MATH 4330/5330, Fourier Analysis
Section 2, Separation of Variables

A classical method for trying to solve partial differential equations is to suppose, at the outset, that there exists a solution of a very special sort, and then try to find out what it is. In our case, we will assume that there exists a solution to the heat equation that is of the form $u(t,x) = g(t) \times h(x)$, that is, the solution has the variables t and x "separated." Supposing there is such a solution to the heat equation, can we determine what it is? Is it unique, for instance?

Plugging this solution $u(t,x) = g(t) \times h(x)$ into the two sides of the heat equation gives

$$\frac{\partial}{\partial t} u(t,x) = g'(t)h(x)$$

and

$$\frac{\partial^2}{\partial x^2} u(t,x) = g(t)h''(x).$$

Setting these derivatives equal to each other, and doing some algebra, we get

$$\frac{g'(t)}{g(t)} = \frac{h''(x)}{h(x)}.$$

The left side of this equation is a function only of the variable t, while the right side is a function only of the variable x. How can a function of t equal a function of x? Only if these functions are constant functions. Therefore, we deduce that there must exist a constant c such that

$$\frac{g'(t)}{g(t)} = c$$

and

$$\frac{h''(x)}{h(x)} = c$$

for all $t > 0$ and all x.

EXERCISE 2.1. Make sure you understand all the calculations above, and in particular make sure you follow the argument that says a function of t equals a function of x only if the functions are constant.

Now, instead of solving the heat equation (PDE) directly, we have reduced our problem to solving two ordinary differential equations:

$$g'(t) = cg(t) \text{ and } h''(x) = ch(x).$$

Are there solutions to these ODEs, are they unique? Existence and uniqueness of solutions to differential equations is a major problem in mathematics.

EXERCISE 2.2. Find two different solutions to the initial value problem:

$$f'(x) = \sqrt{f(x)}\ f(0) = 0.$$

The preceding exercise should scare us a bit. There may not be unique solutions to the heat equation, because there may not be unique solutions to our two ODEs.

Let us attack the first ODE above.

We certainly do know one solution to the equation $g'(t) = cg(t)$. Indeed, if $g(t) = e^{ct}$, then $g'(t)$ does equal $cg(t)$. Is this the only solution to this ODE? No. The function ae^{ct} also satisfies the differential equation. But, are there any other solutions besides these multiples of e^{ct}?

THEOREM 2.1. *Any function f for which $f'(t) = cf(t)$ is of the form $f(t) = ae^{ct}$.*

PROOF. Small trick. Suppose f is such a function, and define another function ϕ by

$$\phi(t) = \frac{f(t)}{e^{ct}}.$$

Then,

$$\begin{aligned}\phi'(t) &= \frac{e^{ct}f'(t) - f(t)ce^{ct}}{e^{2ct}} \\ &= \frac{e^{ct}cf(t) - f(t)ce^{ct}}{e^{2ct}} \\ &= 0.\end{aligned}$$

Therefore $\phi(t)$ must be a constant function, $\phi(t) = a$. So,

$$f(t) = \phi(t)e^{ct} = ae^{ct}$$

as claimed.

EXERCISE 2.3. How do you know that a function whose derivative is 0 must be a constant? This was important in the proof.

Now, let's attack the second ODE

$$h''(x) = ch(x).$$

THEOREM 2.2. *Let k be a square root of the number c. If f is a solution of the equation $f''(x) = cf(x)$, then there exist constants α and β such that*

$$f(x) = \alpha e^{kx} + \beta e^{-kx}.$$

PROOF. Trick 1. Let ϕ be the function defined by

$$\phi(x) = f'(x) + kf(x).$$

Then,

$$\phi'(x) = f''(x) + kf'(x) = cf(x) + kf'(x) = k^2f(x) + kf'(x) = k\phi(x).$$

Hence, using Theorem 2.1,

$$\phi(x) = ae^{kx}$$

for some constant a. That is,

$$f'(x) + kf(x) = ae^{kx}.$$

Trick 2. Multiply both sides of the last equation by e^{kx} to get

$$e^{kx}f'(x) + f(x)ke^{kx} = ae^{2kx}.$$

Observe that the left side of this equation looks like the derivative of a product, in fact the derivative of $e^{kx}f(x)$.

Observe that the right side of the equation is the derivative of the function $(a/2k)e^{2kx}$. Hence, because the derivatives of two functions are equal, it must be that the functions differ by a constant. So,

$$e^{kx}f(x) = \frac{a}{2k}e^{2kx} + b$$

for some constant b.

Finally, dividing, we get that

$$\begin{aligned} f(x) &= \frac{\frac{a}{2k}e^{2kx} + b}{e^{kx}} \\ &= \frac{a}{2k}e^{kx} + be^{-kx}, \end{aligned}$$

and the proof is complete by setting $\alpha = a/2k$ and $\beta = b$.

EXERCISE 2.4. Make sure you follow this argument. By the way, how do we know that if two functions have equal derivatives they must differ by a constant?

EXERCISE 2.5. Try out this separation of variables method to solve the "wave equation," which is

$$\frac{\partial^2}{\partial t^2}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x).$$

We summarize our findings as follows:

THEOREM 2.3. *Suppose $u(t,x)$ is a solution of the heat equation, that $u(t,x) = g(t) \times h(x)$, and that $u(0,x) = f(x)$. Then there exist three constants k, c_1, and c_2 such that*

$$f(x) = c_1e^{kx} + c_2e^{-kx}$$

and

$$u(t,x) = e^{k^2t}f(x) = e^{k^2t}(c_1e^{kx} + c_2e^{-kx}.$$

EXERCISE 2.6. Prove this theorem. Be careful about what the constants are.

EXERCISE 2.7. Are any of the solutions described in the preceding theorem satisfactory for the initial value problem we are considering for the heat equation on the line? That is, does such a $u(t,x)$ satisfy the finite energy condition $\int_{-\infty}^{\infty} |u(t,x)|^2\,dx < \infty$?

Troubling observation: In reality, the function $u(t,x)$ cannot be continually increasing as t goes to infinity. Why not? This would imply that the function e^{k^2t} can't be increasing, and this means the k^2 must be negative. But that means that k itself must be pure imaginary, say $k = i\omega$. But that means the the initial condition $f(x)$ looks like

$$f(x) = c_1e^{i\omega x} + c_2e^{-i\omega x}.$$

What's with these imaginary exponents?