# CHAPTER VII
# THE FUNDAMENTAL THEOREM OF ALGEBRA, AND THE FUNDAMENTAL THEOREM OF ANALYSIS

In this chapter we will discover the incredible difference between the analysis of functions of a single complex variable as opposed to functions of a single real variable. Up to this point, in some sense, we have treated them as being quite similar subjects, whereas in fact they are extremely different in character. Indeed, if $f$ is a differentiable function of a complex variable on an open set $U \subseteq \mathbb{C}$, then we will see that $f$ is actually expandable in a Taylor series around every point in $U$. In particular, a function $f$of a complex variable is guaranteed to have infinitely many derivatives on $U$ if it merely has the first one on $U$. This is in marked contrast with functions of a real variable. See part (3) of Theorem 4.17.

The main points of this chapter are:

(1) **The Cauchy-Riemann Equations** (Theorem 7.1),
(2) **Cauchy's Theorem** (Theorem 7.3),
(3) **Cauchy Integral Formula** (Theorem 7.4),
(4) **A complex-valued function that is differentiable on an open set is expandable in a Taylor series around each point of the set** (Theorem 7.5),
(5) **The Identity Theorem** (Theorem 7.6),
(6) **The Fundamental Theorem of Algebra** (Theorem 7.7),
(7) **Liouville's Theorem** (Theorem 7.8),
(8) **The Maximum Modulus Principle** (corollary to Theorem 7.9),
(9) **The Open Mapping Theorem** (Theorem 7.10),
(10) **The uniform limit of analytic functions is analytic** (Theorem 7.12), and
(11) **The Residue Theorem** (Theorem 7).17.

## CAUCHY'S THEOREM

We begin with a simple observation connecting differentiability of a function of a complex variable to a relation among of partial derivatives of the real and imaginary parts of the function. Actually, we have already visited this point in Exercise 4.8.

**THEOREM 7.1.** (Cauchy-Riemann equations) Let $f = u + iv$ be a complex-valued function of a complex variable $z = x + iy \equiv (x, y)$, and suppose $f$ is differentiable, as a function of a complex variable, at the point $c = (a, b)$. Then the following two partial differential equations, known as the *Cauchy-Riemann Equations*, hold:

$$\frac{\partial u}{\partial x}(a, b) = \frac{\partial v}{\partial y}(a, b),$$

and

$$\frac{\partial u}{\partial y}(a, b) = -\frac{\partial v}{\partial x}(a, b).$$

*PROOF.* We know that

$$f'(c) = \lim h \to 0 \frac{f(c+h) - f(c)}{h},$$

and this limit is taken as the complex number $h$ approaches 0. We simply examine this limit for real $h$'s approaching 0 and then for purely imaginary $h$'s approaching 0. For real $h$'s, we have

$$\begin{aligned}
f'(c) &= f'(a+ib) \\
&= \lim_{h\to 0} \frac{f(a+h+ib)-f(a+ib)}{h} \\
&= \lim h \to 0 \frac{u(a+h,b)+iv(a+h,b)-u(a,b)-iv(a,b)}{h} \\
&= \lim_{h\to 0} \frac{u(a+h,b)-u(a,b)}{h} + i \lim_{h\to 0} \frac{v(a+h,b)-v(a,b)}{h} \\
&= \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b).
\end{aligned}$$

For purely imaginary $h$'s, which we write as $h = ik$, we have

$$\begin{aligned}
f'(c) &= f'(a+ib) \\
&= \lim_{k\to 0} \frac{f(a+i(b+k))-f(a+ib)}{ik} \\
&= \lim_{k\to 0} \frac{u(a,b+k)+iv(a,b+k)-u(a,b)-iv(a,b)}{ik} \\
&= -i \lim_{k\to 0} \frac{u(a,b+k)-u(a,b)}{k} + \frac{v(a,b+k)-v(a,b)}{k} \\
&= -i\frac{\partial u}{\partial y}(a,b) + \frac{\partial v}{\partial y}(a,b).
\end{aligned}$$

Equating the real and imaginary parts of these two equivalent expressions for $f'(c)$ gives the Cauchy-Riemann equations.

As an immediate corollary of this theorem, together with Green's Theorem (Theorem 6.14), we get the following result, which is a special case of what is known as Cauchy's Theorem.

**COROLLARY.** *Let $S$ be a piecewise smooth geometric set whose boundary $C_S$ has finite length. Suppose $f$ is a complex-valued function that is continuous on $S$ and differentiable at each point of the interior $S^0$ of $S$. Then the contour integral $\int_{C_S} f(\zeta)\,d\zeta = 0$.*

**Exercise 7.1.** (a) Prove the preceding corollary. See Theorem 6.12.

(b) Suppose $f = u + iv$ is a differentiable, complex-valued function on an open disk $B_r(c)$ in $\mathbb{C}$, and assume that the real part $u$ is a constant function. Prove that $f$ is a constant function. Derive the same result assuming that $v$ is a constant function.

(c) Suppose $f$ and $g$ are two differentiable, complex-valued functions on an open disk $B_r(c)$ in $\mathbb{C}$. Show that, if the real part of $f$ is equal to the real part of $g$, then there exists a constant $k$ such that $f(z) = g(z) + k$, for all $z \in B_r(c)$.

For future computational purposes, we give the following implications of the Cauchy-Riemann equations. As with Theorem 7.1, this next theorem mixes the notions of differentiability of a function of a complex variable and the partial derivatives of its real and imaginary parts.

**THEOREM 7.2.** *Let $f = u + iv$ be a complex-valued function of a complex variable, and suppose that $f$ is differentiable at the point $c = (a, b)$. Let $A$ be the $2 \times 2$ matrix*

$$A = \begin{pmatrix} u_x(a,b) & v_x(a,b) \\ u_y(a,b) & v_y(a,b) \end{pmatrix}.$$

*Then:*

(1) $|f'(c)|^2 = \det(A)$.

(2) *The two vectors*

$$\vec{V}_1 = (u_x(a,b), u_y(a,b)) \text{ and } \vec{V}_2 = (v_x(a,b), v_y(a,b))$$

*are linearly independent vectors in $\mathbb{R}^2$ if and only if $f'(c) \neq 0$.*

(3) *The vectors*

$$\vec{V}_3 = (u_x(a,b), v_x(a,b)) \text{ and } \vec{V}_4 = (u_y(a,b), v_y(a,b))$$

*are linearly independent vectors in $\mathbb{R}^2$ if and only if $f'(c) \neq 0$.*

*PROOF.* Using the Cauchy-Riemann equations, we see that the determinant of the matrix $A$ is given by

$$\begin{aligned} \det A &= u_x(a,b)v_y(a,b) - u_y(a,b)v_x(a,b) \\ &= (u_x(a,b))^2 + (v_x(a,b))^2 \\ &= (u_x(a,b) + iv_x(a,b))(u_x(a,b) - iv_x(a,b)) \\ &= f'(c)\overline{f'(c)} \\ &= |f'(c)|^2, \end{aligned}$$

proving part (1).

The vectors $\vec{V}_1$ and $\vec{V}_2$ are the columns of the matrix $A$, and so, from elementary linear algebra, we see that they are linearly independent if and only if the determinant of $A$ is nonzero. Hence, part (2) follows from part (1). Similarly, part (3) is a consequence of part (1).

It may come as no surprise that the contour integral of a function $f$ around the boundary of a geometric set $S$ is not necessarily 0 if the function $f$ is not differentiable at each point in the interior of $S$. However, it is exactly these kinds of contour integrals that will occupy our attention in the rest of this chapter, and we shouldn't jump to any conclusions.

**Exercise 7.2.** Let $c$ be a point in $\mathbb{C}$, and let $S$ be the geometric set that is a closed disk $\overline{B}_r(c)$. Let $\phi$ be the parameterization of the boundary $C_r$ of $S$ given by $\phi(t) = c + re^{it}$ for $t \in [0, 2\pi]$. For each integer $n \in \mathbb{Z}$, define $f_n(z) = (z - c)^n$.

(a) Show that $\int_{C_r} f_n(\zeta\, d\zeta = 0$ for all $n \neq -1$.

(b) Show that

$$\int_{C_r} f_{-1}(\zeta)\, d\zeta = \int_{C_r} \frac{1}{\zeta - c}\, d\zeta = 2\pi i.$$

There is a remarkable result about contour integrals of certain functions that aren't differentiable everywhere within a geometric set, and it is what has been called the Fundamental Theorem of Analysis, or Cauchy's Theorem. This theorem has many general statements, but we present one here that is quite broad and certainly adequate for our purposes.

**THEOREM 7.3.** (Cauchy's Theorem, Fundamental Theorem of Analysis) Let $S$ be a piecewise smooth geometric set whose boundary $C_S$ has finite length, and let $\widehat{S} \subseteq S^0$ be a piecewise smooth geometric set, whose boundary $C_{\widehat{S}}$ also is of finite length. Suppose $f$ is continuous on $S \cap \widetilde{\widehat{S}^0}$, i.e., at every point $z$ that is in $S$ but not in $\widehat{S}^0$, and assume that $f$ is differentiable on $S^0 \cap \widetilde{\widehat{S}}$, i.e., at every point $z$ in $S^0$ but not in $\widehat{S}$. (We think of these sets as being the points "between" the boundary curves of these geometric sets.) Then the two contour integrals $\int_{C_S} f(\zeta)\, d\zeta$ and $\int_{C_{\widehat{S}}} f(\zeta)\, d\zeta$ are equal.

*PROOF.* Let the geometric set $S$ be determined by the interval $[a,b]$ and the two bounding functions $u$ and $l$, and let the geometric set $\widehat{S}$ be determined by the subinterval $[\widehat{a},\widehat{b}]$ of $[a,b]$ and the two bounding functions $\widehat{u}$ and $\widehat{l}$. Because $\widehat{S} \subseteq S^0$, we know that $\widehat{u}(t) < u(t)$ and $l(t) < \widehat{l}(t)$ for all $t \in [\widehat{a},\widehat{b}]$. We define four geometric sets $S_1, \ldots, S_4$ as follows:

(1) $S_1$ is determined by the interval $[a,\widehat{a}]$ and the two bounding functions $u$ and $l$ restricted to that interval.
(2) $S_2$ is determined by the interval $[\widehat{a},\widehat{b}]$ and the two bounding functions $u$ and $\widehat{u}$ restricted to that interval.
(3) $S_3$ is determined by the interval $[\widehat{a},\widehat{b}]$ and the two bounding functions $\widehat{l}$ and $l$ restricted to that interval.
(4) $S_4$ is determined by the interval $[\widehat{b},b]$ and the two bounding functions $u$ and $l$ restricted to that interval.

Observe that the five sets $\widehat{S}, S_1, \ldots, S_4$ constitute a partition of the geometric set $S$. The corollary to Theorem 7.1 applies to each of the four geometric sets $S_1, \ldots, S_4$. Hence, the contour integral of $f$ around each of the four boundaries of these geometric sets is 0. So, by Exercise 6.20,

$$\begin{aligned}\int_{C_S} f(\zeta)\, d\zeta &= \int_{C_{\widehat{S}}} f(\zeta)\, d\zeta + \sum_{k=1}^{4} \int_{C_{S_k}} f(\zeta)\, d\zeta \\ &= \int_{C_{\widehat{S}}} f(\zeta)\, d\zeta,\end{aligned}$$

as desired.

**Exercise 7.3.** (a) Draw a picture of the five geometric sets in the proof above and justify the claim that the sum of the four contour integrals around the geometric sets $S_1, \ldots, S_4$ is the integral around $C_S$ minus the integral around $C_{\widehat{S}}$.

(b) Let $S_1, \ldots, S_n$ be pairwise disjoint, piecewise smooth geometric sets, each having a boundary of finite length, and each contained in a piecewise smooth geometric set $S$ whose boundary also has finite length. Prove that the $S_k$'s are some of the elements of a partition $\{\widetilde{S_l}\}$ of $S$, each of which is piecewise smooth and has a boundary of finite length. Show that, by reindexing, $S_1, \ldots, S_n$ can be chosen to be the first $n$ elements of the partition $\{\widehat{S_l}\}$.
HINT: Just carefully adjust the proof of Theorem 5.25.

(c) Suppose $S$ is a piecewise smooth geometric set whose boundary has finite length, and let $S_1, \ldots, S_n$ be a partition of $S$ for which each $S_k$ is piecewise smooth and has a boundary $C_{S_k}$ of finite length. Suppose $f$ is continuous on each of the boundaries $C_{S_k}$ of the $S_k$'s as well as the boundary $C_S$ of $S$, and assume that $f$ is continuous on each of the $S_k$'s, for $1 \leq k \leq m$, and differentiable at each point of their interiors. Prove that

$$\int_{C_S} f(\zeta)\,d\zeta = \sum_{k=m+1}^{n} \int_{C_{S_k}} f(\zeta)\,d\zeta.$$

(d) Prove the following generalization of the Cauchy Theorem: Let $S_1, \dots, S_n$ be pairwise disjoint, piecewise smooth geometric sets whose boundaries have finite length, all contained in the interior of a piecewise smooth geometric set $S$ whose boundary also has finite length. Suppose $f$ is continuous at each point of $S$ that is not in the interior of any of the $S_k$'s, and that $f$ is differentiable at each point of $S^0$ that is not an element of any of the $S_k$'s. Prove that

$$\int_{C_S} f(\zeta)\,d\zeta = \sum_{k=1}^{n} \int_{C_{S_k}} f(\zeta)\,d\zeta.$$

Perhaps the main application of Theorem 7.3 is what's called the Cauchy Integral Formula. It may not appear to be useful at first glance, but we will be able to use it over and over throughout this chapter. In addition to its theoretical uses, it is the basis for a technique for actually evaluating contour integrals, line integrals, as well as ordinary integrals.

**THEOREM 7.4.** (Cauchy Integral Formula) Let $S$ be a piecewise smooth geometric set whose boundary $C_S$ has finite length, and let $f$ be a continuous function on $S$ that is differentiable on the interior $S^0$ of $S$. Then, for any point $z \in S^0$, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_S} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

*REMARK.* This theorem is an initial glimpse at how differentiable functions of a complex variable are remarkably different from differentiable functions of a real variable. Indeed, Cauchy's Integral Formula shows that the values of a differentiable function $f$ at all points in the interior of a geometric set $S$ are completely determined by the values of that function on the boundary of the set. The analogous thing for a function of a real variable would be to say that all the values of a differentiable function $f$ at points in the open interval $(a, b)$ are completely determined by its values at the endpoints $a$ and $b$. This is patently absurd for functions of a real variable, so there surely is something marvelous going on for differentiable functions of a complex variable.

*PROOF.* Let $r$ be any positive number such that $\overline{B}_r(z)$ is contained in the interior $S^0$ of $S$, and note that the close disk $\overline{B}_r(z)$ is a piecewise smooth geometric set $\widehat{S}$ contained in $S^0$. We will write $C_r$ instead of $C_{\widehat{S}}$ for the boundary of this disk, and we will use as a parameterization of the curve $C_r$ the function $\phi : [0, 2\pi] \to C_r$ given by $\phi(t) = z + re^{it}$. Now the function $g(\zeta) = f(\zeta)/(\zeta - z)$ is continuous on $S \cap \widetilde{\widehat{S}^0}$ and differentiable on $S^0 \cap \widetilde{\widehat{S}}$, so that Theorem 7.3 applies to the function $g$.

Hence

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_S}\frac{f(\zeta)}{\zeta-z}\,d\zeta &= \frac{1}{2\pi i}\int_{C_S} g(\zeta)\,d\zeta\\ &= \frac{1}{2\pi i}\int_{C_R} g(\zeta),d\zeta\\ &= \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta-z}\,d\zeta\\ &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z+re^{it})}{z+re^{it}-z}ire^{it}\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt.\end{aligned}$$

Since the equality established above is valid, independent of $r$, we may take the limit as $r$ goes to 0, and the equality will persist. We can evaluate such a limit by replacing the $r$ by $1/n$, in which case we would be evaluating

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f(z+\frac{1}{n}e^{it})\,dt = \lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f_n(t)\,dt,$$

where $f_n(t) = f(z + frac1ne^{it})$. Finally, because the function $f$ is continuous at the point $z$, it follows that the sequence $\{f_n\}$ converges uniformly to the constant function $f(z)$ on the interval $[0, 2\pi]$. So, by Theorem 5.6, we have that

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f_n(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(z)\,dt = f(z).$$

Therefore,

$$\frac{1}{2\pi i}\int_{C_S}\frac{f(\zeta}{\zeta-z}\,d\zeta = \lim_{r\to 0}\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt = f(z),$$

and the theorem is proved.

The next exercise gives two simple but strong consequences of the Cauchy Integral Formula, and it would be wise to spend a few minutes deriving other similar results.

**Exercise 7.4.** (a) Let $S$ and $f$ be as in the preceding theorem, and assume that $f(z) = 0$ for every point on the boundary $C_S$ of $S$. Prove that $f(z) = 0$ for every $z \in S$.

(b) Let $S$ be as in part (a), and suppose that $f$ and $g$ are two continuous functions on $S$, both differentiable on $S^0$, and such that $f(\zeta) = g(\zeta)$ for every point on the boundary of $S$. Prove that $f(z) = g(z)$ for all $z \in S$.

The preceding exercise shows that two differentiable functions of a complex variable are equal everywhere on a piecewise smooth geometric set $S$ if they agree on the boundary of the set. More is true. We will see below in the Identity Theorem that they are equal everywhere on a piecewise smooth geometric set $S$ if they agree just along a single convergent sequence in the interior of $S$.

Combining part (b) of Exercise 7.3, Exercise 6.20, and Theorem 7.3, we obtain the following corollary:

**COROLLARY.** *Let $S_1, \dots, S_n$ be pairwise disjoint, piecewise smooth geometric sets whose boundaries have finite length, all contained in the interior of a piecewise smooth geometric set $S$ whose boundary has finite length. Suppose $f$ is continuous at each point of $S$ that is not in the interior of any of the $S_k$'s, and that $f$ is differentiable at each point of $S^0$ that is not an element of any of the $S_k$'s. Then, for any $z \in S^0$ that is not an element of any of the $S_k$'s, we have*

$$f(z) = \frac{1}{2\pi i}\left(\int_{C_S} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \sum_{k=1}^{n} \int_{C_{S_k}} \frac{f(\zeta)}{\zeta - z}\,d\zeta\right).$$

*PROOF.* Let $r > 0$ be such that $\overline{B}_r(z)$ is disjoint from all the $S_k$'s. By part (b) of Exercise 7.3, let $T_1, \dots, T_m$ be a partition of $S$ such that $T_k = S_k$ for $1 \leq k \leq n$, and $T_{n+1} = \overline{B}_r(z)$. By Exercise 6.20, we know that

$$\int_{C_S} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_{k=1}^{m} \int_{C_{T_k}} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

From the Cauchy Integral Formula, we know that

$$\int_{C_{T_{n+1}}} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 2\pi i f(z).$$

Also, since $f(\zeta)/(\zeta - z)$ is differentiable at each point of the interior of the sets $T_k$ for $k > n+1$, we have from Theorem 7.2 that for all $k > n+1$

$$\int_{C_{t_k}} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 0.$$

Therefore,

$$\int_{C_S} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_{k=1}^{n} \int_{C_{S_k}} \frac{f(\zeta)}{\zeta - z}\,d\zeta + 2\pi i f(z),$$

which completes the proof.

**Exercise 7.5.** Suppose $S$ is a piecewise smooth geometric set whose boundary has finite length, and let $c_1, \dots, c_n$ be points in $S^0$. Suppose $f$ is a complex-valued function that is continuous at every point of $S$ except the $C_k$'s and differentiable at every point of $S^0$ except the $c_k$'s. Let $r_1, \dots, r_n$ be positive numbers such that the disks $\{\overline{B}_{R_k}(c_k)\}$ are pairwise disjoint and all contained in $S^0$.

(a) Prove that

$$\int_{C_S} f(\zeta)\,d\zeta, = \sum_{k=1}^{n} \int_{C_k} f(\zeta)\,d\zeta$$

where $C_k$ denotes the boundary of the disk $\overline{B}_{r_k}(c_k)$.

(b) For any $z \in S^0$ that is not in any of the closed disks $\overline{B}_{r_k}(c_k)$, show that

$$\int_{C_S} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 2\pi i f(z) + \sum_{k=1}^{n} \int_{C_k} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

(c) Specialize part (b) to the case where $S = \overline{B}_r(c)$, and $f$ is analytic at each point of $B_r(c)$ except at the central point $c$. For each $z \neq c$ in $B_r(c)$, and any $0 < \delta < |z - c|$, derive the formula

$$f(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{C_\delta} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

## BASIC APPLICATIONS OF THE CAUCHY INTEGRAL FORMULA

As a major application of the Cauchy Integral Formula, let us show the much alluded to remarkable fact that a function that is a differentiable function of a complex variable on an open set $U$ is actually expandable in a Taylor series around every point in $U$, i.e., is an analytic function on $U$.

**THEOREM 7.5.** *Suppose $f$ is a differentiable function of a complex variable on an open set $U \subseteq \mathbb{C}$, and let $c$ be an element of $U$. Then $f$ is expandable in a Taylor series around $c$. In fact, for any $r > 0$ for which $\overline{B}_r(c) \subseteq U$, we have*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - c)^n$$

*for all $z \in B_r(c)$.*

*PROOF.* Choose an $r > 0$ such that the closed disk $\overline{B}_r(c) \subseteq U$, and write $C_r$ for the boundary of this disk. Note that, for all points $\zeta$ on the curve $C_r$, and any fixed point $z$ in the open disk $B_r(c)$, we have that $|z - c| < r = |\zeta - c|$, whence $|z - c|/|\zeta - c| = |z - c|/r < 1$. Therefore the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{z - c}{\zeta - c}\right)^n \text{ converges to } \frac{1}{1 - \frac{z-c}{\zeta-c}}.$$

Moreover, by the Weierstrass $M$-Test, as functions of the variable $\zeta$, this infinite series converges uniformly on the curve $C_r$. We will use this in the calculation below. Now, according to Theorem 7.4, we have that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - c + c - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta - c)(1 - \frac{z-c}{\zeta-c})}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - c}\sum_{n=0}^{\infty}\left(\frac{z - c}{\zeta - c}\right)^n\,d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r} \sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta - c)^{n+1}}(z - c)^n\,d\zeta \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{C_r} \frac{f(\zeta)}{(\zeta - c)^{n+1}}(z - c)^n\,d\zeta \\
&= \sum_{n=0}^{\infty} a_n (z - c)^n,
\end{aligned}$$

where we are able to bring the summation sign outside the integral by part (3) of Theorem 6.10, and where

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta - c)^{n+1}}\,d\zeta.$$

This proves that $f$ is expandable in a Taylor series around the point $c$, as desired.

Using what we know about the relationship between the coefficients of a Taylor series and the derivatives of the function, together with the Cauchy Integral Theorem, we obtain the following formulas for the derivatives of a differentiable function $f$ of a complex variable. These are sometimes also called the Cauchy Integral Formulas.

**COROLLARY.** *Suppose $f$ is a differentiable function of a complex variable on an open set $U$, and let $c$ be an element of $U$. Then $f$ is infinitely differentiable at $c$, and*

$$f^{(n)}(c) = \frac{n!}{2\pi i}\int_{C_s} \frac{f(\zeta)}{(\zeta - c)^{n+1}}\,d\zeta,$$

*for any piecewise smooth geometric set $S \subseteq U$ whose boundary $C_S$ has finite length, and for which $c$ belongs to the interior $S^0$ of $S$.*

**Exercise 7.6.** (a) Prove the preceding corollary.

(b) Let $f, U$, and $c$ be as in Theorem 7.5. Show that the radius of convergence $r$ of the Taylor series expansion of $f$ around $c$ is at least as large as the supremum of all $s$ for which $B_s(c) \subseteq U$.

(c) Conclude that the radius of convergence of the Taylor series expansion of a differentiable function of a complex variable is as large as possible. That is, if $f$ is differentiable on a disk $B_r(c)$, then the Taylor series expansion of $f$ around $c$ converges on all of $B_r(c)$.

(d) Consider the real-valued function of a real variable given by $f(x) = 1/(1+x^2)$. Show that $f$ is differentiable at each real number $x$. Show that $f$ is expandable in a Taylor series around 0, but show that the radius of convergence of this Taylor series is equal to 1. Does this contradict part (c)?

(e) Let $f$ be the complex-valued function of a complex variable given by $f(z) = 1/(1 + z^2)$. We have just replaced the real variable $x$ of part (d) by a complex variable $z$. Explain the apparent contradiction that parts (c) and (d) present in connection with this function.

**Exercise 7.7.** (a) Let $S$ be a piecewise smooth geometric set whose boundary $C_S$ has finite length, and let $f$ be a continuous function on the curve $C_S$. Define a function $F$ on $S^0$ by

$$F(z) = \int_{C_S} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Prove that $F$ is expandable in a Taylor series around each point $c \in S^0$. Show in fact that $F(z) = \sum a_n(z-c)^n$ for all $z$ in a disk $B_r(c) \subseteq S^0$, where

$$a_n = \frac{n!}{2\pi i}\int_{C_S} \frac{f(\zeta)}{(\zeta - c)^{n+1}}\,d\zeta.$$

HINT: Mimic the proof of Theorem 7.5.

(b) Let $f$ and $F$ be as in part (a). Is $F$ defined on the boundary $C_S$ of $S$? If $z$ belongs to the boundary $C_S$, and $z = \lim z_n$, where each $z_n \in S^0$, Does the sequence $\{F(z_n)\}$ converge, and, if so, does it converge to $f(z)$?

(c) Let $S$ be the closed unit disk $\overline{B}_1(0)$, and let $f$ be defined on the boundary $C_1$ of this disk by $f(z) = \bar{z}$, i.e., $f(x+iy) = x - iy$. Work out the function $F$ of part (a), and then re-think about part (b).

(d) Let $f$ and $F$ be as in part (a). If, in addition, $f$ is continuous on all of $S$ and differentiable on $S^0$, show that $F(z) = 2\pi i f(z)$ for all $z \in S^0$. Think about this "magic" constant $2\pi i$. Review the proof of the Cauchy Integral Formula to understand where this constant comes from.

Theorem 3.14 and Exercise 3.26 constitute what we called the "identity theorem" for functions that are expandable in a Taylor series around a point $c$. An even stronger result than that is actually true for functions of a complex variable.

**THEOREM 7.6.** (Identity Theorem) Let $f$ be a continuous complex-valued function on a piecewise smooth geometric set $S$, and assume that $f$ is differentiable on the interior $S^0$ of $S$. Suppose $\{z_k\}$ is a sequence of distinct points in $S^0$ that converges to a point $c$ in $S^0$. If $f(z_k) = 0$ for every $K$, then $f(z) = 0$ for every $z \in S$.

*PROOF.* It follows from Exercise 3.26 that there exists an $r > 0$ such that $f(z) = 0$ for all $z \in B_r(c)$. Now let $w$ be another point in $S^0$, and let us show that $f(w)$ must equal 0. Using part (f) of Exercise 6.2, let $\phi : [\widehat{a}, \widehat{b}] \to C$ be a piecewise smooth curve, joining $c$ to $w$, that lies entirely in $S^0$. Let $A$ be the set of all $t \in [\widehat{a}, \widehat{b}]$ such that $f(\phi(s)) = 0$ for all $s \in [\widehat{a}, t)$. We claim first that $A$ is nonempty. Indeed, because $\phi$ is continuous, there exists an $\epsilon > 0$ such that $|\phi(s) - c| = |\phi(s) - \phi(\widehat{a})| < r$ if $|s - \widehat{a}| < \epsilon$. Therefore $f(\phi(s)) = 0$ for all $s \in [\widehat{a}, \widehat{a}+\epsilon)$, whence, $\widehat{a}+\epsilon \in A$. Obviously, $A$ is bounded above by $\widehat{b}$, and we write $t_0$ for the supremum of $A$. We wish to show that $t_0 = \widehat{b}$, whence, since $\phi$ is continuous at $\widehat{B}$, $f(w) = f(\phi(\widehat{b})) = f(\phi(t_0)) = 0$. Suppose, by way of contradiction, that $t_0 < \widehat{b}$, and write $z_0 = \phi(t_0)$. Now $z_0 \in S^0$, and $z_0 = \lim \phi(t_0 - 1/k)$ because $\phi$ is continuous at $t_0$. But $f(\phi(t_0 - 1/k)) = 0$ for all $k$. So, again using Exercise 3.26, we know that there exists an $r' > 0$ such that $f(z) = 0$ for all $z \in B_{r'}(z_0)$. As before, because $\phi$ is continuous at $t_0$, there exists a $\delta > 0$ such that $t_0 + \delta < \widehat{b}$ and $|\phi(s) - \phi(t_0)| < r'$ if $|s - t_0| < \delta$. Hence, $f(\phi(s)) = 0$ for all $s \in (t_0 - \delta, t_0 + \delta)$, which implies that $t_0 + \delta$ belongs to $A$. But then $t_0$ could not be the supremum of $A$, and therefore we have arrived at a contradiction. Consequently, $t_0 = \widehat{b}$, and therefore $f(w) = 0$ for all $w \in S^0$. Of course, since every point in $S$ is a limit of points from $S^0$, and since $f$ is continuous on $S$, we see that $f(z) = 0$ for all $z \in S$, and the theorem is proved.

The next exercise gives some consequences of the Identity Theorem. Part (b) may appear to be a contrived example, but it will be useful later on.

**Exercise 7.8.** (a) Suppose $f$ and $g$ are two functions, both continuous on a piecewise smooth geometric set $S$ and both differentiable on its interior. Suppose $\{z_k\}$ is a sequence of elements of $S^0$ that converges to a point $c \in S^0$, and assume that $f(z_k) = g(z_k)$ for all $k$. Prove that $f(z) = g(z)$ for all $z \in S$.

(b) Suppose $f$ is a nonconstant differentiable function defined on the interior of a piecewise smooth geometric set $S$. If $c \in S^0$ and $B_\epsilon(c) \subseteq S^0$, show that there must exist an $0 < r < \epsilon$ for which $f(c) \neq f(z)$ for all $z$ on the boundary of the disk $B_r(c)$.

## THE FUNDAMENTAL THEOREM OF ALGEBRA

We can now prove the Fundamental Theorem of Algebra, the last of our primary goals. One final trumpet fanfare, please!

**THEOREM 7.7.** (Fundamental Theorem of Algebra) Let $p(z)$ be a nonconstant polynomial of a complex variable. Then there exists a complex number $z_0$ such that $p(z_0) = 0$. That is, every nonconstant polynomial of a complex variable has a root in the complex numbers.

*PROOF.* We prove this theorem by contradiction. Thus, suppose that $p$ is a nonconstant polynomial of degree $n \geq 1$, and that $p(z)$ is never 0. Set $f(z) = 1/p(z)$, and observe that $f$ is defined and differentiable at every point $z \in \mathbb{C}$. We will show that $f$ is a constant function, implying that $p = 1/f$ is a constant, and that will give the contradiction. We prove that $f$ is constant by showing that its derivative is identically 0, and we compute its derivative by using the Cauchy Integral Formula for the derivative.

From part (4) of Theorem 3.1, we recall that there exists a $B > 0$ such that $\frac{|c_n|}{2}|z|^n \leq |p(z)|$, for all $z$ for which $|z| \geq B$, and where $c_n$ is the (nonzero) leading coefficient of the polynomial $p$. Hence, $|f(z)| \leq \frac{M}{|z|^n}$ for all $|z| \geq B$, where we write $M$ for $2/|c_n|$. Now, fix a point $c \in \mathbb{C}$. Because $f$ is differentiable on the open set $U = \mathbb{C}$, we can use the corollary to Theorem 7.4 to compute the derivative of $f$ at $c$ by using any of the curves $C_r$ that bound the disks $B_r(c)$, and we choose an $r$ large enough so that $|c + re^{it}| \geq B$ for all $0 \leq t \leq 2\pi$. Then,

$$\begin{aligned}
|f'(c)| &= |\frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta - c)^2}\, d\zeta| \\
&= \frac{1}{2\pi}|\int_0^{2\pi} \frac{f(c + re^{it})}{(c + re^{it} - c)^2} ire^{it}\, dt| \\
&\leq \frac{1}{2\pi r}\int_0^{2\pi} |f(c + re^{it})|\, dt \\
&\leq \frac{1}{2\pi r}\int_0^{2\pi} \frac{M}{|c + re^{it}|^n}\, dt \\
&\leq \frac{M}{rB^n}.
\end{aligned}$$

Hence, by letting $r$ tend to infinity, we get that

$$|f'(c)| \leq \lim_{r\to\infty} \frac{M}{rB^n} = 0,$$

and the proof is complete.

*REMARK.* The Fundamental Theorem of Algebra settles a question first raised back in Chapter I. There, we introduced a number $I$ that was a root of the polynomial $x^2 + 1$. We did this in order to build a number system in which negative numbers would have square roots. We adjoined the "number" $i$ to the set of real numbers to form the set of complex numbers, and we then saw that in fact every complex number $z$ has a square root. However, a fear was that, in order to build a system in which every number has an $n$th root for every $n$, we would continually need to be adjoining new elements to our number system. However, the Fundamental Theorem of Algebra shows that this is not necessary. The set of complex numbers is already rich enough to contain all $n$th roots and even more.

Practically the same argument as in the preceding proof establishes another striking result.

**THEOREM 7.8.** (Liouville) Suppose $f$ is a bounded, everywhere differentiable function of a complex variable. Then $f$ must be a constant function.

**Exercise 7.9.** Prove Liouville's Theorem.

## THE MAXIMUM MODULUS PRINCIPLE

Our next goal is to examine so-called "max/min" problems for coplex-valued functions of complex variables. Since order makes no sense for complex numbers, we will investigate max/min problems for the absolute value of a complex-valued function. For the corresponding question for real-valued functions of real variables, we have as our basic result the First Derivative Test (Theorem 4.8). Indeed, when searching for the poinhts where a differentiable real-valued function $f$ on an interval $[a, b]$ attains its extreme values, we consider first the poinhts where it attains a local max or min, to which purpose end Theorem 4.8 is useful. Of course, to find the absolute minimum and maximum, we must also check the values of the function at the endpoints.

An analog of Theorem 4.8 holds in the complex case, but in fact a much different result is really valid. Indeed, it is nearly impossible for the absolute value of a differentiable function of a complex variable to attain a local maximum or minimum.

**THEOREM 7.9.** *Let $f$ be a continuous function on a piecewise smooth geometric set $S$, and assume that $f$ is differentiable on the interior $S^0$ of $S$. Suppose $c$ is a point in $S^0$ at which the real-valued function $|f|$ attains a local maximum. That is, there exists an $\epsilon > 0$ such that $|f(c)| \geq |f(z)|$ for all $z$ satisfying $|z - c| < \epsilon$. Then $f$ is a constant function on $S$; i.e., $f(z) = f(c)$ for all $z \in S$. In other words, the only differentiable functions of a complex variable, whose absolute value attains a local maximum on the interior of a geometric set, are constant functions on that set.*

*PROOF.* If $f(c) = 0$, then $f(z) = 0$ for all $z \in B_\epsilon(c)$. Hence, by the Identity Theorem (Theorem 7.6), $f(z)$ would equal 0 for all $z \in S$. so, we may as well assume that $f(c) \neq 0$. Let $r$ be any positive number for which the closed disk $\overline{B}_r(c)$ is contained in $B_\epsilon(c)$. We claim first that there exists a point $z$ on the boundary $C_r$ of the disk $\overline{B}_r(c)$ for which $|f(z)| = |f(c)|$. Of course, $|f(z| \leq |f(c)|$ for all $z$ on this boundary by assumption. By way of contradiction, suppose that $|f(\zeta)| < |f(c)|$ for all $\zeta$ on the boundary $C_r$ of the disk. Write $M$ for the maximum value of the function $|f|$ on the compact set $C_r$. Then, by our assumption, $M < |f(c)|$. Now, we use the Cauchy Integral Formula:

$$\begin{aligned}
|f(c)| &= |\frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - c}\,d\zeta| \\
&= \frac{1}{2\pi}|\int_0^{2\pi}\frac{f(c+re^{it})}{re^{it}}ire^{it}\,dt| \\
&\le \frac{1}{2\pi}\int_0^{2\pi}|f(c+re^{it})|\,dt \\
&\le \frac{1}{2\pi}\int_0^{2\pi}M\,dt \\
&= M \\
&< |f(c)|,
\end{aligned}$$

and this is a contradiction.

Now for each natural number $n$ for which $1/n < \epsilon$, let $z_n$ be a point for which $|z_n - c| = 1/n$ and $|f(z_n)| = |f(c)|$. We claim that the derivative $f'(z_n)$ of $f$ at $z_n = 0$ for all $n$. What we know is that the real-valued function $F(x, y) = |f(x + iy)|^2 = (u(x, y))^2 + (v(x, y))^2$ attains a local maximum value at $z_n = (x_n, y_n)$. Hence, by Exercise 4.34, both partial derivatives of $F$ must be 0 at $(x_n, y_n)$. That is

$$2u(x_n, y_n)\frac{\partial u}{\partial x}(x_n, y_n) + 2v(x_n, y_n)\frac{\partial v}{\partial x}(x_n, y_n) = 0$$

and

$$2u(x_n, y_n)\frac{\partial u}{\partial y}(x_n, y_n) + 2v(x_n, y_n)\frac{\partial v}{\partial y}(x_n, y_n) = 0.$$

Hence the two vectors

$$\vec{V}_1 = (\frac{\partial u}{\partial x}(x_n, y_n), \frac{\partial v}{\partial x}(x_n, y_n))$$

and

$$\vec{V}_2 = (\frac{\partial u}{\partial y}(x_n, y_n), \frac{\partial v}{\partial y}(x_n, y_n))$$

are both perpendicular to the vector $\vec{V}_3 = (u(x_n, y_n), v(x_n, y_n))$. But $\vec{V}_3 \neq 0$, because $\|\vec{V}_3\| = |f(z_n)| = |f(c)| > 0$, and hence $\vec{V}_1$ and $\vec{V}_2$ are linearly dependent. But this implies that $f'(z_n) = 0$, according to Theorem 7.2.

Since $c = \lim z_n$, and $f'$ is analytic on $S^0$, it follows from the Identity Theorem that there exists an $r > 0$ such that $f'(z) = 0$ for all $z \in B_r(c)$. But this implies that $f$ is a constant $f(z) = f(c)$ for all $z \in B_r(c)$. And thenm, again using the Identity Theorem, this implies that $f(z) = f(c)$ for all $z \in S$, which completes the proof.

*REMARK.* Of course, the preceding proof contains in it the verification that if $|f|$ attains a maximum at a point $c$ where it is differentiable, then $f'(c) = 0$. This is the analog for functions of a complex variable of Theorem 4.8. But, Theorem 7.9 certainly asserts a lot more than that. In fact, it says that it is impossible for the absolute value of a nonconstant differentiable function of a complex variable to attain a local maximum. Here is the coup d'grâs:

**COROLLARY.** (Maximum Modulus Principle) Let $f$ be a continuous, nonconstant, complex-valued function on a piecewise smooth geometric set $S$, and suppose that $f$ is differentiable on the interior $S^0$ of $S$. Let $M$ be the maximum value of the continuous, real-valued function $|f|$ on $S$, and let $z$ be a point in $S$ for which $|f(z)| = M$. Then, $z$ does not belong to the interior $S^0$ of $S$; it belongs to the boundary of $S$. In other words, $|f|$ attains its maximum value only on the boundary of $S$.

**Exercise 7.10.** (a) Prove the preceding corollary.

(b) Let $f$ be an analytic function on an open set $U$, and let $c \in U$ be a point at which $|f|$ achieves a local minimum; i.e., there exists an $\epsilon > 0$ such that $|f(c)| \leq |f(z)|$ for all $z \in B_\epsilon(c)$. Show that, if $f(c) \neq 0$, then $f$ is constant on $B_\epsilon(c)$. Show by example that, if $f(c) = 0$, then $f$ need not be a constant on $B_\epsilon(c)$.

(c) Prove the "Minimum Modulus Principle:" Let $f$ be a nonzero, continuous, nonconstant, function on a piecewise smooth geometric set $S$, and let $m$ be the minimum value of the function $|f|$ on $S$. If $z$ is a point of $S$ at which this minimum value is atgtained, then $z$ belongs to the boundary $C_S$ of $S$.

## THE OPEN MAPPING THEOREM AND THE INVERSE FUNCTION THEOREM

We turn next to a question about functions of a complex variable that is related to Theorem 4.10, the Inverse Function Theorem. That result asserts, subject to a couple of hypotheses, that the inverse of a one-to-one differentiable function of a real variable is also differentiable. Since a function is only differentiable at points in the interior of its domain, it is necessary to verify that the point $f(c)$ is in the interior of the domain $f(S)$ of the inverse function $f^{-1}$ before the question of differentiability at that point can be addressed. And, the peculiar thing is that it is this point about $f(c)$ being in the interior of $f(S)$ that is the subtle part. The fact that the inverse function is differentiable there, and has the prescribed form, is then only a careful $\epsilon - \delta$ argument. For continuous real-valued functions of real variables, the fact that $f(c)$ belongs to the interior of $f(S)$ boils down to the fact that intervals get mapped onto intervals by continuous functions, which is basically a consequence of the Intermediate Value Theorem. However, for complex-valued functions of complex variables, the situation is much deeper. For instance, the continuous image of a disk is just not always another disk, and it may not even be an open set. Well, all is not lost; we just have to work a little harder.

**THEOREM 7.10.** (Open Mapping Theorem) Let $S$ be a piecewise smooth geometric set, and write $U$ for the (open) interior $S^0$ of $S$. Suppose $f$ is a nonconstant differentiable, complex-valued function on the set $U$. Then the range $f(U)$of $f$ is an open subset of $\mathbb{C}$.

*PROOF.* Let $c$ be in $U$. Because $f$ is not a constant function, there must exist an $r > 0$ such that $f(c) \neq f(z)$ for all $z$ on the boundary $C_r$ of the disk $B_r(c)$. See part (b) of Exercise 7.8. Let $z_0$ be a point in the compact set $C_r$ at which the continuous real-valued function $|f(z) - f(c)|$ attains its minimum value $s$. Since $f(z) \neq f(c)$ for any $z \in C_r$, we must have that $s > 0$. We claim that the disk $B_{s/2}(f(c))$ belongs to the range $f(U)$ of $f$. This will show that the point $f(c)$ belongs to the ihnterior of the set $f(U)$, and that will finish the proof.

By way of contradiction, suppose $B_{s/2}(f(c)$ is not contained in $f(U)$,, and let $w \in B_{s/2}(f(c))$ be a complex number that is not in $f(U)$. We have that $|w - f(c)| <$

$s/2$, which implies that $|w - f(z)| > s/2$ for all $z \in C_r$. Consider the function $g$ defined on the closed disk $\overline{B}_r(c)$ by $g(z) = 1/(w - f(z))$. Then $g$ is continuous on the closed disk $\overline{B}_r(c)$ and differentiable on $B_r(c)$. Moreover, $g$ is not a constant function, for if it were, $f$ would also be a constant function on $B_r(c)$ and therefore, by the Identity Theorem, constant on all of $U$, whichg is not the case by hypothesis. Hence, by the Maximum Modulus Principle, the maximum value of $|g|$ only occurs on the boundary $C_r$ of this disk. That is, there exists a point $z' \in C_r$ such that $|g(z)| < |g(z')|$ for all $z \in B_r(c)$. But then

$$\frac{2}{s} = \frac{1}{s/2} < \frac{1}{|w - f(c)|} < \frac{1}{|w - f(z')|} \le \frac{1}{s},$$

which gives the desired contradiction. Therefore, the entire disk $B_{s/2}(f(c))$ belongs to $f(U)$, and hence the point $f(c)$ belongs to the interior of the set $f(U)$. Since this holds for any point $c \in U$, it follows that $f(U)$ is open, as desired.

Now we can give the version of the Inverse Function Theorem for complex variables.

**THEOREM 7.11.** *Let $S$ be a piecewise smooth geometric set, and suppose $f : S \to \mathbb{C}$ is continuously differentiable at a point $c = a+bi$, and assume that $f'(c) \neq 0$. Then:*

(1) *There exists an $r > 0$, such that $\overline{B}_r(c) \subseteq S$, for which $f$ is one-to-one on $\overline{B}_r(c)$.*
(2) *$f(c)$ belongs to the interior of $f(S)$.*
(3) *If $g$ denotes the restriction of the function $f$ to $B_r(c)$, then $g$ is one-to-one, $g^{-1}$ is differentiable at the point $f(c)$, and ${g^{-1}}'(f(c) = 1/f'(c)$.*

*PROOF.* Arguing by contradiction, suppose that $f$ is not one-to-one on any disk $\overline{B}_r(c)$. Then, for each natural number $n$, there must exist two points $z_n = x_n + iy_n$ and $z'_n = x'_n + iy'_n$ such that $|z_n - c| < 1/n$, $|z'_n - c| < 1/n$, and $f(z_n) = f(z'_n)$. If we write $f = u + iv$, then we would have that $u(x_n, y_n) - u(x'_n, y'_n) = 0$ for all $n$. So, by part (c) of Exercise 4.35, there must exist for each $n$ a point $(\widehat{x}_n, \widehat{y}_n)$, such that $(\widehat{x}_n, \widehat{y}_n)$ is on the line segment joining $z_n$ and $z'_n$, and for which

$$0 = u(x_n, y_n) - u(x'_n, y'_n) = \frac{\partial u}{\partial x}(\widehat{x}_n, \widehat{y}_n)(x_n - x'_n) + \frac{\partial u}{\partial y}(\widehat{x}_n, \widehat{y}_n)(y_n - y'_n).$$

Similarly, applying the same kind of reasoning to $v$, there must exist points $(\widetilde{x}_n, \widetilde{y}_n)$ on the segment joining $z_n$ to $z'_n$ such that

$$0 = \frac{\partial v}{\partial x}(\widetilde{x}_n, \widetilde{y}_n)(x_n - x'_n) + \frac{\partial v}{\partial y}(\widetilde{x}_n, \widetilde{y}_n)(y_n - y'_n).$$

If we define vectors $\vec{U}_n$ and $\vec{V}_n$ by

$$\vec{U}_n = (\frac{\partial u}{\partial x}(\widehat{x}_n, \widehat{y}_n), \frac{\partial u}{\partial y}(\widehat{x}_n, \widehat{y}_n))$$

and

$$\vec{V}_n = (\frac{\partial v}{\partial x}(\widetilde{x}_n, \widetilde{y}_n), \frac{\partial v}{\partial y}(\widetilde{x}_n, \widetilde{y}_n)),$$

then we have that both $\vec{U}_n$ and $\vec{V}_n$ are perpendicular to the nonzero vector $((x_n - x'_n), (y_n - y'_n))$. Therefore, $\vec{U}_n$ and $\vec{V}_n$ are linearly dependent, whence

$$\det(\begin{pmatrix} \frac{\partial u}{\partial x}(\widehat{x}_n, \widehat{y}_n) & \frac{\partial u}{\partial y}(\widehat{x}_n, \widehat{y}_n) \\ \frac{\partial v}{\partial x}(\widetilde{x}_n, \widetilde{y}_n) & \frac{\partial v}{\partial y}(\widetilde{x}_n, \widetilde{y}_n) \end{pmatrix}) = 0.$$

Now, since both $\{\widehat{x}_n + i\widehat{y}_n\}$ and $\{\widetilde{x}_n + i\widetilde{y}_n\}$ converge to the point $c = a + ib$, and the partial derivatives of $u$ and $v$ are continuous at $c$, we deduce that

$$\det(\begin{pmatrix} \frac{\partial u}{\partial x}(a, b) & \frac{\partial u}{\partial y}(a, b) \\ \frac{\partial v}{\partial x}(a, b) & \frac{\partial v}{\partial y}(a, b) \end{pmatrix}) = 0.$$

Now, from Theorem 7.2, this would imply that $f'(c) = 0$, and this is a contradiction. Hence, there must exist an $r > 0$ for which $f$ is one-to-one on $\overline{B}_r(c)$, and this proves part (1).

Because $f$ is one-to-one on $B_r(c)$, $f$ is obviously not a constant function. So, by the Open Mapping Theorem, the point $f(c)$ belongs to the interior of the range of $f$, and this proves part (2).

Now write $g$ for the restriction of $f$ to the disk $B_r(c)$. Then $g$ is one-to-one. According to part (2) of Theorem 4.2, we can prove that $g^{-1}$ is differentiable at $f(c)$ by showing that

$$\lim_{z \to f(c)} \frac{g^{-1}(z) - g^{-1}(f(c))}{z - f(c)} = \frac{1}{f'(c)}.$$

That is, we need to show that, given an $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |z - f(c)| < \delta$ then

$$|\frac{g^{-1}(z) - g^{-1}(f(c))}{z - f(c)} - \frac{1}{f'(c)}| < \epsilon.$$

First of all, because the function $1/w$ is continuous at the point $f'(c)$, there exists an $\epsilon' > 0$ such that if $|w - f'(c)| < \epsilon'$, then

$$|\frac{1}{w} - \frac{1}{f'(c)}| < \epsilon.$$

Next, because $f$ is differentiable at $c$, there exists a $\delta' > 0$ such that if $0 < |y - c| < \delta'$ then

$$|\frac{f(y) - f(c)}{y - c} - f'(c)| < \epsilon'.$$

Now, by Theorem 3.10, $g^{-1}$ is continuous at the point $f(c)$, and therefore there exists a $\delta > 0$ such that if $|z - f(c)| < \delta$ then

$$|g^{-1}(z) - g^{-1}(f(c)| < \delta'.$$

So, if $|z - f(c)| < \delta$, then

$$|g^{-1}(z) - c| = |g^{-1}(z) - g^{-1}(f(c))| < \delta'.$$

But then,

$$\left|\frac{f(g^{-1}(z)) - f(c)}{g^{-1}(z) - c} - f'(c)\right| < \epsilon',$$

from which it follows that

$$\left|\frac{g^{-1}(z) - g^{-1}(f(c))}{z - f(c)} - \frac{1}{f'(c)}\right| < \epsilon,$$

as desired.

## UNIFORM CONVERGENCE OF ANALYTIC FUNCTIONS

Part (c) of Exercise 4.26 gives an example showing that the uniform limit of a sequence of differentiable functions of a real variable need not be differentiable. Indeed, when thinking about uniform convergence of functions, the fundamental result to remember is that the uniform limit of continuous functions is continuous (Theorem 3.17). The functions in Exercise 4.26 were differentiable functions of a real variable. The fact is that, for functions of a complex variable, things are as usual much more simple. The following theorem is yet another masterpiece of Weierstrass.

**THEOREM 7.12.** *Suppose $U$ is an open subset of $\mathbb{C}$, and that $\{f_n\}$ is a sequence of analytic functions on $U$ that converges uniformly to a function $f$. Then $f$ is analytic on $U$. That is, the uniform limit of differentiable functions on an open set $U$ in the complex plane is also differentiable on $U$.*

*PROOF.* Though this theorem sounds impressive and perhaps unexpected, it is really just a combination of Theorem 6.10 and the Cauchy Integral Formula. Indeed, let $c$ be a point in $U$, and let $r > 0$ be such that $\overline{B}_r(c) \subseteq U$. Then the sequence $\{f_n\}$ converges uniformly to $f$ on the boundary $C_r$ of this closed disk. Moreover, for any $z \in B_r(c)$, the sequence $\{f_n(\zeta)/(\zeta - z)\}$ converges uniformly to $f(\zeta)/(\zeta - z)$ on $C_r$. Hence, by Theorem 6.10, we have

$$\begin{aligned} f(z) &= \lim f_n(z) \\ &= \lim_n \frac{1}{2\pi i}\int_{C_r} \frac{f_n(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta. \end{aligned}$$

Hence, by part (a) of Exercise 7.7, $f$ is expandable in a Taylor series around $c$, i.e., $f$ is analytic on $U$.

## ISOLATED SINGULARITIES, AND THE RESIDUE THEOREM

The first result we present in this section is a natural extension of Theorem 7.3. However, as we shall see, its consequences for computing contour integrals can hardly be overstated.

**THEOREM 7.13.** *Let $S$ be a piecewise smooth geometric set whose boundary $C_S$ has finite length. Suppose $c_1, \dots, c_n$ are distinct points in the interior $S^0$ of $S$, and that $r_1, \dots, r_n$ are positive numbers such that the closed disks $\{\overline{B}_{r_k}(c_k)\}$ are contained in $S^0$ and pairwise disjoint. Suppose $f$ is continuous on $S \setminus \cup B_{r_k}(c_k)$, i.e., at each point of $S$ that is not in any of the open disks $B_{r_k}(c_k)$, and that $f$ is differentiable on $S^0 \setminus \cup \overline{B}_{r_k}(c_k)$, i.e., at each point of $S^0$ that is not in any of the closed disks $\overline{B}_{r_k}(c_k)$. Write $C_k$ for the circle that is the boundary of the closed disk $\overline{B}_{r_k}(c_k)$. Then*

$$\int_{C_S} f(\zeta)\, d\zeta = \sum_{k=1}^{n} \int_{C_k} f(\zeta)\, d\zeta.$$

*PROOF.* This is just a special case of part (d) of Exercise 7.3.

Let $f$ be continuous on the punctured disk $\overline{B'}_r(c)$, analytic at each point $z$ in $B'_r(c)$, and suppose $f$ is undefined at the central point $c$. Such points $c$ are called *isolated singularities* of $f$, and we wish now to classify these kinds of points. Here is the first kind:

**DEFINITION.** A complex number $c$ is called a *removable singularity* of an analytic function $f$ if there exists an $r > 0$ such that $f$ is continuous on the punctured disk $\overline{B'}_r(c)$, analytic at each point in $B'_r(c)$, and $\lim_{z \to c} f(z)$ exists.

**Exercise 7.11.** (a) Define $f(z) = \sin z / z$ for all $z \neq 0$. Show that 0 is a removable singularity of $f$.

(b) For $z \neq c$, define $f(z) = (1 - \cos(z - c))/(z - c)$. Show that $c$ is a removable singularity of $f$.

(c) For $z \neq c$, define $f(z) = (1 - \cos(z - c))/(z - c)^2$. Show that $c$ is still a removable singularity of $f$.

(d) Let $g$ be an analytic function on $B_r(c)$, and set $f(z) = (g(z) - g(c))/(z - c)$ for all $z \in B'_r(c)$. Show that $c$ is a removable singularity of $f$.

The following theorem provides a good explanation for the term "removable singularity." The idea is that this is not a "true" singularity; it's just that for some reason the natural definition of $f$ at $c$ has not yet been made.

**THEOREM 7.14.** *Let $f$ be continuous on the punctured disk $\overline{B}'_r(c)$ and differentiable at each point of the open punctured disk $B'_r(c)$, and assume that $c$ is a removable singularity of $f$. Define $\widetilde{f}$ by $\widetilde{f}(z) = f(z)$ for all $z \in B'_r(c)$, and $\widetilde{f}(c) = \lim_{z \to c} f(z)$. Then*

(1) *$\widetilde{f}$ is analytic on the entire open disk $B_r(c)$, whence*

$$f(z) = \sum_{k=0}^{\infty} c_k (z - c)^k$$

*for all $z \in B'_r(c)$.*

(2) *For any piecewise smooth geometric set $S \subseteq B_r(c)$, whose boundary $C_S$ has finite length, and for which $c \in S^0$,*

$$\int_{C_S} f(\zeta)\, d\zeta = 0.$$

*PROOF.* As in part (a) of Exercise 7.7, define $F$ on $B_r(c)$ by

$$F(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Then, by that exercise, $F$ is analytic on $B_r(c)$. We show next that $F(z) = \widetilde{f}(z)$ on $B_r(c)$, and this will complete the proof of part (1).

Let $z$ be a point in $B_r(c)$ that is not equal to $c$, and let $\epsilon > 0$ be given. Choose $\delta > 0$ such that $\delta < |z - c|/2$ and such that $|\widetilde{f}(\zeta) - \widetilde{f}(c)| < \epsilon$ if $|\zeta - c| < \delta$. Then, using part (c) of Exercise 7.5, we have that

$$\begin{aligned}
\widetilde{f}(z) &= f(z)\\
&= \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{C_\delta} \frac{f(\zeta)}{\zeta - z}\, d\zeta\\
&= F(z) - \frac{1}{2\pi i}\int_{C_\delta} \frac{f(\zeta) - \widetilde{f}(c)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{C_\delta} \frac{\widetilde{f}(c)}{\zeta - z}\, d\zeta\\
&= F(z) - \frac{1}{2\pi i}\int_{C_\delta} \frac{\widetilde{f}(\zeta) - \widetilde{f}(c)}{\zeta - z}\, d\zeta,
\end{aligned}$$

where the last equality holds because the function $\widetilde{f}(c)/(\zeta - z)$ is an analytic function of $\zeta$ on the disk $B_\delta(c)$, and hence the integral is 0 by Theorem 7.3. So,

$$\begin{aligned}
|\widetilde{f}(z) - F(z)| &= |\frac{1}{2\pi i}\int_{C_\delta} \frac{\widetilde{f}(\zeta) - \widetilde{f}(c)}{\zeta - z}\, d\zeta|\\
&\leq \frac{1}{2\pi}\int_{C_\delta} \frac{|\widetilde{f}(\zeta) - \widetilde{f}(c)|}{|\zeta - z|}\, ds\\
&\leq \frac{1}{2\pi}\int_{C_\delta} \frac{\epsilon}{\delta/2}\, ds\\
&= \frac{2\epsilon}{\delta} \times \delta\\
&= 2\epsilon.
\end{aligned}$$

Since this holds for arbitrary $\epsilon > 0$, we see that $\widetilde{f}(z) = F(z)$ for all $z \neq c$ in $B_r(c)$.

Finally, since

$$\widetilde{f}(c) = \lim_{z\to c} \widetilde{f}(z) = \lim z \to cF(z) = F(c),$$

the equality of $F$ and $\widetilde{f}$ on all of $B_r(c)$ is proved. This finishes the proof of part (1).

**Exercise 7.12.** Prove part (2) of the preceding theorem.

Now, for the second kind of isolated singularity:

**DEFINITION.** A complex number $c$ is called a *pole* of a function $f$ if there exists an $r > 0$ such that $f$ is continuous on the punctured disk $\overline{B'}_r(c)$, analytic at each point of $B'_r(c)$, the point $c$ is not a removable singularity of $f$, and there exists a positive integer $k$ such that the analytic function $(z-c)^k f(z)$ has a removable singularity at $c$.

A pole $c$ of $f$ is said to be *of order n*, if $n$ is the smallest positive integer for which the function $\widetilde{f}(z) \equiv (z-c)^n f(z)$ has a removable singularity at $c$.

**Exercise 7.13.** (a) Let $c$ be a pole of order $n$ of a function $f$, and write $\widetilde{f}(z) = (z-c)^n f(z)$. Show that $\widetilde{f}$ is analytic on some disk $B_r(c)$.

(b) Define $f(z) = \sin z/z^3$ for all $z \neq 0$. Show that 0 is a pole of order 2 of $f$.

**THEOREM 7.15.** *Let $f$ be continuous on a punctured disk $\overline{B'}_r(c)$, analytic at each point of $B'_r(c)$, and suppose that $c$ is a pole of order $n$ of $f$. Then*

(1) *For all $z \in B'_r(c)$,*

$$f(z) = \sum_{k=-n}^{\infty} a_k (z-c)^k.$$

(2) *The infinite series of part (1) converges uniformly on each compact subset $K$ of $B'_r(c)$.*

(3) *For any piecewise smooth geometric set $S \subseteq B_r(c)$, whose boundary $C_S$ has finite length, and satisfying $c \in S^0$,*

$$\int_{C_S} f(\zeta)\, d\zeta = 2\pi i a_{-1},$$

*where $A_{-1}$ is the coefficient of $(z-c)^{-1}$ in the series of part (1).*

*PROOF.* For each $z \in B'_r(c)$, write $\widetilde{f}(z) = (z-c)^n f(z)$. Then, by Theorem 7.14, $\widetilde{f}$ is analytic on $B_r(c)$, whence

$$\begin{aligned} f(z) &= \frac{\widetilde{f}(z)}{(z-c)^n} \\ &= \frac{1}{(z-c)^n} \sum_{k=0}^{\infty} c_k (z-c)^k \\ &= \sum_{k=-n}^{\infty} a_k (z-c)^k, \end{aligned}$$

where $a_k = c_{n+k}$. This proves part (1).

We leave the proof of the uniform convergence of the series on each compact subset of $B'_r(c)$, i.e., the proof of part (2), to the exercises.

Part (3) follows from Cauchy's Theorem (Theorem 7.3) and the computations in Exercise 7.2. Thus:

$$\begin{aligned} \int_{C_S} f(\zeta)\, d\zeta &= \int_{C_r} f(\zeta)\, d\zeta \\ &= \int_{C_r} \sum_{k=-n}^{\infty} a_k (z-c)^k \, d\zeta \\ &= \sum_{k=-n}^{\infty} a_k \int_{C_r} (\zeta - c)^k \, d\zeta \\ &= a_{-1} 2\pi i, \end{aligned}$$

as desired. The summation sign comes out of the integral because of the uniform convergence of the series on the compact circle $C_r$.

**Exercise 7.14.** (a) Complete the proof to part (2) of the preceding theorem. That is, show that the infinite series $\sum_{k=-n}^{\infty} a_k(z-c)^k$ converges uniformly on each compact subset $K$ of $B'_r(c)$.
HINT: Use the fact that the Taylor series $\sum_{n=0}^{\infty} c_n(z-c)^n$ for $\widetilde{f}$ converges uniformly on the entire disk $\overline{B}_r(c)$, and that if $c$ is not in a compact subset $K$ of $B_r(c)$, then there exists a $\delta > 0$ such that $|z-c| > \delta$ for all $z \in K$.

(b) Let $f, c$, and $\widetilde{f}$ be as in the preceding proof. Show that

$$a_{-1} = \frac{\widetilde{f}^{(n-1)}(c)}{(n-1)!}.$$

(c) Suppose $g$ is a function defined on a punctured disk $B'_r(c)$ that is given by the formula

$$g(z) = \sum_{k=-n}^{\infty} a_k(z-c)^k$$

for some positive integer $n$ and for all $z \in B'_r(c)$. Suppose in addition that the coefficient $a_{-n} \neq 0$. Show that $c$ is a pole of order $n$ of $g$.

Having defined two kinds of isolated singularities of a function $f$, the removable ones and the polls of finite order, there remain all the others, which we collect into a third type.

**DEFINITION.** Let $f$ be continuous on a punctured disk $\overline{B'}_r(c)$, and analytic at each point of $B'_r(c)$. The point $c$ is called an *essential singularity* of $f$ if it is neither a removable singularity nor a poll of any finite order. Singularities that are either poles or essential singularities are called *nonremovable singularities.*

**Exercise 7.15.** For $z \neq 0$, define $f(z) = e^{1/z}$. Show that 0 is an essential singularity of $f$.

**THEOREM 7.16.** *Let $f$ be continuous on a punctured disk $\overline{B'}_r(c)$, analytic at each point of $B'_r(c)$, and suppose that $c$ is an essential singularity of $f$. Then*

(1) *For all $z \in B'_r(c)$,*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-c)^k,$$

*where the sequence $\{a_k\}_{-\infty}^{\infty}$ has the property that for any negative integer $N$ there is a $k < N$ such that $a_k \neq 0$.*

(2) *The infinite series in part (1) converges uniformly on each compact subset $K$ of $B'_r(c)$. That is, if $F_n$ is defined by $F_n(z) = \sum_{k=-n}^{n} a_k(z-c)^k$, then the sequence $\{F_n\}$ converges uniformly to $f$ on the compact set $K$.*

(3) *For any piecewise smooth geometric set $S \subseteq B_r(c)$, whose boundary $C_S$ has finite length, and satisfying $c \in S^0$, we have*

$$\int_{C_S} f(\zeta)\, d\zeta = 2\pi i a_{-1},$$

*where $a_{-1}$ is the coefficient of $(z-c)^{-1}$ in the series of part (1).*

*PROOF.* Define numbers $\{a_k\}_{-\infty}^{\infty}$ as follows.

$$a_k = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta - c)^{k+1}}\,d\zeta.$$

Note that for any $0 < \delta < r$ we have from Cauchy's Theorem that

$$a_k = \frac{1}{2\pi i}\int_{C_\delta} \frac{f(\zeta)}{(\zeta - c)^{k+1}}\,d\zeta,$$

where $C_\delta$ denotes the boundary of the disk $\overline{B}_\delta(c)$.

Let $z \neq c$ be in $B_r(c)$, and choose $\delta > 0$ such that $\delta < |z - c|$. Then, using part (c) of Exercise 7.5, and then mimicking the proof of Theorem 7.5, we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{C_\delta}\frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{(\zeta - c) - (z - c)}\,d\zeta + \frac{1}{2\pi i}\int_{C_\delta}\frac{f(\zeta)}{(z - c) - (\zeta - c)}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - c}\frac{1}{1 - \frac{z-c}{\zeta-c}}\,d\zeta + \frac{1}{2\pi i}\int_{C_\delta}\frac{f(\zeta)}{z - c}\frac{1}{1 - \frac{\zeta-c}{z-c}}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - c}\sum_{k=0}^{\infty}(\frac{z - c}{\zeta - c})^k\,d\zeta + \frac{1}{2\pi i}\int_{C_\delta}\frac{f(\zeta)}{z - c}\sum_{j=0}^{\infty}(\frac{\zeta - c}{z - c})^j\,d\zeta \\
&= \sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{(\zeta - c)^{k+1}}\,d\zeta(z - c)^k + \sum_{j=0}^{\infty}\frac{1}{2\pi i}\int_{C_\delta} f(\zeta)(\zeta - c)^j\,d\zeta(z - c)^{-j-1} \\
&= \sum_{k=0}^{\infty} a_k(z - c)^k + \sum_{k=-\infty}^{-1}\frac{1}{2\pi i}\int_{C_\delta}\frac{f(\zeta)}{(\zeta - c)^{k+1}}\,d\zeta(z - c)^k \\
&= \sum_{k=0}^{\infty} a_k(z - c)^k + \sum_{k=-\infty}^{-1}\frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{(\zeta - c)^{k+1}}\,d\zeta(z - c)^k \\
&= \sum_{k=-\infty}^{\infty} a_k(z - c)^k,
\end{aligned}$$

which proves part (1).

We leave the proofs of parts (2) and (3) to the exercises.

**Exercise 7.16.** (a) Justify bringing the summation signs out of the integrals in the calculation in the preceding proof.

(b) Prove parts (2) and (3) of the preceding theorem. Compare this with Exercise 7.14.

*REMARK.* The representation of $f(z)$ in the punctured disk $B'_r(c)$ given in part (1) of Theorems 7.15 and 7.16 is called the *Laurent expansion* of $f$ around the singularity $c$. Of course it differs from a Taylor series representation of $f$, as this one contains negative powers of $z - c$. In fact, which negative powers it contains indicates what kind of singularity the point $c$ is.

Non removable isolated singularities of a function $f$ share the property that the integral of $f$ around a disk centered at the singularity equals $2\pi i a_{-1}$, where the number $a_{-1}$ is the coefficient of $(z-c)^{-1}$ in the Laurent expansion of $f$ around $c$. This number $2\pi i a_{-1}$ is obviously significant, and we call it the *residue of f at c,* and denote it by $R_f(c)$.

Combining Theorems 7.13, 7.15, and 7.16, we obtain:

**THEOREM 7.17.** (Residue Theorem) Let $S$ be a piecewise smooth geometric set whose boundary has finite length, let $c_1, \ldots, c_n$ be points in $S^0$, and suppose $f$ is a complex-valued function that is continuous at every point $z$ in $S$ except the $c_k$'s, and differentiable at every point $z \in S^0$ except at the $c_k$'s. Assume finally that each $c_k$ is a nonremovable isolated singularity of $f$. Then

$$\int_{C_S} f(\zeta)\, d\zeta = \sum_{k=1}^{n} R_f(c_k).$$

That is, the contour integral around $C_S$ is just the sum of the residues inside $S$.

**Exercise 7.17.** Prove Theorem 7.17.

**Exercise 7.18.** Use the Residue Theorem to compute $\int_{C_S} f(\zeta)\, d\zeta$ for the functions $f$ and geometric sets $S$ given below. That is, determine the poles of $f$ inside $S$, their orders, the corresponding residues, and then evaluate the integrals.

(a) $f(z) = \sin(3z)/z^2$, and $S = \overline{B}_1(0)$.
(b) $f(z) = e^{1/z}$, and $S = \overline{B}_1(0)$.
(c) $f(z) = e^{1/z^2}$, and $S = \overline{B}_1(0)$.
(d) $f(z) = (1/z(z-1))$, and $S = \overline{B}_2(0)$.
(e) $f(z) = ((1-z^2)/z(1+z^2)(2z+1)^2)$, and $S = \overline{B}_2(0)$.
(f) $f(z) = 1/(1+z^4) = (1/(z^2-i)(z^2+i))$, and $S = \overline{B}_r(0)$ for any $r > 1$.

The Residue Theorem, a result about contour integrals of functions of a complex variable, can often provide a tool for evaluating integrals of functions of a real variable.

**EXAMPLE 1.** Consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx.$$

Let us use the Residue Theorem to compute this integral.

Of course what we need to compute is

$$\lim_{B\to\infty} \int_{-B}^{B} \frac{1}{1+x^4}\, dx.$$

The first thing we do is to replace the real variable $x$ by a complex variable $Z$, and observe that the function $f(z) = 1/(1+z^4)$ is analytic everywhere except at the four points $\pm e^{i\pi/4}$ and $\pm e^{3i\pi/4}$. See part (f) of the preceding exercise. These are the four points whose fourth power is $-1$, and hence are the poles of the function $f$.

Next, given a positive number $B$, we consider the geometric set (rectangle) $S_B$ that is determined by the interval $[-B, B]$ and the two bounding functions $l(x) = 0$

and $u(x) = B$. Then, as long as $B > 1$, we know that $f$ is analytic everywhere in $S^0$ except at the two points $c_1 = e^{i\pi/4}$ and $c_2 = e^{3i\pi/4}$, so that the contour integral of $f$ around the boundary of $S_B$ is given by

$$\int_{C_{S_B}} \frac{1}{1+\zeta^4}\, d\zeta = R_f(c_1) + R_f(c_2).$$

Now, this contour integral consists of four parts, the line integrals along the bottom, the two sides, and the top. The magic here is that the integrals along the sides, and the integral along the top, all tend to 0 as $B$ tends to infinity, so that the integral along the bottom, which after all is what we originally were interested in, is in the limit just the sum of the residues inside the geometric set.

**Exercise 7.19.** Verify the details of the preceding example.

(a) Show that

$$\lim_{B\to\infty} \int_0^B \frac{1}{1+(B+it)^4}\, dt = 0.$$

(b) Verify that

$$\lim_{B\to\infty} \int_{-B}^B \frac{1}{1+(t+iB)^4}\, dt = 0.$$

(c) Show that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx = \pi\sqrt{2}.$$

Methods similar to that employed in the previous example and exercise often suffice to compute integrals of real-valued functions. However, the method may have to be varied. For instance, sometimes the appropriate geometric set is a rectangle below the $x$-axis instead of above it, sometimes it should be a semicircle instead of a rectangle, etc. Indeed, the choice of contour (geometric set) can be quite subtle. The following exercise may shed some light.

**Exercise 7.20.** (a) Compute

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^4}\, dx$$

and

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{1+x^4}\, dx.$$

(b) Compute

$$\int_{-\infty}^{\infty} \frac{\sin(-x)}{1+x^3}\, dx$$

and

$$\int_{-\infty}^{\infty} \frac{\sin x}{1+x^3}\, dx.$$

**EXAMPLE 2.** An historically famous integral in analysis is $\int_{-\infty}^{\infty} \sin x/x\,dx$. The techniques described above don't immediately apply to this function, for, even replacing the $x$ by a $z$, this function has no poles, so that the Residue Theorem wouldn't seem to be much help. Though the point 0 is a singularity, it is a removable one, so that this function $\sin z/z$ is essentially analytic everywhere in the complex plane. However, even in a case like this we can obtain information about integrals of real-valued functions from theorems about integrals of complex-valued functions.

Notice first that $\int_{-\infty}^{\infty} \sin x/x\,dx$ is the imaginary part of $\int_{-\infty}^{\infty} e^{ix}/x\,dx$, so that we may as well evaluate the integral of this function. Let $f$ be the function defined by $f(z) = e^{iz}/z$, and note that 0 is a pole of order 1 of $f$, and that the residue $R_f(0) = 2\pi i$. Now, for each $B > 0$ and $\delta > 0$ define a geometric set $S_{B,\delta}$, determined by the interval $[-B, B]$, as follows: The upper bounding function $u_{B,\delta}$ is given by $u_{B,\delta}(x) = B$, and the lower bounding function $l_{B,\delta}$ is given by $l_{B,\delta}(x) = 0$ for $-B \le x \le -\delta$ and $\delta \le x \le B$, and $l_{B,\delta}(x) = \delta e^{i\pi x/\delta}$ for $-\delta < x < \delta$. That is, $S_{B,\delta}$ is just like the rectangle $S_B$ in Example 1 above, except that the lower boundary is not a straight line. Rather, the lower boundary is a straight line from $-B$ to $-\delta$, a semicircle below the $x$-axis of radius $\delta$ from $-\delta$ to $\delta$, and a straight line again from $\delta$ to $B$.

By the Residue Theorem, the contour integral

$$\int_{C_{S_{B,\delta}}} f(\zeta)\,d\zeta = R_f(0) = 2\pi i.$$

As in the previous example, the contour integrals along the two sides and across the top of $S_{B,\delta}$ tend to 0 as $B$ tends to infinity. Finally, according to part (e) of Exercise 6.15, the contour integral of $f$ along the semicircle in the lower boundary is $\pi i$ independent of the value of $\delta$. So,

$$\lim_{B\to\infty} \lim_{\delta\to 0} \int_{\mathrm{graph}(l_{B,\delta})} \frac{e^{i\zeta}}{\zeta}\,d\zeta = \pi i,$$

implying then that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi.$$

**Exercise 7.21.** (a) Justify the steps in the preceding example. In particular, verify that

$$\lim_{B\to\infty} \int_0^B \frac{e^{i(B+it)}}{B+it}\,dt = 0,$$

$$\lim_{B\to\infty} \int_{-B}^B \frac{e^{i(t+iB)}}{t+iB}\,dt = 0,$$

and

$$\int_{C_\delta} \frac{e^{i\zeta}}{\zeta}\,d\zeta = \pi i,$$

where $C_\delta$ is the semicircle of radius $\delta$, centered at the origin and lying below the $x$-axis.

(b) Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx.$$