MATH 4330/5330, Fourier Analysis
Section 11, The Discrete Fourier Transform

Now, instead of considering functions defined on a continuous domain, like the interval $[0,1)$ or the whole real line $\mathbb{R}$, we wish to study functions defined on a discrete set of points. This is of course how engineers actually use Fourier analysis. We begin with the discrete set (group) $\mathbb{Z}$ of all integers. This is certainly a group with respect to the binary operation of $+$. What should be the definition of the Fourier transform $\widehat{f}$ of a function f on this set? To begin with, what should the domain of the function $\widehat{f}$ be? By analogy with what we did earlier, let us determine the set of homomorphisms of the group $\mathbb{Z}$ into the circle group $\mathbb{T}$. Remember, this is the way we decided what the domain of the Fourier transform on the real line should be.

THEOREM 11.1. *Let ϕ be a function on $\mathbb{Z}$ that maps into the circle group $\mathbb{T}$ and satisfies the law of exponents $\phi(k+l)=\phi(k)\phi(l)$. (ϕ is a homomorphism of the additive group $\mathbb{Z}$ into the multiplicative group $\mathbb{T}$.) Then there exists a unique element $x \in [0,1)$ such that*

$$\phi(k)=e^{2\pi ikx}$$

for all integers k.

PROOF. Notice first that $\phi(0)=\phi(0+0)=\phi(0)\phi(0)=\phi(0)^2$, so that $\phi(0)$ must be equal to 1. (Why couldn't $\phi(0)$ be 0?)

Now let λ be the element of $\mathbb{T}$ for which $\lambda=\phi(1)$, and let x be the unique point in the interval $[0,1)$ such that $\lambda=e^{2\pi ix}$. (Why is there only one such number x?) We have then that $\phi(1)=\lambda=e^{2\pi ix}$. Then,

$$\phi(2)=\phi(1+1)=\phi(1)\times\phi(1)=\lambda^2=e^{2\pi i2x},$$

$$\phi(3)=\lambda^3=e^{2\pi i3x},\ \phi(4)=\lambda^4=e^{2\pi i4x},\dots .$$

That is, for any $0\le k$, we must have $\phi(k)=\lambda^k=e^{2\pi ikx}$. Now, for $-k<0$, we know that

$$1=\phi(0)=\phi(k-k)=\phi(k)\phi(-k)=\lambda^k\phi(-k),$$

showing that

$$\phi(-k)=\frac{1}{\lambda^k}=\lambda^{-k}=e^{2\pi i(-k)x}.$$

This proves the theorem.

The preceding theorem shows that the set of homomorphisms of the group $\mathbb{Z}$ into the circle $BbbT$ are parameterized by the numbers x in the interval $[0,1)$. That is, for each element $x\in[0,1)$, there is a homomorphism ϕ_x of $\mathbb{Z}$ into $\mathbb{T}$ defined by $\phi_x(k)=e^{2\pi ikx}$, and these are all the homomorphisms of $\mathbb{Z}$ into $\mathbb{T}$. In other words, the set of homomorphisms of the group $\mathbb{Z}$ into the group $\mathbb{T}$ is parameterized by the points in the interval $[0,1)$. So, following our earlier development, we should define the Fourier transform of a function f on $\mathbb{Z}$ to be the function $\widehat{f}$ on $[0,1)$ given by

$$\widehat{f}(x)=\sum_{k=-\infty}^{\infty}f(k)\overline{\phi_x}(k)=\sum_{k=-\infty}^{\infty}f(k)e^{-2\pi ikx}.$$

REMARK. Notice that, just as in the case of the real line, we are only able to define the Fourier transform of a function f on $\mathbb{Z}$ if it is absolutely summable. that is, only if $\sum_{k=-\infty}^{\infty} |f(k)| < \infty$.

EXERCISE 11.1. (a) Let f be the function on $\mathbb{Z}$ given by $f(k) = 0$ unless $k = \pm 1$, and $f(1) = f(-1) = 1$. What is the Fourier transform of f?

(b) Fix a positive integer N, and let f be the function on $\mathbb{Z}$ given by $f(k) = 1$ if $-N \leq k \leq N$ and 0 otherwise. Find the Fourier transform of f.

What should Fourier's Theorem be in this case? that is, how can we recover the function f on $\mathbb{Z}$ from the function $\widehat{f}$ on $[0, 1)$. Before giving the answer, we make the following observations:

EXERCISE 11.2. Let f be an absolutely summable function on the group $\mathbb{Z}$ of integers, and write $\widehat{f}$ for the function on $[0, 1)$ defined by

$$\widehat{f}(x) = \sum_{k=-\infty}^{\infty} f(k) e^{-2\pi i k x}.$$

(a) Use the Weierstrass M-Test to show that $\widehat{f}$ is a continuous function on $[0, 1)$. The Weierstrass M-Test says the following: Suppose $\{u_k\}$ is an infinite sequence of continuous functions, and suppose that $\{m_k\}$ is a corresponding sequence of nonnegative numbers for which $|u_k(x)| \leq m_k$ for all x, and for which the infinite series $\sum m_k$ converges. Then the infinite series $\sum u_k(x)$ converges to a continuous function.

(b) Conclude that $\widehat{f}$ is a square-integrable function on $[0, 1)$.

Here is Fourier's Theorem in this context:

THEOREM 11.2. *Let f be an absolutely summable function on $\mathbb{Z}$. Then, for each integer n, we have*

$$f(n) = \int_0^1 \widehat{f}(x) e^{2\pi i n x}\, dx.$$

PROOF.

$$\begin{aligned}
\int_0^1 \widehat{f}(x) e^{2\pi i n x}\, dx &= \int_0^1 \sum_{k=-\infty}^{\infty} f(k) e^{-2\pi i k x} e^{2\pi i n x}\, dx \\
&= \sum_{k=-\infty}^{\infty} \int_0^1 f(k) e^{2\pi i (n-k) x}\, dx \\
&= \sum_{k=-\infty}^{\infty} f(k) \int_0^1 e^{2\pi i (n-k) x}\, dx \\
&= \sum_{-\infty}^{\infty} f(k) \delta_{n,k} \\
&= f(n),
\end{aligned}$$

which proves the theorem.

REMARK. This whole thing looks awfully familiar. It surely must be related to the Fourier transform on the circle. Indeed, is this just the inverse of that transform?

EXERCISE 11.3. (a) If f is a square-integrable function on $[0,1)$, what is the formula for its Fourier transform, and what is the formula for the inverse transform?

(b) If g is a function on $\mathbb{Z}$, what is the formula for its Fourier transform, and what is the formula for the inverse transform?

(c) Exactly how are parts (a) and (b) related?

Of course, we could now try to do all the same kinds of things we did with the other Fourier transforms. It may be a little less interesting this time, primarily because functions on the discrete set $\mathbb{Z}$ are not continuous, and they don't have derivatives. The kinds of problems one encounters with these functions are **not** differential equations. Perhaps we could work on "difference equations."

One thing that can be defined in this case is convolution.

DEFINITION. Let f and g be two absolutely summable functions on $\mathbb{Z}$. Define the *convolution* $f*g$ of f and g to be the function on $\mathbb{Z}$ given by

$$f*g(n)=\sum_{k=-\infty}^{\infty}f(n-k)g(k).$$

EXERCISE 11.4. (a) Show that, if f and g are absolutely summable, then so is $f*g$.

(b) Prove the convolution theorem in this case:

$$\widehat{f*g}(x)=\widehat{f}(x)\widehat{g}(x).$$

(c) Show that, in this case, there is an identity for convolution. That is, there is a function e on $\mathbb{Z}$ for which $f=f*e$ for every f.

(d) Show that

$$f*g(n)=\sum_{k,l:k+l=n}f(k)g(l).$$

The Finite Fourier Transform

Fix a positive integer N, and let G be the finite set $0,1,2,\dots,N-1$. The set G is a group (actually a ring), where addition and multiplication are computed mod N. It is frequently denoted by $\mathbb{Z}_N$.

EXERCISE 11.5. Let $N=64$.

(a) Compute $41+52$ in this group $\mathbb{Z}_{64}$.

(b) Compute 32×56 in this ring.

We wish to define a Fourier transform on this group. We follow our standard procedure.

THEOREM 11.3. *Let ϕ be a function on $G=\mathbb{Z}_N$ that maps into the circle group $\mathbb{T}$ and satisfies the law of exponents $\phi(k+l)=\phi(k)\phi(l)$. (ϕ is a homomorphism of the additive group $\mathbb{Z}_N$ into the multiplicative group $\mathbb{T}$.) Then there exists a unique integer $0\le j\le N-1$ such that*

$$\phi(k)=e^{2\pi i\frac{jk}{N}}$$

for all $k \in G$. That is, these homomorphisms are parameterized by the numbers $j = 0, 1, 2, \dots, N-1$.

PROOF. Notice first that $\phi(0) = \phi(0+0) = \phi(0)^2$, so that $\phi(0)$ must be equal to 1.

Now, let $z = \phi(1)$. Then,

$$\phi(2) = \phi(1+1) = \phi(1) \times \phi(1) = z^2,$$

$$\phi(3) = z^3,\ \phi(4) = z^4, \dots .$$

That is, for any $0 \leq k \leq N-1 <$ we must have $\phi(k) = z^k$. Now, because N is the same as 0 in the group G, we must have that $z^N = \phi(N) = \phi(0) = 1$. That is, z^N must equal 1. Write $z = e^{2\pi i x}$ for some $x \in [0, 1)$. We must have that

$$e^{2\pi i N x} = (e^{2\pi i x})^N = \phi(1)^N = \phi(N) = 1,$$

implying that xN is equal to some integer j between 0 and $N-1$. (Why?)

Hence, $x = j/N$, and $\phi(k) = z^k = e^{2\pi i x k}$ for all $k \in G$, as desired.

EXERCISE 11.6. For each integer $0 \leq j \leq N-1$, define a function ϕ_j on $G = \mathbb{Z}_N$ by

$$\phi_j(k) = e^{2\pi i \frac{jk}{N}}.$$

Prove that ϕ_j is a homomorphism of the group G into the circle group $\mathbb{T}$, and use Theorem 11.3 to conclude that these are all such homomorphisms of G.

Now we proceed just as we did in the case of $\mathbb{Z}$. Having determined the set of homomorphisms of our group into the circle group $\mathbb{T}$, and having seen that they are parameterized by the numbers $0 \leq j \leq N-1$, we define the Fourier transform of a function on $\mathbb{Z}_N$. This is a finite set, so we won't need any kind of absolute summability assumption. We can define the Fourier transform for any function on this group.

DEFINITION. Let V be the set (vector space) of all complex-valued functions defined on the set $G = \mathbb{Z}_N$. If f is an element of the vector space V, we define the *finite Fourier transform* $\widehat{f}$ of f to be the function, also defined on the set $0, 1, 2, \dots, N-1$, by

$$\begin{aligned}\widehat{f}(j) &= \sum_{k=0}^{N-1} f(k)\overline{\phi_j}(k) \\ &= \sum_{k=0}^{N-1} f(k) e^{-2\pi i \frac{jk}{N}}.\end{aligned}$$

EXERCISE 11.7. (a) Let f be define on G by $f(k) = 1/2 - k$. Find the finite Fourier transform $\widehat{f}$ of f. Is this anything like the Fourier transform of the function $1/2 - x$ on the circle?

(b) Suppose $N > 8$, and let $f(k) = 1$ for $0 \leq k \leq 7$, and $f(k) = 0$ otherwise. Find $\widehat{f}$.

EXERCISE 11.8. For f and g in V, define

$$\langle f \mid g\rangle = \sum_{k=0}^{N-1} f(k)\overline{g(k)}.$$

(a) Prove that V is a complex inner product space with respect to the above definition. That is, show that

(1) $\langle c_1 f_1 + c_2 f_2 \mid g\rangle = c_1\langle f_1 \mid g\rangle + c_2\langle f_2 \mid g\rangle$ for all elements $f, g \in V$ and all complex numbers c_1 and c_2.
(2) $\langle f \mid g\rangle = \overline{\langle g \mid f\rangle}$ for all $f, g \in V$.
(3) $\langle f \mid f\rangle \geq 0$ for all $f \in V$, and $\langle f \mid f\rangle = 0$ if and only if f is the 0 element of V.

(b) Let $f_0, f_1, \ldots, f_{N-1}$ be the functions in V defined as follows: $f_n(k) = \delta_{n,k}$. That is, f_n is the function that equals 1 on the integer n and equals 0 on all other integers k. Prove that the collection $f_0, \ldots, f_{N-1}$ is an orthonormal set in V. That is, show that $\langle f_n \mid f_k\rangle = \delta_{n,k}$.

(c) Prove that the functions $f_0, \ldots, f_{N-1}$ is a basis for the vector space V. That is, show that each $f \in V$ can be written in a unique way as a linear combination of the vectors (functions) f_n. Conclude then that $f_0, \ldots, f_{N-1}$ is an orthonormal basis for V.

(d) If T is a linear transformation of V into itself, recall from Linear Algebra that the entries of the matrix A that represents the transformation T with respect to this orthonormal basis are given by

$$A_{j,k} = \langle T(f_k) \mid f_j\rangle,$$

where both k and j run from 0 to $N-1$.

THEOREM 11.4. *The Fourier transform on V is a linear transformation from the N-dimensional vector space V into itself. If $f_0, \ldots, f_{N-1}$ is the orthonormal basis of the preceding exercise, then the entries of the matrix A that represents the Fourier transform with respect to this basis are given by*

$$A_{j,k} = e^{-2\pi i \frac{jk}{N}},$$

j and k both running from 0 to $N-1$.

EXERCISE 11.9. (a) Use Exercise 11.8 to prove this theorem. Of course, you will need to compute the Fourier transforms of the functions $f_0, \ldots, f_{N-1}$.

(b) Suppose f is a function (thought of as a column vector) in the vector space V. Verify that the jth component of the column vector $A \times f$ is given by the formula

$$(A \times f)_j = \widehat{f}(j).$$

Here is Fourier's Theorem in this case. It should be easier than the other cases, for this time the Fourier transform is a linear transformation of a finite dimensional vector space into itself. If it is 1-1, there should be an inverse given by the inverse of the matrix A.

THEOREM 11.5. *The finite Fourier transform on V is invertible, and in fact we can recover f from $\widehat{f}$ as follows:*

$$f(k) = \frac{1}{N}\sum_{j=0}^{N-1} \widehat{f}(j)e^{2\pi i \frac{jk}{N}}.$$

PROOF. Let B be the $N \times N$ matrix whose entries are given by

$$B_{l,m} = \frac{1}{N}e^{2\pi i \frac{lm}{N}}.$$

We claim that B is the inverse of the matrix A that represents the finite Fourier transform on V. Thus let us show that BA is the identity matrix. If $j = k$, then

$$\begin{aligned}
(AB)_{j,j} &= \sum_{m=0}^{N-1} A_{j,m}B_{m,j} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} e^{-2\pi i \frac{jm}{N}} e^{2\pi i \frac{jm}{N}} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} 1 \\
&= 1.
\end{aligned}$$

Therefore, down the diagonal of AB we have 1's.

Now, if $j \neq k$, we have

$$\begin{aligned}
(AB)_{j,k} &= \sum_{m=0}^{N-1} A_{j,m}B_{m,k} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} e^{-2\pi i \frac{jm}{N}} e^{2\pi i \frac{mk}{N}} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} e^{2\pi i \frac{m(k-j)}{N}} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} (e^{2\pi i \frac{k-j}{N}})^m \\
&= \frac{1}{N}\frac{1 - e^{2\pi i \frac{(k-j)N}{N}}}{1 - e^{2\pi i \frac{k-j}{N}}} \\
&= 0,
\end{aligned}$$

where we have used the formula for the sum of a geometric series at the end of this calculation. Hence, AB has 0's off the diagonal, and so AB is the identity matrix, and B is A^{-1}.

Therefore, $f = B(A(f)$, or

$$\begin{aligned} f(k) &= (B \times A \times f)_k \\ &= \frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi i \frac{jk}{N}} (A \times f)_j \\ &= \frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi i \frac{jk}{N}} \widehat{f}(j), \end{aligned}$$

as claimed.

EXERCISE 11.10. Let f be a function in V.

(a) Suppose N is even, say $N = 2M$. Show that

$$f(k) = a_0 + \sum_{j=1}^{M}(a_j \cos(2\pi \frac{jk}{N}) + b_j \sin(2\pi \frac{jk}{N})),$$

where

$$a_0 = \frac{1}{2M}(\widehat{f}(0) - \widehat{f}(M)),$$

and for $1 \leq j \leq M$,

$$a_j = \frac{1}{2M}(\widehat{f}(j) + \widehat{f}(N-j))$$

and

$$b_j = \frac{i}{2M}(\widehat{f}(j) - \widehat{f}(N-j))).$$

(b) Deduce that, from the point of view of frequency analysis, the highest frequencies present in a signal (function) f on a finite set of $N = 2M$ elements is $M - 1$.

Just as in the other cases we have studied, there is a notion of convolution on the space V.

DEFINITION. If f and g are functions in V, define a function $f * g$ on G by

$$f * g(k) = \sum_{l=0}^{N-1} f(l)g(k-l).$$

THEOREM 11.6. *Let f and g be elements of V. Then*

$$\widehat{f * g}(j) = \widehat{f}(j)\widehat{g}(j)$$

for all $0 \leq j \leq N-1$.

EXERCISE 11.11. Prove the preceding theorem

The Fast Fourier Transform

Computer designers tell us that all a computer really does is binary additions. Multiplications are just enormous addition calculations. Therefore, when estimating the cost of any computer computation, the designers count the number of multiplications and ignore any simple additions. If f is an element of V, how many multiplications must be carried out to compute the finite Fourier transform $\widehat{f}$, i.e., $\widehat{f}(j)$ for all $0 \leq j \leq N-1$? The formula for $\widehat{f}(j)$ is

$$\widehat{f}(j) = \sum_{k=0}^{N-1} f(k) e^{-2\pi i \frac{jk}{N}}.$$

So, for each j, we must compute N products. Hence, to compute the entire transform $\widehat{f}$ it appears that we must compute $N \times N = N^2$ products.

Remarkably, and marvelously, this number N^2 multiplications can be reduced to $N \log N$ multiplications if we choose N to be a power of 2. (Here, the logarithm is meant to be taken with respect to the base 2.) This reduction is enormous if N is large. For example if $N = 1024 = 2^{10}$, then N^2 is on the order of 10^6, a million, while $N \log N$ is on the order of 10^4.

THEOREM 11.7. *Suppose $N = 2^p$. Then the finite Fourier transform of any element f in V can be computed using at most $N \log N = p2^p$ multiplications.*

PROOF. We must compute all products of the form

$$f(k) e^{-2\pi i \frac{jk}{2^p}}$$

for both j and k ranging from 0 to $2^p - 1$. The trick is that some of these products are automatically equal to others, so that a single multiplication can suffice to compute two different products, thereby cutting our work roughly in half. Let's explore this more carefully. Note the following two things:

1. If k is an even integer, say $k = 2l$, then

$$e^{-2\pi i \frac{(j+2^{p-1})k}{2^p}} = e^{-2\pi i \frac{(jk+2^P l}{2^p}} = e^{-2\pi i \frac{jk}{2^p}},$$

so that

$$f(k) e^{-2\pi i \frac{jk}{2^p}} = f(k) e^{-2\pi i \frac{(j+2^{p-1})k}{2^p}},$$

and therefore, for even numbers k, we need only compute the product $f(k)e^{-2\pi i \frac{jk}{2^p}}$ for $0 \leq j \leq 2^{p-1} - 1$, i.e., for approximately half the j's. Each of the remaining products equals one of these.

Next, if k is odd, say $k = 2l + 1$, then

$$e^{-2\pi i \frac{(j+2^{p-1})k}{2^p}} = e^{-2\pi i \frac{jk}{2^p}} e^{-2\pi i \frac{2^{p-1} \diamond (2l+1)}{2^p}} = e^{-2\pi i \frac{jk}{2^p}} e^{-\pi i} = -e^{-2\pi i \frac{jk}{2^p}},$$

so that, if $k = 2l + 1$ is an odd number, again we need only compute the product $f(k)e^{-2\pi i \frac{jk}{2^p}}$ for $0 \leq j \leq 2^{p-1} - 1$. The remaining half of the products are just the negatives of these, and the computer guys tell us that it costs nothing to change the sign of a number.

Moreover, for even numbers $k = 2l$, the sums of products we must compute to find the finite Fourier transform of f are of the form

$$\sum_{l=0}^{2^{p-1}-1} f(2l)e^{-2\pi i \frac{2lj}{2^p}} = \sum_{l=0}^{2^{p-1}-1} f(2l)e^{-2\pi i \frac{jl}{2^{p-1}}},$$

so that if a function g is defined on the set $0, 1, \ldots, 2^{p-1} - 1$ by $g(l) = f(2l)$, then computing the part of $\widehat{f}$ having to do with even numbers $k = 2l$ is exactly the same as if we were computing the discrete Fourier transform on the set $0, 1, 2, \ldots, 2^{p-1}-1$ of g. (You can probably see a mathematical induction argument coming up here.)

In the case when k is odd, the sums of products we must compute look like

$$\begin{aligned}\sum_{l=0}^{2^{p-1}-1} f(2l+1)e^{-2\pi i \frac{j(2l+1)}{2^p}} &= \sum_{l=0}^{2^{p-1}-1} f(2l+1)e^{-2\pi i \frac{jl}{2^{p-1}}} e^{-2\pi i \frac{j}{2^p}} \\ &= \Big(\sum_{l=0}^{2^{p-1}-1} f(2l+1)e^{-2\pi i \frac{jl}{2^{p-1}}}\Big)e^{-2\pi i \frac{j}{2^p}},\end{aligned}$$

and, unlike the case when k was even, this is a bit more complicated than just computing the discrete Fourier transform of a function on the set $0, 1, \ldots, 2^{p-1} - 1$. We again could let h be the function on the set $0, 1, 2, \ldots, 2^{p-1} - 1$ defined by $h(l) = f(2l+1)$. What we are doing first is computing the discrete Fourier transform of the function h, and then we are multiplying each value $\widehat{h}(j)$ by the number $e^{-2\pi i \frac{j}{2^p}}$, which would be an additional 2^{p-1} multiplications.

Finally, let M_p be the number of multiplications required to compute the discrete Fourier transform when $N = 2^p$. The theorem asserts that $M_p \leq p2^p$. We have seen above that

$$M_p = M_{p-1} + M_{p-1} + 2^{p-1}.$$

So, by mathematical induction, if we assume that $M_{p-1} \leq 2^{p-1} \times (p - 1)$, then

$$\begin{aligned}M_p &\leq m_{p-1} + m_{p-1} + 2^{p-1} \\ &\leq (p-1)2^{p-1} + (p-1)2^{p-1} + 2^{p-1} \\ &= (2p - 2 + 1)2^{p-1} \\ &< 2p2^{p-1} \\ &= p2^p,\end{aligned}$$

as desired.