MATH 4330/5330, Fourier Analysis
Section 1, The Heat Equation on the Line

We imagine we have an infinitely long rod and that we have fixed a point on the rod that we call 0. We suppose that a function u of two variables t and x is such that the value $u(t,x)$ is the temperature at the point x on the rod at time t. We would like to be able to predict how this temperature function changes with time. That is, we consider the following so-called "initial value problem."

We suppose that we know the values $u(0,x) \equiv f(x)$ for all points x. These are the initial values (temperatures). Is that enough information for us to be able to figure out the values $u(t,x)$ for a later time t? That is, is the evolution of the temperature function uniquely determined by what it is at a starting time? Can we find out what the temperature was at an earlier time, given what it is now? Is there an explicit formula for $u(t,x)$ in terms of this initial function f? Moreover, can we see what happens as time tends to infinity, i.e., the long-term behavior? Or, can we determine what happened at $t = -\infty$? That is, can we analyze backwards to figure out what the temperature was at the very beginning of time?

REMARK. Physicists think that the temperature at a point on the rod is proportional to the velocity of the molecule at that point in the rod. Since the square of the velocity V^2 is proportional to the kinetic energy $mV^2/2$, we presume that the function $|u(t,x)|^2$ is proportional to the "instantaneous" energy at the point x, and so $\int_{-\infty}^{\infty} |u(t,x)|^2\,dx$ should represent the total energy at time t. This leads us to our first discovery about the function $u(t,x)$. It must satisfy $\int_{-\infty}^{\infty} |u(t,x)|^2\,dx < \infty$ for every time t. The total energy must be finite at any given time.

Mathematicians say that these functions of x *belong to* L^2.

Said precisely, a function ϕ is said to belong to L^p if $\int_{-\infty}^{\infty} |\phi(x)|^p\,dx < \infty$.

Physicists also believe that this temperature function u must satisfy the following partial differential equation, called the *heat equation*.

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x).$$

EXERCISE 1.1. Can you think of any solutions to this partial differential equation? How about $u(t,x) = 0$ or $u(t,x) = 1$, or $u(t,x) = 2t + x^2$? How about $u(t,x) = e^{k^2 t} \times e^{kx}$? Do any of these functions satisfy the finite energy (L^2) requirement? Can you think of any other solutions of the heat equation?

EXAMPLE 1.1. For all real numbers x and all positive t, define

$$k(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

EXERCISE 1.2. Verify that this function k satisfies the heat equation. Notice that this function is not defined for $t = 0$. Can you figure out what happens to $k(t,x)$ as t approaches 0?

DEFINITION. The function $k(t,x)$ is called the *fundamental solution* of the heat equation on the line, and it is also frequently referred to as the *heat kernel*.

EXERCISE 1.3. Let $f(x) = e^{-\pi x^2}$, and let $C = \int_{-\infty}^{\infty} f(x)\,dx$. Justify the following computations

$$\begin{aligned}
C^2 &= C \int_{-\infty}^{\infty} f(x)\,dx \\
&= \int_{-\infty}^{\infty} Cf(x)\,dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)\,dy f(x)\,dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi y^2}\,dy f(x)\,dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-\pi y^2}\,dy\,dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-\pi y^2}\,dydx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}\,dydx \\
&= \int_0^{2\pi} \int_0^{\infty} e^{-\pi r^2} r\,drd\theta \\
&= \int_0^{2\pi} \frac{-1}{2\pi} \int_0^{\infty} e^{-\pi r^2}(-2\pi r)\,drd\theta \\
&= \frac{-1}{2\pi} \int_0^{2\pi} e^{-\pi r^2} [_0^\infty \,d\theta \\
&= \frac{-1}{2\pi} \int_0^{2\pi} (0-1)\,d\theta \\
&= 1.
\end{aligned}$$

Conclude that

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1.$$

EXERCISE 1.4. (a) For any positive number a, compute $\int_{-\infty}^{\infty} e^{-ax^2}\,dx$. (Use Exercise 1.3, and make a change of variables.)

(b) Let $k(t,x)$ be the function in Example 1.1 above. Compute $\int_{-\infty}^{\infty} |k(t,x)|\,dx$ and $\int_{-\infty}^{\infty} |k(t,x)|^2\,dx$, and investigate what happens to this total energy as t tends to infinity. Also, what happens to this energy as t approaches 0?

(c) For any real number b, compute $\int_{-\infty}^{\infty} e^{-(x+b)^2}\,dx$.

(d) For a, b, and c real numbers, with $a > 0$, compute $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,dx$. HINT: Complete the square and use earlier parts of this exercise.

One of our main goals is to figure out where this funny function $k(t,x)$ comes from. You surely couldn't have guessed that this would be a solution to the heat equation. It may not be any good anyhow, because it doesn't really fit our initial value problem. Why do we even consider this function?