MATH 4330/5330, Fourier Analysis
Section 8, The Fourier Transform on the Line

What makes the Fourier transform on the circle work? What is it about the functions $\phi_n(x) = e^{2\pi inx}$ that underlies their importance for analyzing other functions in terms of them? Here's one very special property that these exponential functions have.

THEOREM 8.1. *If f is a complex-valued, differentiable, periodic function for which*

(1) *The values $f(x)$ of x belong to the unit circle $\mathbb{T}$; i.e., $|f(x)| = 1$ for all x, and*

(2) *f satisfies the law of exponents, i.e., $f(x+y) = f(x)f(y)$ for all x and y,*

Then there must exist a unique integer n_0 such that $f(x) = e^{2\pi in_0x}$ for all x. That is, the functions $e^{2\pi inx}$ are precisely the (differentiable) homomorphisms of the group of real numbers $\mathbb{R}$ (under addition mod 1) into the circle group $\mathbb{T}$ (under multiplication).

PROOF. Suppose f is a function that satisfies (1) and (2). Since the values of f belong to the unit circle, f is bounded and hence square-integrable. Writing f in its Fourier series, we have

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx}$$

for all x. We then have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} e^{2\pi iny}\widehat{f}(n)e^{2\pi inx} &= \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi in(x+y)} \\ &= f(x+y) \\ &= f(x)f(y) \\ &= f(y)f(x) \\ &= f(y)\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx} \\ &= \sum_{n=-\infty}^{\infty} f(y)\widehat{f}(n)e^{2\pi inx},\end{aligned}$$

which implies that

$$e^{2\pi iny}\widehat{f}(n) = f(y)\widehat{f}(n)$$

for all n. Hence, if n is any integer for which $\widehat{f}(n) \neq 0$, we must have $f(y) = e^{2\pi iny}$ for every y. Because f is not the 0 function, there must be at least one integer n for which $\widehat{f}(n) \neq 0$. Obviously the equation above for $f(y)$ can not hold for more than one integer n, and we write n_0 for the unique integer for which $f(y) = e^{2\pi in_0y}$ for all y. The proof of the theorem is now complete.

Let us now consider the real line $\mathbb{R}$. What are the analogous homomorphisms f on $\mathbb{R}$ into $\mathbb{T}$? That is, what are the differentiable homomorphisms of the group $\mathbb{R}$ under addition into the group $\mathbb{T}$ under multiplication? Here's the answer.

THEOREM 8.2. *Let f be a (complex-valued) differentiable function on $\mathbb{R}$ into $\mathbb{T}$ for which $f(x+y) = f(x)f(y)$ for all x and y in $\mathbb{R}$. Then there exists a real number ω (not necessarily an integer) such that*

$$f(x) = e^{2\pi i\omega x}$$

for all real numbers x.

PROOF. first of all, if $f(x) = 1$ for every x, then we set $\omega = 0$, and obviously

$$f(x) = 1 = e^{2\pi i\omega x}$$

as desired.

If $f(x)$ is not identically 1, let p be a positive number for which $f(p) = 1$. (How do we know there is such a positive number? See the exercise below.) Define a function h by $h(x) = f(px)$. Then, h is a differentiable, periodic, square-integrable function that satisfies $h(x+y) = h(x)h(y)$ for all x and y. (Check these claims out.) So, by Theorem 8.1, let n_0 be the unique integer such that $h(x) = e^{2\pi i n_0 x}$. But then

$$f(x) = f(p\frac{1}{p}x) = h(\frac{1}{p}x) = e^{2\pi i n_0 \frac{1}{p}x} = e^{2\pi i\omega x},$$

where $\omega = n_0/p$.

EXERCISE 8.1. Suppose f is as in the preceding theorem, and assume that $f(x)$ is not identically 1. Use the outline below to show that there must exist a positive number p such that $f(p) = 1$.

(a) First, show that there exists a positive number y such that $f(y) \neq 1$.

(b) Next, define a function g by

$$g(x) = \frac{f(x) - \overline{f(x)}}{2i}.$$

If we write the element $f(x)$ of the unit circle as $f(x) = e^{i\theta(x)} = \cos\theta(x) + i\sin\theta(x)$ for some angle $\theta(x)$ between 0 and 2π, then note that $g(x) = \sin\theta(x)$. Show that, if $g(y) = 0$, then $\theta(y)$ is either 0 or π, and therefore $f(y) = \pm 1$, and so $f(2y) = 1$. So, in this case we may take p to be $2y$.

(c) Next, suppose that $g(y) \neq 0$, i.e., that $\sin\theta(y) \neq 0$. Show that there exists an integer n such that $\sin\theta(y)$ and $\sin(n\theta(y))$ are of opposite signs, one positive and the other negative. In other words, $g(y)$ and $g(ny)$ are of opposite signs. Use the Intermediate Value theorem to conclude that there exists a number z between y and ny for which $g(z) = 0$. Conclude then that $f(z) = \pm 1$, and therefore that $f(2z) = 1$, so that we may take p to be $2z$ in this case.

EXERCISE 8.2. If ω is any real number, show that the function $\phi_\omega(x) = e^{2\pi i\omega x}$ satisfies the two hypotheses of Theorem 8.2. Conclude that there is a 1-1 correspondence between the set of all real numbers ω and the set of all differentiable homomorphisms of the group $\mathbb{R}$ of real numbers into the group $\mathbb{T}$ of complex numbers of absolute value 1.

By analogy with what we did for functions on the circle, together with Theorem 8.2, it is tempting to define the Fourier transform of a function f on the real line to be another function $\widehat{f}$, also on the real line, and which is given by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}\,dt = \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}\,dt.$$

One problem immediately shows up here that did not appear before. What do we mean by the integral over the entire real line? Of course we no doubt mean an improper Riemann integral. That is,

$$\widehat{f}(\omega) = \lim_{B\to\infty} \int_{-B}^{B} f(t)e^{-2\pi i\omega t}\,dt.$$

To make good sense of this definition of the Fourier transform, we would be forced to assume that all the improper Riemann integrals, a different one for each number ω, exist. Not only is this a funny assumption to make, it would be horrible to check. We solve this particular dilemma by defining the Fourier transform only for *absolutely integrable* functions. These are functions f for which $\lim_{B\to\infty} \int_{-B}^{B} |f(t)|\,dt$ exists. Because $|e^{-2\pi i\omega t}| = 1$ for every t and ω, it follows that if f is absolutely integrable, then each of the integrals in our definition of the Fourier transform will exist.

Also, in the case of the circle, we worked for the most part with square-integrable functions instead of integrable ones. The difference between these concepts for functions on the circle is not so great, but on the whole real line these notions are quite distinct, as the following exercise shows.

EXERCISE 8.3. (a) Define f on $\mathbb{R}$ by $f(x) = 1/x$ if $1 \le x < \infty$, and $f(x) = 0$ otherwise. Show that f is square-integrable over the whole real line but not integrable over the whole real line.

(b) Define f on $\mathbb{R}$ by $f(x) = x^{-1/2}$ if $0 < x < 1$ and $f(x) = 0$ otherwise. Show that f is integrable over the whole real line but not square-integrable over the whole real line.

(c) Define $f(t) = \sin t/t$. Recall that

$$\int_{-\infty}^{\infty} \frac{\sin t}{t}\,dt = \lim_{B\to\infty} \int_{-B}^{B} \frac{\sin t}{t}\,dt = \pi.$$

Show that f is not absolutely integrable; i.e., show that

$$\lim_{B\to\infty} \int_{-B}^{B} |\frac{\sin t}{t}|\,dt = \infty.$$

HINT: Verify that

$$
\begin{aligned}
\int_0^\infty |\frac{\sin t}{t}|\,dt &= \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} |\frac{\sin t}{t}|\,dt \\
&= \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} |\sin t|\frac{1}{t}\,dt \\
&\geq \sum_{n=0|}^\infty \int_{n\pi}^{(n+1)\pi} |\sin t|\frac{1}{(n+1)\pi}\,dt \\
&= \sum_{n=0}^\infty \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin t|\,dt \\
&= \sum_{n=0}^\infty \frac{1}{(n+1)\pi} \int_0^1 |\sin s + n\pi|\,ds \\
&= \sum_{n=0}^\infty \frac{1}{(n+1)\pi} \int_0^1 |\sin s|\,ds \\
&= \sum_{n=0}^\infty \frac{1}{(n+1)\pi} \int_0^1 \sin s\,ds \\
&= 2\sum_{n=0}^\infty \frac{1}{(n+1)\pi} \\
&= \infty.
\end{aligned}
$$

(d) Conclude that, just because a function is (improperly) integrable, it may not be absolutely integrable; i.e., its absolute value may not be integrable.

Here is our formal definition of the Fourier transform on the line.

DEFINITION. Let f be an absolutely integrable function on $\mathbb{R}$. In other words, let f be an element of $L^1(\mathbb{R})$. By the *Fourier transform* $\widehat{f}$ of f we mean the function $\widehat{f}$, also defined on $\mathbb{R}$, given by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}\,dt = \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}\,dt.$$

What would be the analog of Fourier's Theorem in this context? That is, what should the inverse of the Fourier transform be? How do we recover the function f from its transform $\widehat{F}$?

THEOREM 8.3. (Fourier's Theorem on the line) If f is an absolutely integrable function on $\mathbb{R}$, then

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\omega)e^{2\pi i\omega x}\,d\omega.$$

REMARK. As it was for functions on the circle, this broad assertion is not quite right. Hypotheses will have to be added to make it true. For instance, it looks here as if we might very well need (or hope) that the Fourier transform $\widehat{f}$ also should be absolutely integrable for Fourier's formula to hold. Also, as we will see later, the L^2 theory will again be more perfect than the pointwise theory.

EXERCISE 8.4. Compute the Fourier transforms of the following functions.

(a) $f(x) = 1$ if $-1/2 \leq x < 1/2$, and $f(x) = 0$ otherwise. You should get

$$\widehat{f}(\omega) = \frac{\sin(\pi\omega)}{\pi\omega}.$$

Notice that, although f is absolutely integrable, $\widehat{f}$ is not.

(b) $f(x) = 1$ if $-B/2 \leq x < B/2$, and $f(x) = 0$ otherwise. You should get

$$\widehat{f}(\omega) = \frac{\sin(B\pi\omega)}{\pi\omega}.$$

(c) $f(x) = e^{-|x|}$. You should get

$$\widehat{f}(\omega) = \frac{2}{1 + 4\pi^2\omega^2}.$$

Here is one of the most interesting calculations in Fourier theory.

THEOREM 8.4. *Let $f(x) = e^{-\pi x^2}$. Then $\widehat{f} = f$, i.e., $\widehat{f}(\omega) = e^{-\pi\omega^2}$.*

PROOF. This calculation begins by computing the derivative of $\widehat{f}$. Be alert for mathematical details!

$$\begin{aligned} \widehat{f}'(\omega &= \frac{d}{d\omega} \int_{\mathbb{R}} f(t) e^{-2\pi i\omega t}\, dt \\ &= \int_{\mathbb{R}} f(t)(-2\pi i t) e^{-2\pi i\omega t}\, dt \\ &= i \int_{\mathbb{R}} e^{-\pi t^2}(-2\pi t) e^{-2\pi i\omega t}\, dt. \end{aligned}$$

Now, we integrate this by parts, obtaining

$$\begin{aligned} i \int_{\mathbb{R}} e^{-\pi t^2}(-2\pi t) e^{-2\pi i\omega t}\, dt = (ie^{-\pi t^2} e^{-2\pi i\omega t})[_{-\infty}^{\infty} \\ + i(2\pi i\omega) \int_{\mathbb{R}} e^{-\pi t^2} e^{-2\pi i\omega t}\, dt \\ = 0 - 0 - 2\pi\omega \widehat{f}(\omega. \end{aligned}$$

Hence, we see that the function $\widehat{f}$ satisfies the differential equation

$$\widehat{f}'(\omega) = -2\pi\omega \widehat{f}(\omega).$$

It follows that $\widehat{f}(\omega) = ce^{-\pi\omega^2}$, where c is a constant. We can evaluate c by setting $\omega = 0$. Thus,

$$c = \widehat{f}(0) = \int_{\mathbb{R}} e^{-\pi t^2}\, dt = 1.$$

What precisely does Fourier's Theorem (8.3) say analytically? (See if you can spot the subtle mathematical points in the following computation. Also note the similarity between this calculation and the first calculation in Section 5.)

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}} \widehat{f}(\omega)e^{2\pi i\omega x}\,d\omega \\
&= \lim_{B\to\infty} I_B(x) \\
&= \lim_{B\to\infty} \int_{-B}^{B} \widehat{f}(\omega)e^{2\pi i\omega x}\,d\omega \\
&= \lim_{B\to\infty} \int_{-B}^{B} \int_{\mathbb{R}} f(t)e^{-2\pi i t\omega}\,dt e^{2\pi i\omega x}\,d\omega \\
&= \lim_{B\to\infty} \int_{-B}^{B} \int_{\mathbb{R}} f(t)e^{2\pi i(x-t)\omega)}\,dt d\omega \\
&= \lim_{B\to\infty} \int_{\mathbb{R}} \int_{-B}^{B} e^{2\pi i(x-t)\omega}\,d\omega\, f(t)\,dt \\
&= \lim_{B\to\infty} \int_{\mathbb{R}} \frac{e^{2\pi i(x-t)B} - e^{-2\pi i(x-t)B}}{2\pi i(x-t)} f(t)\,dt \\
&= \lim_{B\to\infty} \int_{\mathbb{R}} f(t)\frac{\sin(B2\pi(x-t))}{\pi(x-t)}\,dt \\
&= \lim_{B\to\infty} \int_{\mathbb{R}} f(t)K_B(x-t)\,dt,
\end{aligned}$$

where K_B is the function (kernel) given by

$$K_B(t) = \frac{\sin(2\pi Bt)}{\pi t}.$$

Obviously, the kernel K_B is playing the role here that the Dirichlet kernel D_N played for the circle, and the "partial" integral

$$I_B(x) = \int_{-B}^{B} \widehat{f}(\omega)e^{2\pi i\omega x}\,d\omega = \int_{\mathbb{R}} f(t)K_B(x-t)\,dt$$

is playing the role of the partial sums of Fourier series.

Note that K_B is the Fourier transform of a rather simple function. (See part (b) of Exercise 8.4.)

EXERCISE 8.5. Here is an important theorem from Lebesgue integration theory. It is called *Fubini's Theorem,* and it is what's needed to justify interchanging the order of integration in a double integral.

THEOREM: If f is a function of two variables x and y, then

$$\int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy$$

providing that

$$\int_a^b \int_c^d |f(x,y)|\,dydx \text{ is finite..}$$

Show that the interchange of the order of integration in the preceding computation is justified by appealing to Fubini's Theorem.

Here are some properties of the kernel K_B that you should recognize as similar to corresponding properties of the Dirichlet kernel D_N.

EXERCISE 8.6. (a) Prove that $\int_{\mathbb{R}} K_B(t)\,dt = 1$ for all B.
(b) Show that, for any $\delta > 0$,

$$\lim_{B\to\infty} \int_{-\delta}^{\delta} K_B(t)\,dt = 1.$$

Conclude that

$$\lim_{B\to\infty} \int_{\delta}^{\infty} K_B(t)\,dt = 0.$$

(This kernel K_B is some version of the Dirac δ function.)

Though the initial observations indicate that the Fourier transform on the real line are quite similar to the Fourier transform on the circle, it is not entirely the case. The transform on the real line is a good bit more subtle. As a first step toward understanding Fourier analysis in this real line context, we have the following analog to Theorem 6.1. We will again rely heavily on the Riemann-Lebesgue Lemma.

EXERCISE 8.7. Revisit the statement and proof of the Riemann-Lebesgue Lemma: If f is an absolutely integrable function, then

$$\lim_{B\to\infty} \int_{\mathbb{R}} f(t)\sin(Bt)\,dt = 0.$$

First assume that $f(x) = 0$ unless x belongs to a closed interval $[a, b]$, suppose that f is differentiable on the open interval (a, b), and that its derivative f' is bounded there. Then prove the Riemann-Lebesgue Lemma by integrating by parts, etc. Compare with Theorem 5.2 and Exercise 5.3.

Recall that the general statement of the Riemann-Lebesgue Lemma follows from this special case by Lebesgue integration theory.

THEOREM 8.5. *If f is an absolutely integrable function on $\mathbb{R}$, and if f is differentiable at a point x, then Fourier's Theorem holds. That is,*

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\omega)e^{2\pi i\omega x}\,d\omega = \lim_{B\to\infty} I_B(x).$$

PROOF. We know from above that the partial integral $I_B(x) = \int_{-B}^{B} \widehat{f}(\omega)e^{2\pi i\omega x}\,d\omega$ is given in an integral form by

$$I_B(x) = \int_{\mathbb{R}} f(t)K_B(x-t)\,dt.$$

So, changing variables, and using the fact that the function K_B is an even function, we get

$$I_B(x) = \int_{\mathbb{R}} f(x+t)K_B(t)\,dt$$

and

$$I_B(x) = \int_{\mathbb{R}} f(x-t) K_B(t)\, dt.$$

Now, $f(x) = f(x) \int_{\mathbb{R}} K_B(t)\, dt$, and so

$$\begin{aligned} I_B(x) - f(x) &= \int_{\mathbb{R}} \frac{f(x+t)+f(x-t)}{2} K_B(t)\, dt - \int_{\mathbb{R}} f(x) K_B(t)\, dt \\ &= \int_{\mathbb{R}} \frac{f(x+t)+f(x-t)-2f(x)}{2} K_B(t)\, dt \\ &= \int_{\mathbb{R}} \frac{f(x+t)+f(x-t)-2f(x)}{2\pi t} \sin(2B\pi t)\, dt \\ &= \int_{\mathbb{R}} g(t) \sin(2B\pi t)\, dt, \end{aligned}$$

where $g(t) = (f(x+t)+f(x-t)-2f(x))/(2\pi t)$. So $\lim_{B\to\infty}(I_B(x) - f(x)) = 0$, i.e., $f(x)$ equals $\lim_{B\to\infty} I_B(x)$ if we can show that the integral above tends to 0 as B tends to ∞.

Because f is assumed to be differentiable at the point x, there must exist a $\delta > 0$ so that the two differential quotients $(f(x+t)-f(x))/t$ and $(F(x-t)-f(x))/t$ are both bounded if $|t| < \delta$. We show that the limit of the integral above is 0 by breaking the integral into three integrals over the three intervals $(-\infty, -\delta)$, $(-\delta, \delta)$, and (δ, ∞), and showing that each of these three integrals tends to 0.

For the middle interval $(-\delta, \delta)$, we write

$$g(t) = \frac{1}{2\pi}(\frac{f(x+t)-f(x)}{t} + \frac{f(x-t)-f(x)}{t}),$$

which is a bounded function, and so is an integrable function on the finite interval $(-\delta, \delta)$. Hence, that integral tends to 0 by the Riemann-Lebesgue Lemma.

For the interval (δ, ∞), we write

$$\int_\delta^\infty g(t) \sin(2B\pi t)\, dt = \int_\delta^\infty \frac{f(x+t)+f(x-t}{2\pi t} \sin(2B\pi t)\, dt - -2f(x) \int_\delta^\infty K_B(t\, dt.$$

Note that the first integral on the right tends to 0 by the Riemann-Lebesgue Lemma, since the function $(f(x+t)+f(x-t))/(2\pi t)$ is integrable on that interval. Finally, the second integral on the right tends to 0 by part (b) of Exercise 8.6. The proof is now complete.

EXERCISE 8.8. (a) State and prove a theorem for functions in $L^1(\mathbb{R})$ that is an analog to Theorem 6.3 for functions on the circle.

(b) Use the theorem you stated in part (a) to prove that, for every real number x,

$$e^{-|x|} = \int_{\mathbb{R}} \frac{2}{1+4\pi^2\omega^2} e^{2\pi i \omega x}\, d\omega = \int_{\mathbb{R}} \frac{2}{1+4\pi^2\omega^2} \cos(2\pi\omega x)\, d\omega.$$

(c) Let $f(x) = e^{-|x|}$. Prove that $f = \widehat{\widehat{f}}$.

Before developing any more results about Fourier's Theorem on the real line, we must develop some additional properties of the Fourier transform on L^1.

THEOREM 8.6. *Let f and g be elements of $L^1(\mathbb{R})$, and let a be any real number. Then*

(1) *The Fourier transform on $L^1(\mathbb{R})$ is a linear transformation. That is, $\widehat{f+g} = \widehat{f} + \widehat{g}$, and $\widehat{cf} = c\widehat{f}$ for any complex number c.*

(2) *Each function $\widehat{f}$ is bounded. In fact,*

$$|\widehat{f}(\omega)| \leq \int_{\mathbb{R}} |f(t)|\, dt.$$

(3) *If f_a is the function given by $f_a(t) = f(t+a)$, Then*

$$\widehat{f}_a(\omega) = e^{2\pi i\omega a}\widehat{f}(\omega).$$

Under the Fourier transform, the translation operator is converted into a multiplication operator.

(4) *Suppose f is a differentiable function, and assume that f' is also in L^1. Then*

$$\widehat{f'}(\omega) = 2\pi i\omega\widehat{f}(\omega).$$

The differentiation operator is converted into a multiplication operator.

(5) *(The Hat Trick.)*

$$\int_{\mathbb{R}} \widehat{f}(x)g(x)\, dx = \int_{\mathbb{R}} f(x)\widehat{g}(x)\, dx.$$

PROOF. We leave parts (1), (2), and (3) to an exercise.

To prove part (4), we integrate by parts:

$$\begin{aligned}
\widehat{f'}(\omega) &= \lim_{B\to\infty} \int_{-B}^{B} f'(t)e^{-2\pi i\omega t}\, dt \\
&= \lim_{B\to\infty} f(t)e^{-2\pi i\omega t}[_{-B}^{B} - \lim_{B\to\infty} \int_{-B}^{B} f(t)(-2\pi i\omega)e^{-2\pi i\omega t}\, dt \\
&= \lim_{B\to\infty} (f(B)e^{-2\pi i\omega B} - f(-B)e^{2\pi i\omega B}) + 2\pi i\omega \lim_{B\to\infty} \int_{-B}^{B} f(t)e^{-2\pi i\omega t}\, dt \\
&= 0 - 0 + 2\pi i\omega\widehat{f}(\omega).
\end{aligned}$$

(Why do the integrated terms go away?)

The proof to part (5) depends on interchanging the order of integration in a double integral. Fubini's Theorem is what's required to justify this interchange.

$$\begin{aligned}
\int_{\mathbb{R}} \widehat{f}(x)g(x)\, dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)e^{-2\pi ixt}\, dt g(x)\, dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(x)e^{-2\pi ixt}\, dtdx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)g(x)e^{-2\pi ixt}\, dxdt \\
&= \int_{\mathbb{R}} f(t)\int_{\mathbb{R}} g(x)e^{-2\pi ixt}\, dxdt \\
&= \int_{\mathbb{R}} f(t)\widehat{g}(t)\, dt \\
&= \int_{\mathbb{R}} f(x)\widehat{g}(x)\, dx,
\end{aligned}$$

which proves part (5).

EXERCISE 8.9. (a) Prove parts (1), (2), and (3) of the preceding theorem.

(b) Use Fubini's Theorem to justify the interchange of the order of integration in the preceding proof.