intermediate Value Theorem, there exists $x_1 \in (a, \alpha)$ such that $v = f(x_1)$, and there exists an $x_2 \in (\alpha, \beta)$ such that $v = f(x_2)$. But this contradicts the hypothesis that f is 1-1, since $x_1 \neq x_2$. A similar argument leads to a contradiction in the second case $f(\beta) < f(b)$. (See the following exercise.) Hence, there can exist no such α and β, implying that f is strictly increasing on $[a, b]$.

Exercise 3.15. Derive a contradiction from the assumption that $f(\beta) < f(b)$ in the preceding proof.

POWER SERIES FUNCTIONS

The class of functions that we know are continuous includes, among others, the polynomials, the rational functions, and the nth root functions. We can combine these functions in various ways, e.g., sums, products, quotients, and so on. We also can combine continuous functions using composition, so that we know that nth roots of rational functions are also continuous. The set of all functions obtained in this manner is called the class of "algebraic functions." Now that we also have developed a notion of limit, or infinite sum, we can construct other continuous functions.

We introduce next a new kind of function. It is a natural generalization of a polynomial function. Among these will be the exponential function and the trigonometric functions. We begin by discussing functions of a complex varible, although totally analogous definitions and theorems hold for functions of a real variable.

DEFINITION. Let $\{a_n\}_0^\infty$ be a sequence of real or complex numbers. By the *power series function* $f(z) = \sum_{n=0}^\infty a_n z^n$ we mean the function $f : S \to \mathbb{C}$ where the domain S is the set of all $z \in \mathbb{C}$ for which the infinite series $\sum a_n z^n$ converges, and where f is the rule that assigns to such a $z \in S$ the sum of the series.

The numbers $\{a_n\}$ defining a power series function are called the *coefficients* of the function.

We associate to a power series function $f(z) = \sum_{n=0}^\infty a_n z^n$ its sequence $\{S_N\}$ of partial sums. We write

$$S_N(z) = \sum_{n=0}^N a_n z^n.$$

Notice that polynomial functions are very special cases of power series functions. They are the power series functions for which the coefficients $\{a_n\}$ are all 0 beyond some point. Note also that each partial sum S_N for any power series function is itself a polynomial function of degree less than or equal to N. Moreover, if f is a power series function, then for each z in its domain we have $f(z) = \lim_N S_N(z)$. Evidently, every power series function is a "limit" of a sequence of polynomials.

Obviously, the domain $S \equiv S_f$ of a power series function f depends on the coefficients $\{a_n\}$ determining the function. Our first goal is to describe this domain.

THEOREM 3.12. *Let f be a power series function: $f(z) = \sum_{n=0}^\infty a_n z^n$ with domain S. Then:*

(1) *0 belongs to S.*
(2) *If a number t belongs to S, then every number u, for which $|u| < |t|$, also belongs to S.*
(3) *S is a disk of radius r around 0 in $\mathbb{C}$ (possibly open, possibly closed, possibly neither, possibly infinite). That is, S consists of the disk $B_r(0) = \{z : |z| < r\}$ possibly together with some of the points z for which $|z| = r$.*
(4) *The radius r of the disk in part (3) is given by the Cauchy-Hadamard formula:*

$$r = \frac{1}{\limsup |a_n|^{1/n}},$$

which we interpret to imply that $r = 0$ if and only if the limsup on the right is infinite, and $r = \infty$ if and only if that limsup is 0.

PROOF. Part (1) is clear.

To see part 2, assume that t belongs to S and that $|u| < |t|$. We wish to show that the infinites series $\sum a_n u^n$ converges. In fact, we will show that $\sum |a_n u^n|$ is convergent, i.e., that $\sum a_n u^n$ is absolutely convergent. We are given that the infinite series

$\sum a_n t^n$ converges, which implies that the terms $a_n t^n$ tend to 0. Hence, let B be a number such that $|a_n z^n| \leq B$ for all n, and set $\alpha = |u|/|t|$. Then $\alpha < 1$, and therefore the infinite series $\sum B\alpha^n$ is convergent. Finally, $|a_n u^n| = |a_n t^n|\alpha^n \leq B\alpha^n$, which, by the Comparison Test, implies that $\sum |a_n u^n|$ is convergent, as desired.

Part (3) follows, with just a little thought, from part 2.

To prove part (4), note that $\limsup |a_n|^{1/n}$ either is finite or it is infinite. assume first that the sequence $\{|a_n|^{1/n}\}$ is not bounded; i.e., that $\limsup |a_n|^{1/n} = \infty$. Then, given any number p, there are infinitely many terms $|a_n|^{1/n}$ that are larger than p. So, for any $z \neq 0$, there exist infinitely many terms $|a_n|^{1/n}$ that are larger than $1/|z|$. But then $|a_n z^n| > 1$ for all such terms. Therefore the infinite series $\sum a_n z^n$ is not convergent, since $\lim a_n z^n$ is not zero. So no such z is in the domain S. This shows that if $\limsup |a_n|^{1/n} = \infty$, then $r = 0 = 1/\limsup |a_n|^{1/n}$.

Now, suppose the sequence $\{|a_n|^{1/n}\}$ is bounded, and let L denote its limsup. We must show that $1/r = L$. We will show the following two claims: (a) if $1/|z| > L$, then $z \in S$, and (b) if $1/|z| < L$, then $z \notin S$. (Why will these two claims complete the proof?) Thus, suppose that $1/|z| > L$. Let β be a number satisfying $L < \beta < 1/|z|$, and let $\alpha = \beta|z|$. Then $0 < \alpha < 1$. Now there exists a natural number N so that $|a_n|^{1/n} < \beta$ for all $n \geq N$, or equivalently $|a_n| \leq \beta^n$ for all $n \geq N$. (See part (a) of Exercise 2.17.) This means that for all $n \geq N$ we have $|a_n z^n| = |a_n/\beta^n||\beta z|^n \leq \alpha^n$. This implies by the Comparison Test that the power series $\sum a_n z^n$ is absolutely convergent, whence convergent. Hence, $z \in S$, and this proves claim (a) above. Incidentally, note also that if $L = 0$, this argument shows that $r = \infty$, as desired.

To verify claim (b), suppose that $1/|z| < L$. Then there are infinitely many terms of the sequence $\{|a_n|^{1/n}\}$ that are greater than $1/|z|$. (Why?) For each such term, we would then have $|a_n z^n| \geq 1$. This means that the infinite series $\sum a_n z^n$ is not convergent and $z \notin S$, which shows claim b.

Hence, in all cases, we have that $r = 1/\limsup |a_n|^{1/n}$, as desired.

DEFINITION. If f is a power series function, the number r of the preceding theorem is called the *radius of convergence* of the power series. The disk S of radius r around 0, denoted by $B_r(0)$, is called the *disk of convergence.*

Exercise 3.16. Compute directly the radii of convergence for the following power series functions, i.e., without using the Cauchy-Hadamard formula. Then, when possible, verify that the Cauchy-Hadamard formula agrees with your computation.

(a) $f(z) = \sum z^n$.
(b) $f(z) = \sum n^2 z^n$.
(c) $f(z) = \sum (-1)^n (1/(n+1)) z^n$.
(d) $f(z) = \sum (1/(n+1)) z^{3n+1}$.
(e) $f(z) = \sum_{n=0}^{\infty} z^n/n!$.

Exercise 3.17. (a) Use part (e) of Exercise 3.1 to show that a power series function p is an even function if and only if its only nonzero coefficients are even ones, i.e., the a_{2k}'s. Show also that a power series function is an odd function if and only if its only nonzero coefficients are odd ones, i.e., the a_{2k+1}'s.

(b) Suppose $f(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}$ is a power series function that is an even function. Show that

$$f(iz) = \sum_{k=0}^{\infty} (-1)^k a_{2k} z^{2k} = f^a(z),$$

where f^a is the power series function obtained from f by alternating the signs of its coefficients. We call this function f^a the alternating version of f.

(c) If $g(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}$ is a power series function that is an odd function, show that

$$g(iz) = i\sum_{k=0}^{\infty}(-1)^k a_{2k+1} z^{2k+1} = ig^a(z),$$

where again g^a is the power series function obtained from g by alternating the signs of its coefficients.

(d) If f is any power series function, show that

$$f(iz) = f_e(iz) + f_o(iz) = f_e^a(z) + if_o^a(z),$$

and hence that $f_e(iz) = f_e^a(z)$ and $f_o(iz) = ip_o^a(z)$.

The next theorem will not come as a shock, but its proof is not so simple.

THEOREM 3.13. *Let $f(z) = \sum a_n z^n$ be a power series function with radius of convergence r. Then f is continuous at each point in the open disk $B_r(0)$, i.e., at each point z for which $|z| < r$.*

PROOF. Let $z \in B_r(0)$ be given. We must make some auxiliary constructions before we can show that f is continuous at z. First, choose a z' such that $|z| < |z'| < r$. Next, set $b_n = |na_n|$, and define $g(z) = \sum b_n z^n$. By the Cauchy-Hadamard formula, we see that the power series function g has the same radius of convergence as the power series function f. Indeed, $\limsup |b_n|^{1/n} = \limsup n^{1/n}|a_n| = \lim n^{1/n} \limsup |a_n|$. Therefore, z' belongs to the domain of g. Let M be a number such that each partial sum of the series $g(z') = \sum_{n=0}^{N} b_n z'^n$ is bounded by M.

Now, let $\epsilon > 0$ be given, and choose δ to be the minimum of the two positive numbers $\epsilon|z'|/M$ and $|z'| - |z|$. We consider any y for which $|y - z| < \delta$. Then $y \in B_r(0)$, $|y| < |z'|$, and

$$\begin{aligned}
|f(y) - f(z)| &= \lim |S_N(y) - S_N(z)| \\
&= \lim |\sum_{n=0}^{N} a_n(y^n - z^n)| \\
&\le \lim_N \sum_{n=0}^{N} |a_n||y^n - z^n| \\
&= \lim_N \sum_{n=1}^{N} |a_n||y - z| \sum_{j=0}^{n-1} |y^j||z^{n-1-j}| \\
&\le \lim_N \sum_{n=1}^{N} |a_n||y - z| \sum_{j=0}^{n-1} |z'|^{n-1} \\
&\le \lim_N |y - z|(1/|z'|) \sum_{n=0}^{N} n|a_n||z'|^n \\
&\le |y - z| \lim_N \frac{M}{|z'|} \\
&< \delta \lim_N \frac{M}{|z'|} \\
&\le \epsilon.
\end{aligned}$$

This completes the proof.

Exercise 3.18. (a) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series function, and let $p(z) = \sum_{k=0}^{m} b_k z^k$ be a polynomial function. Prove that $f + p$ and fp are both power series functions. Express the coefficients for $f + p$ and fp in terms of the a_n's and b_k's.

(b) Suppose f and g are power series functions. Prove that $f + g$ is a power series function. What is its radius of convergence? What about cf? What about fg? What about f/g? What about $|f|$?

Exercise 3.19. (a) Prove that every polynomial is a power series function with infinite radius of convergence.

(b) Prove that $1/z$ and $(1/(z-1)(z+2))$ are not power series functions. (Their domains aren't right.)

(c) Define $f(z) = \sum_{n=0}^{\infty}(-1)^n z^{2n+1}$. Prove that the radius of convergence of this power series function is 1, and that $f(z) = \frac{z}{1+z^2}$ for all $z \in B_1(0)$. Conclude that the rational function $z/(1 + z^2)$ agrees with a power series function on the disk $B_1(0)$. But, they are **not** the same function.

HINT: Use the infinite geometric series.

Theorem 3.13 and Exercises 3.18 and 3.19 raise a very interesting and subtle point. Suppose $f(z) = \sum a_n z^n$ is a power series function having finite radius of convergence $r > 0$. Theorem 3.13 says that f is continuous on the open disk, but it does not say anything about the continuity of f at points on the boundary of this disk that are in the domain of f, i,e., at points z_0 for which $|z_0| = r$. and $\sum a_n z_0^n$ converges. Suppose $g(z)$ is a continuous function whose domain contains the open disk $B_r(0)$ and also a point z_0, and assume that $f(z) = g(z)$ for all $z \in B_r(0)$. Does $f(z_0)$ have to agree with $g(z_0)$? It's worth some thought to understand just what this question means. It amounts to a question of the equality of two different kinds of limits. $f(z_0)$ is the sum of an infinite series, the limit of a sequence of partial sums, while, because g is continuous at z_0, $g(z_0 = \lim_{z\to z_0} g(z)$. At the end of this chapter, we include a theorem of Abel that answers this question.

The next theorem is the analog for power series functions of part (2) of Theorem 3.1 for polynomials. We call it the "Identity Theorem," but it equally well could be known as the "Uniqueness of Coefficients Theorem," for it implies that different coefficients mean different functions.

THEOREM 3.14. (Identity Theorem) Let $f(z) = \sum a_n z^n$ be a power series function with positive radius of convergence r. Suppose $\{z_k\}$ is a sequence of nonzero distinct numbers in the domain of f such that:

(1) $\lim z_k = 0$.
(2) $f(z_k) = 0$ for all k.

Then f is identically 0 ($f(z) \equiv 0$ for all $z \in S$). Moreover, each coefficient a_n of f equals 0.

PROOF. Arguing by induction on n, let us prove that all the coefficients a_n are 0. First, since f is continuous at 0, and since $\lim z_k = 0$, we have that a_0, which equals $f(0)$, $= \lim f(z_k) = 0$.

Assume then that $a_0 = a_1 = \ldots = a_{n-1} = 0$. Then

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$$
$$= z^n \sum_{j=0}^{\infty} b_j z^j,$$

where $b_j = a_{n+j}$. If g is the power series function defined by $g(z) = \sum b_j z^j$, then, by the Cauchy-Hadamard Formula, we have that the radius of convergence for g is the same as that for f. (Why does $\limsup |b_j|^{1/j} = \limsup |a_k|^{1/k}$?) We have that $f(z) = z^n g(z)$ for all z in the common disk of convergence of these functions f and g. Since, for each k, $z_k \neq 0$ and $f(z_k) = z_k^n g(z_k) = 0$, it follows that $g(z_k) = 0$ for every k. Since g is continuous at 0, it then follows as in the argument above that $g(0) = 0$. But, $g(0) = b_0 = a_n$. Hence $a_n = 0$, and so by induction all the coefficients of the power series function f are 0. Clearly this implies that $f(z)$ is identically 0.

COROLLARY. *Suppose f and g are two power series functions, that $\{z_k\}$ is a sequence of nonzero points that converges to 0, and that $f(z_k) = g(z_k)$ for all k. Then f and g have the same coefficients, the same radius of convergence, and hence $f(z) = g(z)$ for all z in their common domain.*

Exercise 3.20. (a) Prove the preceding corollary. (Compare with the proof of Theorem 3.1.)

(b) Use the corollary, and the power series function $g(z) = z$, to prove that $f(z) = |z|$ is not a power series function.

(c) Show that there are power series functions that are not polynomial functions.

(d) Let $f(z) = \sum a_n z^n$ be a power series function with infinite radius of convergence, all of whose coefficients are positive. Show that there is no rational function $r = p/q$ for which $f(z) = r(z)$ for all complex numbers z. Conclude that the collection of power series functions provides some **new** functions.

HINT: Use the fact that for any n we have that $f(x) > a_n x^n$ for all positive x. Then, by choosing n appropriately, derive a contradiction to the resulting fact that $|p(x)/q(x)| > a_n x^n$ for all positive x. See part (b) of Exercise 3.2.

THE ELEMENTARY TRANSCENDENTAL FUNCTIONS

Having introduced a class of new functions (power series functions), we might well expect that some of these will have interesting and unexpected properties. So, which sets of coefficients might give us an exotic new function? Unfortunately, at this point in our development, we haven't much insight into this question. It is true, see Exercise 3.16, that most power series functions that we naturally write down have finite radii of convergence. Such functions may well be new and fascinating, but as a first example, we would prefer to consider a power series function that is defined everywhere, i.e., one with an infinite radius of convergence. Again revisiting Exercise 3.16, let us consider the coefficients $a_n = 1/n!$. This may seem a bit ad hoc, but let's have a look.

DEFINITION. Define a power series function, denoted by exp, as follows:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We will call this function, with 20-20 hindsight, the *exponential function.*

What do we know about this function, apart from the fact that it is defined for all complex numbers? We certainly do not know that it has anything to do with the function e^z; that will come in the next chapter. We do know what the number e is, but we do not know how to raise that number to a complex exponent.

All of the exponential function's coefficients are positive, and so by part (d) of Exercise 3.20 exp is not a rational function; it really is something new. It is natural to consider the even and odd parts $\exp_e$ and $\exp_o$ of this new function. And then, considering the constructions in Exercise 3.17, to introduce the alternating versions $\exp_e^a$ and $\exp_o^a$ of them.

DEFINITION. Define two power series functions cosh (hyperbolic cosine) and sinh (hyperbolic sine) by

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2} \text{ and } \sinh(z) = \frac{\exp(z) - \exp(z)}{2},$$

and two other power series functions cos (cosine) and sin (sine) by

$$\cos(z) = \cosh(iz) = \frac{\exp(iz) + \exp(-iz)}{2}$$

and

$$\sin(z) = -i\sinh(iz) = \frac{\exp(iz) - \exp(-iz)}{2i}.$$

The five functions just defined are called the *elementary transcendental functions*, the sinh and cosh functions are called the basic *hyperbolic functions,* and the sine and cosine functions are called the basic *trigonometric* or *circular functions.* The connections between the hyperbolic functions and hyperbolic geometry, and the connection between the trigonometric functions and circles and triangles, will only emerge in the next chapter. From the very definitions, however, we can see a connection between the hyperbolic functions and the trigonometric functions. It's something like interchanging the roles of the real and imaginary axes. This is probably worth some more thought.

Exercise 3.21. (a) Verify the following equations:

$$\begin{aligned}\exp(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^k}{k!} + \ldots, \\ &= \cosh(z) + \sinh(z).\end{aligned}$$

$$\begin{aligned}\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots + (-1)^k \frac{z^{2k+1}}{(2k+1)!} + \ldots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!},\end{aligned}$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots + (-1)^k \frac{z^{2k}}{(2k)!} + \ldots$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!},$$

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!},$$

and

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

(These expressions for the elementary transcendental functions are perhaps the more familiar ones from a calculus course.)

(b) Compute the radii of convergence for the elementary transcendental functions.

HINT: Do not use the Cauchy-Hadamard formula. Just figure out for which z's the functions are defined.

(c) Verify that $\exp(0) = 1$, $\sin(0) = \sinh(0) = 0$, and $\cos(0) = \cosh(0) = 1$.

(d) Prove that all five of the elementary transcendental functions are not rational functions.

(e) Can you explain why $\sin^2(z) + \cos^2(z) \equiv 1$? What about the " Addition Formula"

$$\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w).$$

Exercise 3.22. (a) Show that the elementary transcendental functions map real numbers to real numbers. That is, as functions of a real variable, they are real-valued functions.

(b) Show that the exponential function exp is not bounded above. Show in fact that, for each nonnegative integer n, $\exp(x)/x^n$ is unbounded. Can you show that $\exp(x) = e^x$? What, in fact, does e^x mean if x is an irrational or complex number?

At this point, we probably need a little fanfare!

THEOREM 3.14159. (Definition of π) There exists a smallest positive number x for which $\sin(x) = 0$. We will denote this distinguished number x by the symbol π.

PROOF. First we observe that $\sin(1)$ is positive. Indeed, the infinite series for $\sin(1)$ is alternating. It follows from the alternating series test (Theorem 2.18) that $\sin(1) > 1 - 1/6 = 5/6$.

Next, again using the alternating series test, we observe that $\sin(4) < 0$. Indeed,

$$\sin(4) < 4 - \frac{4^3}{3!} + \frac{4^5}{5!} - \frac{4^7}{7!} + \frac{4^9}{9!} \approx -0.4553 < 0.$$

Hence, by the intermediate value theorem, there must exist a number c between 1 and 4 such that $\sin(c) = 0$. So, there is at least one positive number x such that $\sin(x) = 0$. However, we must show that there is a smallest positive number satisfying this equation.

Let A be the set of all $x > 0$ for which $\sin(x) = 0$. Then A is a nonempty set of real numbers that is bounded below. Define $\pi = \inf A$. We need to prove that $\sin(\pi) = 0$, and that $\pi > 0$. Clearly then it will be the smallest positive number x for which $\sin(x) = 0$.

By Exercise 2.20, there exists a sequence $\{x_k\}$ of elements of A such that $\pi = \lim x_k$. Since sin is continuous at π, it follows that $\sin(\pi) = \lim \sin(x_k) = \lim 0 = 0$. Finally, if π were equal to 0, then by the Identity Theorem, Theorem 3.14, we would have that $\sin x = 0$ for all x. Since this is clearly not the case, we must have that $\pi > 0$.

Hence, π is the smallest (minimum) positive number x for which $\sin(x) = 0$.

As hinted at earlier, the connection between this number π and circles is not at all evident at the moment. For instance, you probably will not be able to answer the questions in the next exercise.

Exercise 3.23. (a) Can you see why $\sin(x + 2\pi) \equiv \sin(x)$? That is, is it obvious that sin is a periodic function?

(b) Can you prove that $\cos(\pi) = -1$?

REMARK. Defining π to be the smallest positive zero of the sine function may strike many people as very much "out of the blue." However, the zeroes of a function are often important numbers. For instance, a zero of the function $x^2 - 2$ is a square root of 2, and that number we know was exztremely important to the Greeks as they began the study of what real numbers are. A zero of the function $z^2 + 1$ is something whose square is -1, i.e., negative. The idea of a square being negative was implausible at first, but is fundamental now, so that the zero of this particular function is critical for our understanding to numbers. Very likely, then, the zeroes of any "new" function will be worth studying. For instance, we will soon see that, perhaps disappointingly, there are no zeroes for the exponential function: $\exp(z)$ is never 0. Maybe it's even more interesting then that there are zeroes of the sine function.

The next theorem establishes some familiar facts about the trigonometric functions.

THEOREM 3.15.

(1) $\exp(iz) = \cos(z) + i\sin(z)$ *for all* $z \in \mathbb{C}$.

(2) *Let* $\{z_k\}$ *be a sequence of complex numbers that converges to 0. Then*

$$\lim \frac{\sin(z_k)}{z_k} = 0.$$

(3) *Let* $\{z_k\}$ *be a sequence of complex numbers that converges to 0. Then*

$$\lim \frac{1 - \cos(z_k)}{z_k^2} = \frac{1}{2}.$$

Exercise 3.24. Prove Theorem 3.15.
HINT: For parts (2) and (3), use Theorem 3.13.

ANALYTIC FUNCTIONS AND TAYLOR SERIES

DEFINITION. Let S be a subset of $\mathbb{C}$, let $f : S \to \mathbb{C}$ be a complex-valued function, and let c be a point of S. Then f is said to be *expandable in a Taylor series around* c with radius of convergence r if there exists an $r > 0$ such that $B_r(c) \subseteq S$, and $f(z)$ is given by the formula

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

for all $z \in B_r(c)$.

Let S be a subset of $\mathbb{R}$, let $f : S \to \mathbb{R}$ be a real-valued function on S, and let c be a point of S. Then f is said to be *expandable in a Taylor series around* c with radius of convergence r if there exists an $r > 0$ such that the interval $(c-r, c+r) \subseteq S$, and $f(x)$ is given by the formula

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

for all $x \in (c-r, c+r)$.

Suppose S is an open subset of $\mathbb{C}$. A function $f : S \to \mathbb{C}$ is called *analytic on* S if it is expandable in a Taylor series around every point c of S.

Suppose S is an open subset of $\mathbb{R}$. A function $f : S \to \mathbb{C}$ is called *real analytic on* S if it is expandable in a Taylor series around every point c of S.

THEOREM 3.16. *Suppose S is a subset of $\mathbb{C}$, that $f : S \to \mathbb{C}$ is a complex-valued function and that c belongs to S. Assume that f is expandable in a Taylor series around c with radius of convergence r. Then f is continuous at each $z \in B_r(c)$.*

Suppose S is a subset of $\mathbb{R}$, that $f : S \to \mathbb{R}$ is a real-valued function and that c belongs to S. Assume that f is expandable in a Taylor series around c with radius of convergence r. Then f is continuous at each $x \in (c-r, c+r)$.

PROOF. If we let g be the power series function given by $g(z) = \sum a_n z^n$, and T be the function defined by $T(z) = z - c$, then $f(z) = g(T(z))$, and this theorem is a consequence of Theorems 3.3 and 3.13.

Exercise 3.25. Prove that $f(z) = 1/z$ is analytic on its domain.
HINT: Use $r = |c|$, and then use the infinite geometric series.

Exercise 3.26. State and prove an Identity Theorem, analogous to Theorem 3.14, for functions that are expandable in a Taylor series around a point c.

Exercise 3.27. (a) Prove that every polynomial is expandable in a Taylor series around every point c.
HINT: Use the binomial theorem.

(b) Is the exponential function expandable in a Taylor series around the number -1?

UNIFORM CONVERGENCE

We introduce now two different notions of the limit of a sequence of functions. Let S be a set of complex numbers, and let $\{f_n\}$ be a sequence of complex-valued functions each having domain S.

DEFINITION. We say that the sequence $\{f_n\}$ *converges* or *converges pointwise* to a function $f : S \to \mathbb{C}$ if for every $x \in S$ and every $\epsilon > 0$ there exists a natural number N, depending on x and ϵ, such that for every $n \geq N$, $|f_n(x) - f(x)| < \epsilon$. That is, equivalently, $\{f_n\}$ converges pointwise to f if for every $x \in S$ the sequence $\{f_n(x)\}$ of numbers converges to the number $f(x)$.

We say that the sequence $\{f_n\}$ *converges uniformly* to a function f if for every $\epsilon > 0$, there exists an N, depending only on ϵ, such that for every $n \geq N$ and every $x \in S$, $|f_n(x) - f(x)| < \epsilon$.

If $\{u_n\}$ is a sequence of functions defined on S, we say that the infinite series $\sum u_n$ *converges uniformly* if the sequence $\{S_N = \sum_{n=0}^N u_n\}$ of partial sums converges uniformly.

These two definitions of convergence of a sequence of functions differ in subtle ways. Study the word order in the definitions.

Exercise 3.28. (a) Prove that if a sequence $\{f_n\}$ of functions converges uniformly on a set S to a function f then it converges pointwise to f.

(b) Let $S = (0, 1)$, and for each n define $f_n(x) = x^n$. Prove that $\{f_n\}$ converges pointwise to the zero function, but that $\{f_n\}$ does not converge uniformly to the zero function. Conclude that pointwise convergence does **not** imply uniform convergence.

HINT: Suppose the sequence does converge uniformly. Take $\epsilon = 1/2$, let N be a corresponding integer, and consider x's of the form $x = 1 - h$ for tiny h's.

(c) Suppose the sequence $\{f_n\}$ converges uniformly to f on S, and the sequence $\{g_n\}$ converges uniformly to g on S. Prove that the sequence $\{f_n + g_n\}$ converges uniformly to $f + g$ on S.

(d) Suppose $\{f_n\}$ converges uniformly to f on S, and let c be a constant. Show that $\{cf_n\}$ converges uniformly to cf on S.

(e) Let $S = \mathbb{R}$, and set $f_n(x) = x + (1/n)$. Does $\{f_n\}$ converge uniformly on S? Does $\{f_n^2\}$ converge uniformly on S? What does this say about the limit of a product of uniformly convergent sequences versus the product of the limits?

(f) Suppose a and b are nonnegative real numbers and that $|a - b| < \epsilon^2$. Prove that $|\sqrt{a} - \sqrt{b}| < 2\epsilon$.

HINT: Break this into cases, the first one being when both $\sqrt{a}$ and $\sqrt{b}$ are less than ϵ.

(g) Suppose $\{f_n\}$ is a sequence of nonnegative real-valued functions that converges uniformly to f on S. Use part (f) to prove that the sequence $\{\sqrt{f_n}\}$ converges uniformly to $\sqrt{f}$.

(h) For each positive integer n, define f_n on $(-1, 1)$ by $f_n(x) = |x|^{1+1/n}$. Prove that the sequence $\{f_n\}$ converges uniformly on $(-1, 1)$ to the function $f(x) = |x|$.

HINT: Let $\epsilon > 0$ be given. Consider $|x|$'s that are $< \epsilon$ and $|x|$'s that are $\geq \epsilon$. For $|x| < \epsilon$, show that $|f_n(x) - f(x)| < \epsilon$ for all n. For $|x| \geq \epsilon$, choose N so that $|\epsilon^{1/n} - 1| < \epsilon$. How?

Exercise 3.29. Let $\{f_n\}$ be a sequence of functions on a set S, let f be a function on S, and suppose that for each n we have $|f(x) - f_n(x)| < 1/n$ for all $x \in S$. Prove that the sequence $\{f_n\}$ converges uniformly to f.

We give next four important theorems concerning uniform convergence. The first of these theorems is frequently used to prove that a given function is continuous.

The theorem asserts that if f is the uniform limit of a sequence of continuous functions, then f is itself continuous.

THEOREM 3.17. (The uniform limit of continuous functions is continuous.) Suppose $\{f_n\}$ is a sequence of continuous functions on a set $S \subseteq \mathbb{C}$, and assume that the sequence $\{f_n\}$ converges uniformly to a function f. Then f is continuous on S.

PROOF. This proof is an example of what is called by mathematicians a "3ϵ argument."

Fix an $x \in S$ and an $\epsilon > 0$. We wish to find a $\delta > 0$ such that if $y \in S$ and $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

We use first the hypothesis that the sequence converges uniformly. Thus, given this $\epsilon > 0$, there exists a natural number N such that if $n \geq N$ then $|f(z) - f_n(z)| < \epsilon/3$ for all $z \in S$. Now, because f_N is continuous at x, there exists a $\delta > 0$ such that if $y \in S$ and $|y - x| < \delta$ then $|f_N(y) - f_N(x)| < \epsilon/3$. So, if $y \in S$ and $|y - x| < \delta$, then

$$\begin{aligned}|f(y) - f(x)| &= |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)| \\ &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

This completes the proof.

REMARK. Many properties of functions are preserved under the taking of uniform limits, e.g., continuity, as we have just seen. However, not all properties are preserved under this limit process. Differentiability is not, integrability is sometimes, being a power series function is, and so on. We must be alert to be aware of when it works and when it does not.

THEOREM 3.18. (Weierstrass M-Test) Let $\{u_n\}$ be a sequence of complex-valued functions defined on a set $S \subseteq \mathbb{C}$. Write S_N for the partial sum $S_N(x) = \sum_{n=0}^{N} u_n(x)$. Suppose that, for each n, there exists an $M_n > 0$ for which $|u_n(x)| \leq M_n$ for all $x \in S$. Then

(1) If $\sum M_n$ converges, then the sequence $\{S_N\}$ converges uniformly to a function S. That is, the infinite series $\sum u_n$ converges uniformly.
(2) If each function u_n is continuous, and $\sum M_n$ converges, then the function S of part (1) is continuous.

PROOF. Because $\sum M_n$ is convergent, it follows from the Comparison Test that for each $x \in S$ the infinite series $\sum_{n=0}^{\infty} u_n(x)$ is absolutely convergent, hence convergent. Define a function S by $S(x) = \sum_{n=0}^{\infty} u_n(x) = \lim S_N(x)$.

To show that $\{S_N\}$ converges uniformly to S, let $\epsilon > 0$ be given, and choose a natural number N such that $\sum_{n=N+1}^{\infty} M_n < \epsilon$. This can be done because $\sum M_n$

converges. Now, for any $x \in S$ and any $m \geq N$, we have

$$\begin{aligned}
|S(x) - S_m(x)| &= |\lim_{k\to\infty} S_k(x) - S_m(x)| \\
&= |\lim_{k\to\infty} (S_k(x) - S_m(x))| \\
&= \lim_{k\to\infty} |S_k(x) - S_m(x)| \\
&= \lim_{k\to\infty} |\sum_{n=m+1}^{k} u_n(x)| \\
&\leq \lim_{k\to\infty} \sum_{n=m+1}^{k} |u_n(x)| \\
&\leq \lim_{k\to\infty} \sum_{n=m+1}^{k} M_n \\
&= \sum_{n=m+1}^{\infty} M_n \\
&\leq \sum_{n=N}^{\infty} M_n \\
&< \epsilon.
\end{aligned}$$

This proves part (1).

Part (2) now follows from part (1) and Theorem 3.17, since the S_N's are continuous.

THEOREM 3.19. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series function with radius of convergence $r > 0$, and let $\{S_N(z)\}$ denote the sequence of partial sums of this series:*

$$S_N(z) = \sum_{n=0}^{N} a_n z^n.$$

If $0 < r' < r$, then the sequence $\{S_N\}$ converges uniformly to f on the disk$B_{r'}(0)$.

PROOF. Define a power series function g by $g(z) = \sum_{n=0}^{\infty} |a_n| z^n$, and note that the radius of convergence for g is the same as that for f, i.e., r. Choose t so that $r' < t < r$. Then, since t belongs to the disk of convergence of the power series function g, we know that $\sum_{n=0}^{\infty} |a_n| t^n$ converges. Set $m_n = |a_n| t^n$, and note that $\sum m_n$ converges. Now, for each $z \in B_{r'}(0)$, we have that

$$|a_n z^n| \leq |a_n| {r'}^n \leq |a_n| t^n = m_n,$$

so that the infinite series $\sum a_n z^n$ converges uniformly on $B_{r'}(0)$ by the Weierstrass M-Test.

Exercise 3.30. Let $f(z) = \sum_{n=0}^{\infty} z^n$. Recall that the radius of convergence for f is 1. Verify that the sequence $\{S_N\}$ of partial sums of this power series function fails to converge uniformly on the full open disk of convergence $B_1(0)$, so that the requirement that $r' < r$ is necessary in the preceding theorem.

The next theorem shows that continuous, real-valued functions on closed bounded intervals are uniform limits of step functions. Step functions have not been mentioned lately, since they aren't continuous functions, but this next theorem will be crucial for us when we study integration in Chapter V.

THEOREM 3.20. *Let f be a continuous real-valued function on the closed and bounded interval $[a,b]$. Then there exists a sequence $\{h_n\}$ of step functions on $[a,b]$ that converges uniformly to f.*

PROOF. We use the fact that a continuous function on a compact set is uniformly continuous (Theorem 3.9).

For each positive integer n, let δ_n be a positive number satisfying $|f(x)-f(y)| < 1/n$ if $|x-y| < \delta_n$. Such a δ_n exists by the uniform continuity of f on $[a,b]$. Let $P_n = \{x_0 < x_1 < \ldots < x_{m_n}\}$ be a partition of $[a,b]$ for which $x_i - x_{i-1} < \delta_n$ for all $1 \leq i \leq m_n$. Define a step function h_n on $[a,b]$ as follows: If $x_{i-1} \leq x < x_i$, then $h_n(x) = f(x_{i-1})$. This defines $h_n(x)$ for every $x \in [a,b)$, and we complete the definition of h_n by setting $h_n(b) = f(b)$. It follows immediately that h_n is a step function.

Now, we claim that $|f(x)-h_n(x)| < 1/n$ for all $x \in [a,b]$. This is clearly the case for $x = b$, since $f(b) = h_n(b)$ for all n. For any other x, let i be the unique index such that $x_{i-1} \leq x < x_i$. Then

$$|f(x) - h_n(x)| = |f(x) - f(x_{i-1})| < 1/n$$

because $|x - x_{i-1}| < \delta_n$.

So, we have defined a sequence $\{h_n\}$ of step functions, and the sequence $\{h_n\}$ converges uniformly to f by Exercise 3.29.

We close this chapter with a famous theorem of Abel concerning the behavior of a power series function on the boundary of its disk of convergence. See the comments following Exercise 3.19.

THEOREM 3.21. (Abel) Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series function having finite radius of convergence $r > 0$, and suppose there exists a point z_0 on the boundary of $B_r(0)$ that is in the domain of f; i.e., $\sum a_n z_0^n$ converges to $f(z_0)$. Suppose g is a continuous function whose domain contains the open disk $B_r(0)$ as well as the point z_0, and assume that $f(z) = g(z)$ for all z in the open disk $B_r(0)$. Then $f(z_0)$ must equal $g(z_0)$.

PROOF. For simplicity, assume that $r = 1$ and that $z_0 = 1$. See the exercise that follows this proof. Write S_n for the partial sum of the a_n's: $S_n = \sum_{n=0}^{n} a_n$. In the following computation, we will use the Abel Summation Formula in the form

$$\sum_{n=0}^{N} a_n z^n = S_N z^N + \sum_{n=0}^{N-1} S_n (z^n - z^{n+1}).$$

See Exercise 2.30. Let ϵ be a positive number. Then, for any $0 < t < 1$ and any positive integer N, we have

$$
\begin{aligned}
|g(1)-f(1)| &= |g(1)-f(t)+f(t)-\sum_{n=0}^{N} a_n t^n + \sum_{n=0}^{N} a_n t^n - f(1)| \\
&\leq |g(1)-g(t)| + |f(t)-\sum_{n=0}^{N} a_n t^n| + |\sum_{n=0}^{N} a_n t^n - f(1)| \\
&\leq |g(1)-g(t)| + |f(t)-\sum_{n=0}^{N} a_n t^n| \\
&\qquad + |S_N t^N + \sum_{n=0}^{N-1} S_n (t^n - t^{n+1}) - f(1)| \\
&= |g(1)-g(t)| + |f(t)-\sum_{n=0}^{N} a_n t^n \\
&\qquad + |S_N t^N + \sum_{n=0}^{N-1} (S_n - S_N)(t^n - t^{n+1}) + S_N \sum_{n=0}^{N-1} (t^n - t^{n+1}) - f(1)| \\
&= |g(1)-g(t)| + |f(t)-\sum_{n=0}^{N} a_n t^n \\
&\qquad + |\sum_{n=0}^{N-1} (S_n - S_N)(t^n - t^{n+1}) + S_N (t^N + \sum_{n=0}^{N-1} (t^n - t^{n+1})) - f(1)| \\
&\leq |g(1)-g(t)| + |f(t)-\sum_{n=0}^{N} a_n t^n| \\
&\qquad + |\sum_{n=0}^{N-1} (S_n - S_N)(t^n - t^{n+1})| + |S_N - f(1)| \\
&\leq |g(1)-g(t)| + |f(t)-\sum_{n=0}^{N} a_n t^n| \\
&\qquad + |\sum_{n=0}^{P} (S_n - S_N)(t^n - t^{n+1})| + |\sum_{n=P+1}^{N-1} (S_n - S_N)(t^n - t^{n+1})| + |S_N - f(1)| \\
&\leq |g(1)-g(t)| + |f(t)-\sum_{n=0}^{N} a_n t^n| \\
&\qquad + |\sum_{n=0}^{P} (S_n - S_N)(t^n - t^{n+1})| + \sum_{n=P+1}^{N-1} |S_n - S_N|(t^n - t^{n+1}) + |S_N - f(1)| \\
&= t_1 + t_2 + t_3 + t_4 + t_5.
\end{aligned}
$$

First, choose an integer M_1 so that if P and N are both larger than M_1, then $t_4 < \epsilon$. (The sequence $\{S_k\}$ is a Cauchy sequence, and $\sum(t^k - t^{k+1}$ is telescoping.)

Fix such a $P > M_1$. Then choose a $\delta > 0$ so that if $1 > t > 1 - \delta$, then both t_1 and $t_3 < \epsilon$. How?

Fix such a t. Finally, choose a N, greater than M_1, and also large enough so that both t_2 and t_5 are less than ϵ. (How?)

Now, $|g(1) - f(1)| < 5\epsilon$. Since this is true for every $\epsilon > 0$, it follows that $f(1) = g(1)$, and the theorem is proved.

Exercise 3.31. Let f, g, r, and z_0 be as in the statement of the preceding theorem. Define $\hat{f}(z) = f(z_0 z)$ and $\hat{g}(z) = g(z_0 z)$.

(a) Prove that $\hat{f}$ is a power series function $\hat{f}(z) = \sum_{n=0}^\infty b_n z^n$, with radius of convergence equal to 1, and such that $\sum_{n=0}^\infty b_n$ converges to $\hat{f}(1)$; i.e., 1 is in the domain of $\hat{f}$.

(b) Show that $\hat{g}$ is a continuous function whose domain contains the open disk $B_1(0)$ and the point $z = 1$.

(c) Show that, if $\hat{f}(1) = \hat{g}(1)$, then $f(z_0) = g(z_0)$. Deduce that the simplification in the preceding proof is justified.

(d) State and prove the generalization of Abel's Theorem to a function f that is expandable in a Taylor series around a point c.