MATH4330/5330, Fourier Analysis
Section 3
COMPLEX NUMBERS, TRIGONOMETRY, AND EULER'S THEOREM

DEFINITION. Let i denote a "number" satisfying $I^2 = -1$. By the complex numbers we mean the set $\mathbb{C}$ of all objects of the form $a+bi$, where a and b are real numbers. Two complex numbers $a+bi$ and $a'+b'i$ are *equal* if and only if $a=a'$ and $b=b'$.

We add and multiply complex numbers according to the following formulas:

(1) $(a+bi)+(c+di) = a+c+(b+d)i$, and
(2) $(a+bi)\times(c+di) = ac+bic+adi+bidi = ac-bd+(ad+bc)i.$

We write 0 for the complex number $0+0i$ and 1 for the complex number $1+0i$. Complex numbers of the form $a+0i$ are called *real* numbers, and those of the form $0+bi$ *purely imaginary* numbers. If $z=a+bi$ is a complex number, we say that the *real part* of z is a, and the *imaginary part* of z is b. Denote the real part of z by the symbol $\Re(z)$ and the imaginary part by the symbol $\Im(z)$.

If $z=a+bi$ is a complex number, define the *conjugate* of z, which we denote by $\overline{z}$, by $\overline{z}=a-bi$.

Define the *absolute value* of the complex number $z=a+bi$ by $|z|=\sqrt{a^2+b^2}$.

Note that (prove that) $0+z=z$ for all complex numbers z, and $1\times z=z$ for all complex numbers.

THEOREM 3.1. *The set $\mathbb{C}$, equipped with the operations of addition and multiplication defined above, is a field. That is, both addition and multiplication are commutative and associative, multiplication is distributive over addition, and every nonzero element in $\mathbb{C}$ has a multiplicative inverse; i.e., if $z\neq 0$ then there exists a w such that $zw=1$.*

EXERCISE 3.1. (a) If $z=a+bi$ is not the 0 element in $\mathbb{C}$, show that the multiplicative inverse of z is given by

$$\frac{1}{z}=\frac{\overline{z}}{|z|^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i.$$

(b) Prove that $z=(-1/2)+(\sqrt{3}/2)i$ is a cube root of 1; i.e., that $z^3=1$.

(c) Show that $\mathbb{C}$ can be identified with the Cartesian plane $\mathbb{R}^2$ by corresponding the complex number $x+yi$ with the ordered pair (x,y). Show that under this identification the real numbers are along the x-axis and the purely imaginary complex numbers are along the y-axis.

EXERCISE 3.2. (a) Show that $\overline{z}\times z$ is always ≥ 0, and in fact equals $|z|^2$.

(b) Prove that $\overline{z+w}=\overline{z}+\overline{w}$, and $\overline{zw}=\overline{z}\ ,\overline{w}$. (Just do the algebra.)

(c) Show that $|\Re(z)|\leq |z|$ and $|\Im(z)|\leq |z|$.

(d) Show that $\Re(z)=\frac{z+\overline{z}}{2}$ and $\Im(z)=\frac{z-\overline{z}}{2}$. Conclude that $z\overline{w}+\overline{z}w=2\Re(z\overline{w})$.

(e) Show that the absolute value satisfies the triangle inequality:

$$|z+w|\leq |z|+|w|.$$

(f) Using the fact that $z=z-w+w$, derive the backwards triangle inequality:

$$|z-w|\geq ||z|-|w||.$$

(g) Let $\mathbb{T}$ be the set of complex numbers having absolute value equal to 1. Prove that $\mathbb{T}$ coincides with the unit circle in the plane, and show also that $\mathbb{T}$ forms a group under the operation of multiplication. That is, show that if both z and w belong to $\mathbb{T}$, then so do zw and $1/z$.

TRIGONOMETRY

For each positive real number t, think of traveling a distance t counterclockwise around the unit circle $\mathbb{T}$, starting at the point $(1, 0)$. Obviously, to each such t, there corresponds a point $(x(t), y(t))$ representing the point on the unit circle we have reached after traveling this distance of t. We call the number $x(t)$ the *cosine* of t and the number $y(t)$ the *sine* of t. By construction, the point $(\cos t, \sin t)$lies on the unit circle; i.e., $\cos^2 t + \sin^2 t = 1$. If t is a negative real number, we make the same kind of construction, except we travel in a clockwise direction around the unit circle.

The "functions" cos and sin of the real variable t are called the *trigonometric functions.*

We have identified the Euclidean plane with the complex plane, i.e., the set of all complex numbers. So, by the above discussion, for every real number t, there exists a point $\cos t + i \sin t$ in the complex plane. This is nothing more than realizing that there is a perfect 1-1 correspondence between the set of all ordered pairs (a, b) and the set of all complex numbers $a + ib$.

DEFINITION. For each real number t, we use the shorthand notation e^{it} for the complex number $\cos t + i \sin t$.

This definition in other contexts is called Euler's Theorem:

$$e^{it} = \cos t + i \sin t.$$

We will justify the use of this exponential notation a bit later.

Recall the following properties of the trigonometric functions as well as the accompanying trigonometric identities:

(1) $\cos 0 = 1$, and $\sin 0 = 0$.
(2) $\cos(\pi/2) = 0$, and $\sin(\pi/2) = 1$.
(3) $\cos \pi) = -1$, and $\sin \pi = 0$.
(4) $\cos(t + 2\pi) = \cos t$, and $\sin(t + 2\pi) = \sin t$.
(5) $\cos(2n\pi) = 1$ and $\sin(2n\pi) = 0$ for all integers n.
(6) $\cos(-t) = \cos t$.
(7) $\sin(-t) = -\sin t$.

EXERCISE 3.3. Recall from basic trigonometry the following "addition formulas" for the trig functions:

(1) $\sin(x + y) = \sin x \cos y + \sin y \cos x$, and
(2) $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

(a) Derive the double angle formulas

$$\sin(2x) = 2 \sin(x) \cos(x) \text{ and } \cos(2x) = \cos^2(x) - \sin^2(x).$$

(b) Derive the half angle formulas: $\sin(x/2) = \sqrt{1 - \cos x}/\sqrt{2}$, and $\cos(x/2) = \sqrt{1 + \cos x}/\sqrt{2}$.

Here is the justification for the exponential notation e^{it} we are using.

THEOREM 3.2. *For any two real numbers t and s, we have*

$$e^{i(t+s)} = e^{it}e^{is}.$$

That is, the function e^{it} satisfies the law of exponents.

EXERCISE 3.4. (a) Prove the preceding theorem. Notice that it boils down to showing that

$$\cos(t+s) + i\sin(t+s) = (\cos t + i\sin t)(\cos s + i\sin s),$$

which can be done by doing the algebra and then equating real parts on both sides and imaginary parts on both sides.

(b) Show that $e^{2\pi in} = 1$ for every integer n.

EXERCISE 3.5. Derive the following relations among the trigonometric functions and the function e^{it}.

$$\cos t = \frac{e^{it} + e^{-it}}{2}$$

and

$$\sin t = \frac{e^{it} - e^{-it}}{2i}.$$

EXERCISE 3.6. Suppose $u(t, x)$ is a solution of the heat equation, that $u(t, x) = g(t) \times h(x)$, and that $u(0, x) = f(x)$. Show there exist three constants ω, a, and b such that

$$f(x) = a\cos(\omega x) + b\sin(\omega x)$$

and

$$u(t, x) = e^{-\omega^2 t}(a\cos(\omega x) + b\sin(\omega x)).$$

Compare with Theorem 2.3.

Derivatives and Integrals of Trig Functions

Recall that the derivative of cos is $-\sin$, and the derivative of sin is cos. And, the antiderivative of cos is sin and the antiderivative of sin is $-\cos$.

EXERCISE 3.7. (a) Show that the derivative of e^{it} is ie^{it}.

(b) Show that the antiderivative of e^{it} is e^{it}/i.

(c) Find an antiderivative of $\cos(at)$, $\sin(at)$, and e^{iat}, where a is a real number.

(d) Evaluate

$$\int_a^b e^{ict}\,dt.$$

(e) Evaluate

$$\int_0^1 e^{2\pi int}\,dt$$

where n is any integer.

(f) Evaluate

$$\int_s^{s+1} e^{2\pi int}\,dt,$$

where n is an integer and s is any real number.