MATH 4330/5330, Fourier Analysis
Section 6, Proof of Fourier's Theorem for Pointwise Convergence

First, some comments about integrating periodic functions. If g is a periodic function, $g(x+1) = g(x)$ for all real x, then

(1) $\int_0^1 g(t)\,dt = \int_c^{c+1} g(t)\,dt$ for all real numbers c.
(2) $\int_0^1 g(t+a)\,dt = \int_0^1 g(t)\,dt$. for all real numbers a.
(3) $\int_0^1 g(-t)\,dt = \int_0^1 g(t)\,dt$.

EXERCISE 6.1. Verify that the above integration formulas are correct.

What can Fourier really prove? It's not quite as wonderful as it may have sounded. Here's a first step:

THEOREM 6.1. *Let f be a square-integrable, periodic function, and suppose that f is differentiable at a point x. Then*

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x},$$

where of course

$$\widehat{f}(n) = c_n = \int_0^1 f(t) e^{-2\pi i n t}\,dt.$$

REMARK. So, Fourier's Theorem may not hold for all functions and all points. For instance, we are assuming here that f is in fact differentiable at the point in question, and clearly not every function is necessarily differentiable at every point.

PROOF. We know from Section 5 that the partial sum

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$$

of the Fourier series for f is given in an integral form by

$$S_N(x) = \int_0^1 f(t) D_N(x-t)\,dt,$$

where D_N is the Dirichlet kernel. Now, using the integration formulas of Exercise 6.1, we see that

$$S_N(x) = \int_0^1 f(x+t) D_N(t)\,dt$$

and also

$$S_N(x) = \int_0^1 f(x-t) D_N(t)\,dt.$$

Hence,

$$S_N(x) = \int_0^1 \frac{f(x+t) + f(x-t)}{2} D_N(t)\,dt.$$

And then, since the integrand here is an even function, we get that

$$S_N(x) = 2\int_0^{\frac{1}{2}} \frac{f(x+t)+f(x-t)}{2} D_N(t)\,dt = \int_0^{\frac{1}{2}} (f(x+t)+f(x-t))D_N(t)\,dt.$$

Now,

$$f(x) = f(x) \times \int_0^1 D_N(t)\,dt = 2\int_0^{\frac{1}{2}} f(x)D_N(t)\,dt.$$

To show that $f(x)$ is the sum of its Fourier series, we must show that $f(x)$ is the limit of the partial sums of the Fourier series. That means we must show that $\lim_{N\to\infty}(f(x) - S_N(x)) = 0$. We have

$$\begin{aligned} f(x) - S_N(x) &= \int_0^{\frac{1}{2}} [2f(x) - f(x+t) - f(x-t)]D_N(t)\,dt \\ &= \int_0^{\frac{1}{2}} \frac{2f(x) - f(x+t) - f(x-t)}{\sin(\pi t)} \sin(2\pi(N+\frac{1}{2})t)\,dt, \end{aligned}$$

so, by the Riemann-Lebesgue Lemma, $\lim_{N\to\infty} f(x) - S_N(x)$ will be 0 if the function g defined by

$$g(t) = \frac{2f(x) - f(x+t) - f(x-t)}{\sin(\pi t)}$$

is integrable.

Now, because f is assumed to be differentiable at x, there exists a $\delta > 0$ such that the differential quotient $(f(x) - f(x+t))/t$ is close to $f'(x)$, implying that this quotient is bounded by a number M, for all $0 < |t| < \delta$. We will prove that the function $g(t)$ is integrable by showing that it is integrable on the interval $[0, \delta]$ and also integrable on the interval $[\delta, 1/2]$.

On the interval $[\delta, 1/2]$, the three terms in the numerator of the function g, $f(x), f(x+t)$, and $f(x-t)$ are all integrable functions of t, so the sum is also integrable. And, on that interval $[\delta, 1/2]$, the denominator $\sin(\pi t)$ is bounded away from 0, so that the reciprocal $1/\sin(\pi t)$ is bounded. Hence, the entire function g, on the interval $[\delta, 1/2]$ must be integrable, being the product of an integrable function and a bounded function.

Now, on the interval $[0, \delta]$, write

$$\begin{aligned} g(t) &= \frac{2f(x) - f(x+t) - f(x-t)}{t} \times \frac{t}{\sin(\pi t)} \\ &= (\frac{f(x) - f(x+t)}{t} + \frac{f(x) - f(x-t)}{t})\frac{t}{\sin(\pi t)}. \end{aligned}$$

Recall that the function $t/\sin(\pi t)$ is bounded on the interval $[0, \delta]$, (Why is that?), and since both the differential quotients are bounded by M on that interval, the function g must be bounded on the interval $[0, \delta)$ and hence integrable.

This completes the proof.

EXERCISE 6.2. (a) Let f be the periodic function defined by $f(x) = x$ on $[0, 1)$. Compute the Fourier coefficients $c_n = \widehat{f}(n)$ for f, and then, for $0 < x < 1$, derive the formula:

$$x = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin(2\pi n x).$$

(Consult your computations in Exercise 4.8.) Check this equation out for $x = 1/2$ and for $x = 1/4$. Derive the formula

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

(b) Now let f be the periodic function defined by $f(x) = x^2$. Compute the Fourier coefficients for f, and derive a formula analogous to the one in part (a).

(c) Let h be the periodic function defined by $h(x) = -1$ for $0 \leq x < 1/2$, and $h(x) = 1$ for $1/2 \leq x < 1$. Compute the fourier coefficients for h, and examine the corresponding formulas. What happens to this series for the discontinuity points $x = 1/2$ or $x = 0$?

REMARK. Fourier's Theorem certainly would imply that the Fourier transform T has an inverse. The theorem says that we can recover the function f from the transform $\widehat{f}$. Theorem 6.1 establishes Fourier's Theorem for certain functions, but we don't yet really know that the Fourier transform has an inverse. However, we can use Theorem 6.1 to prove this.

THEOREM 6.2. *The Fourier transform T is 1-1 on $L^2([0,1))$. That is, it has an inverse.*

PROOF. Since T is a linear transformation from one vector space into another, we can prove it is 1-1 by showing that its kernel N is trivial, i.e., the only element of N is the 0 function. Thus, suppose f is a function that satisfies $T(f) = 0$, and let us show that f must be the zero function. We know then that $\widehat{f}(n) = 0$ for every n.

Write F for the function defined by

$$F(x) = \int_0^x f(t)\,dt.$$

Then, it is known that F is continuous on $[0,1]$ and differentiable (almost everywhere), and $F'(x) = f(x)$. Also, note that $F(0) = 0$ and

$$F(1) = \int_0^1 f(t)\,dt = \widehat{f}(0) = 0,$$

so that F can be extended to a continuous periodic function.

Now Theorem 6.1 applies to the function F, so that for almost every x

$$F(x) = \sum_{n=-\infty}^{\infty} \widehat{F}(n) e^{2\pi i n x}.$$

But, by the exercise below, we know that $\widehat{F}(n) = 0$ for all $n \neq 0$. So, $F(x) = \widehat{F}(0)$ for all x, i.e., F is a constant function. But then, f must be the 0 function, since it is the derivative of the constant function F. This completes the proof.

EXERCISE 6.3. Let f and F be as in the preceding theorem. Use integration by parts to show that, for all nonzero n,

$$\widehat{F}(n) = \frac{1}{2\pi i n} \widehat{f}(n) = 0.$$

Convergence of Fourier Series at Jump Discontinuities

Theorem 6.1 asserts that the Fourier series for a function f converges at each point x where f is differentiable. Something analogous occurs if f is differentiable near a point x, but has a jump discontinuity exactly at x.

THEOREM 6.3. *Suppose f is a square-integrable, periodic function, and suppose that f satisfies the following additional assumptions.*

(1) *There exists a point x such that both a left and a right limit as t approaches x exist:*

$$r = \lim_{t \to x+} f(t) \text{ and } l = \lim_{t \to x-} f(t).$$

(2) *f has both a left and a right derivate at the point x; i.e.,*

$$\lim_{h \to 0+} \frac{f(x+h) - r}{h} \text{ exists,}$$

and

$$\lim_{h \to 0-} \frac{f(x+h) - l}{h} \text{ exists.}$$

Then, the Fourier series for f at x converges to the average of the right and left limits at x :

$$\lim_{N \to \infty} S_N(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x} = \frac{r+l}{2}.$$

PROOF. Just as in the proof of Theorem 6.1, we have that

$$S_N(x) = \int_0^{\frac{1}{2}} (f(x+t) + f(x-t)) D_N(t)\, dt.$$

Now,

$$r = r \times \int_0^1 D_N(t)\, dt = 2 \int_0^{\frac{1}{2}} r D_N(t)\, dt$$

and

$$l = l \times \int_0^1 D_N(t)\, dt = 2 \int_0^{\frac{1}{2}} l D_N(t)\, dt.$$

So,

$$\begin{aligned} \frac{l+r}{2} - S_N(x) &= \int_0^{\frac{1}{2}} [2\frac{l+r}{2} - f(x+t) - f(x-t)] D_N(t)\, dt \\ &= \int_0^{\frac{1}{2}} \frac{l+r - f(x+t) - f(x-t)}{\sin(\pi t)} \sin(2\pi(N+\frac{1}{2})t)\, dt, \end{aligned}$$

so, by the Riemann-Lebesgue Lemma, $(l+r)/2$ will equal the limit of $S_N(x)$ if the function g defined by

$$g(t) = \frac{l + r - f(x+t) - f(x-t)}{\sin(\pi t)}$$

is integrable on the interval $[0, \frac{1}{2}]$.

Now, from the assumptions about the existence of left and right derivates of f at x, we know that there must exist a $\delta > 0$ such that the two quotients $(r - f(x+t))/t$ and $(l - f(x-t)/t$ are bounded by a number M for all $0 < |t| < \delta$. We will prove that the function $g(t)$ is integrable by showing that it is integrable on the interval $[0, \delta]$ and also integrable on the interval $[\delta, 1/2]$.

On the interval $[\delta, 1/2]$, the four terms in the numerator of g, $l, r, f(x+t)$, and $f(x-t)$ are all integrable functions of t, so the sum is also integrable. And, on that interval $[\delta, 1/2]$, the denominator $\sin(\pi t)$ is bounded away from 0, so that the reciprocal $1/\sin(\pi t)$ is bounded. Hence, the entire function g, on the interval $[\delta, 1/2]$ must be integrable, being the product of an integrable function and a bounded function.

Now, on the interval $[0, \delta]$, write

$$\begin{aligned} g(t) &= \frac{l + r - f(x+t) - f(x-t)}{t} \times \frac{t}{\sin(\pi t)} \\ &= (\frac{r - f(x+t)}{t} + \frac{l - f(x-t)}{t}) \frac{t}{\sin(\pi t)}. \end{aligned}$$

So, since the function $t/\sin(\pi t)$ is bounded on the interval $[0, \delta]$, and since both of the quotients in the above expression are bounded by M on that interval, the function g must be bounded on that interval and hence integrable.

This completes the proof.

The Gibb's Phenomenon

We begin our discussion of the Gibb's Phenomenon with a special example.

EXERCISE 6.4. Let f be the periodic function defined on $[0, 1)$ by $f(t) = 1/2 - t$.

(a) Use the results from Exercise 6.2 to write down a formula for the partial sums S_N of the Fourier series for f. You should get

$$S_N(x) = \sum_{n=1}^{N} \frac{\sin(2\pi n x)}{n\pi}.$$

(b) Justify the following computations:

$$\begin{aligned} \sum_{n=1}^{N} \frac{\sin(2\pi n x)}{\pi n} &= \sum_{n=1}^{N} 2 \int_0^x \cos(2\pi n t)\, dt \\ &= \int_0^x 2 \sum_{n=1}^{N} \cos(2\pi n t)\, dt \\ &= \int_0^x (D_N(t) - 1)\, dt \\ &= \int_0^x D_N(t)\, dt - x. \end{aligned}$$

(c) Use parts (a) and (b) to conclude that

$$S_N(x) = \int_0^x D_N(t)\, dt - x,$$

where $S_N(x)$ is the Nth partial sum of the Fourier series for the function $f(x) = 1/2 - x$.

(d) Verify that both the hypotheses and the conclusion of Theorem 6.3 hold for this function. That is, show that $\lim_{N\to\infty} \int_0^x D_N(t)\,dt - x = \frac{1}{2} - x$ for every $x \in [0,1)$.

REMARK. The Gibb's Phenomenon concerns the manner in which the partial sums $S_N(x)$ of a Fourier series converge as the point x varies. For instance, how does the rate of convergence depend on the point x. In the function of the preceding exercise, $f(x) = 1/2 - x$, we see that the function values $f(x)$ are always between $-1/2$ and $+1/2$. One would expect that the values of the partial sums $S_N(x)$ would tend, in the limit at least, also to be between $-1/2$ and $+1/2$. The fact that this is not uniformly the case near a jump discontinuity point is an example of what's called the *Gibb's Phenomenon.* The next theorem demonstrates this explicitly for the function $1/2 - x$ and the discontinuity point $x = 0$.

First, some preliminary calculations and estimates:

EXERCISE 6.5. (a) Estimate the number $\int_0^\pi \sin(t)/t\,dt$. Show that it is approximately 1.8.

(b) Define the *Gibb's constant* G by

$$G = \frac{2}{\pi}\int_0^\pi \frac{\sin(t)}{t}\,dt.$$

Show that G is approximately 1.18.

THEOREM 6.4. *Let f be the periodic function of Exercise 6.5. Then*

$$\lim_{N\to\infty} S_N(\frac{1}{2N+1}) = G\frac{1}{2} = \frac{G}{2} > \frac{1}{2} \times 1.18,$$

and

$$\lim_{N\to\infty} S_N(1 - \frac{1}{2N+1}) = -G\frac{1}{2} = -\frac{G}{2} < -\frac{1}{2} \times 1.18,$$

where G is the Gibb's constant.

REMARK. What this shows is that no matter how large N gets, the values of the partial sums of this Fourier series are not uniformly squeezing down to the interval $[-1/2, 1/2)$, even though the function $1/2 - x$ itself is confined to that interval.

PROOF. We will make use of the function

$$h(t) = \frac{1}{\sin(\pi t)} - \frac{1}{\pi t}$$

from Section 5. Using the results from Exercises 6.4 and 6.5, we have

$$\begin{aligned}
S_N(\frac{1}{2N+1}) &= \int_0^{\frac{1}{2N+1}} D_N(t)\,dt - \frac{1}{2N+1} \\
&= \int_0^{\frac{1}{2N+1}} \frac{\sin(2\pi(N+\frac{1}{2})t)}{\sin(\pi t)}\,dt - \frac{1}{2N+1} \\
&= \int_0^{\frac{1}{2N+1}} \frac{\sin(2\pi(N+\frac{1}{2})t)}{\pi t}\,dt \\
&\quad + \int_0^{\frac{1}{2N+1}} \sin(2\pi(N+\frac{1}{2})t)h(t)\,dt - \frac{1}{2N+1} \\
&= \int_0^{\pi} \frac{\sin(t)}{\pi t}\,dt + \int_0^{\frac{1}{2N+1}} \sin(2\pi(N+\frac{1}{2})t)h(t)\,dt - \frac{1}{2N+1}.
\end{aligned}$$

Now the first term is exactly equal to $G/2$, and the limits of the second and third terms are both 0. (Why?) This proves the first assertion of the theorem.

The second assertion is easy now, because the partial sums S_N are periodic and also are odd functions.

$$S_N(1-\frac{1}{2N+1}) = S_N(-\frac{1}{2N+1}) = -S_N(\frac{1}{2N+1}),$$

so that

$$\lim_{N\to\infty} S_N(1-\frac{1}{2N+1}) = -\lim_{N\to\infty} S_N(\frac{1}{2N+1}) = -\frac{G}{2}.$$

EXERCISE 6.6. Sketch the graph of the function $f(x) = 1/2 - x$, and then sketch the graphs of the partial sums $S_N(x)$. Convince yourself that, as x approaches the jump discontinuity, the graphs of the partial sums "overshoot" or "undershoot" the proper values.

Here is the general statement of the Gibb's Phenomenon.

THEOREM 6.5. *Suppose f is a square-integrable, periodic function, and suppose that f satisfies the following additional assumptions.*

(1) *There exists a point x such that both a left and a right limit as t approaches x exist:*

$$r = \lim_{t\to x+} f(t) \text{ and } l = \lim_{t\to x-} f(t).$$

(2) *f has both a left and a right derivate at the point x; i.e.,*

$$\lim_{h\to 0+} \frac{f(x+h)-r}{h} \text{ exists,}$$

and

$$\lim_{h\to 0-} \frac{f(x+h)-l}{h} \text{ exists.}$$

Then, the Fourier series for f at x converges to the average of the right and left limits at x :

$$\lim_{N\to\infty} S_N(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx} = \frac{r+l}{2}.$$

Moreover, there exist two sequences $\{x_N\}$ and $\{y_N\}$ converging to x for which

$$\lim_{N\to\infty} S_N(x_N) = \frac{l+r}{2} + \frac{G}{2}|l-r|$$

and

$$\lim_{N\to\infty} S_N(y_N) = \frac{l+r}{2} - \frac{G}{2}|l-r|.$$