CHAPTER I

THE RIESZ REPRESENTATION THEOREM

We begin our study by identifying certain special kinds of linear functionals on certain special vector spaces of functions. We describe these linear functionals in terms of more familiar mathematical objects, i.e., as integrals against measures. We have labeled Theorem 1.3 as the Riesz Representation Theorem. However, each of Theorems 1.2, 1.3, 1.4 and 1.5 is often referred to by this name, and a knowledge of this nontrivial theorem, or set of theorems, is fundamental to our subject. Theorem 1.1 is very technical, but it is the cornerstone of this chapter.

DEFINITION. A *vector lattice* of functions on a set X is a vector space L of real-valued functions on X which is closed under the binary operations of maximum and minimum. That is:

(1) $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$ implies that $\alpha f + \beta g \in L$.
(2) $f, g \in L$ implies that $\max(f, g) \in L$ and $\min(f, g) \in L$.

REMARKS. The set of all continuous real-valued functions on a topological space X clearly forms a vector lattice, indeed the prototypical one. A nontrivial vector lattice certainly contains some nonnegative functions (taking maximum of f and 0). If a vector lattice does not contain any nonzero constant function, it does not follow that the minimum of an $f \in L$ and the constant function 1 must belong to L. The set of all scalar multiples of a fixed positive nonconstant function is a counterexample.

Stone's axiom for a vector lattice L is as follows: If f is a nonnegative function in L, then $\min(f, 1)$ is an element of L.

EXERCISE 1.1. Let L be a vector lattice of functions on a set X, and suppose L satisfies Stone's axiom.

(a) Show that $\min(f, c) \in L$ whenever f is a nonnegative function in L and $c \geq 0$.

(b) (A Urysohn-type property) Let E and F be disjoint subsets of X, $0 \leq a < b$, and let $f \in L$ be a nonnegative function such that $f(x) \geq b$ on F and $f(x) \leq a$ on E. Show that there exists an element $g \in L$ such that $0 \leq g(x) \leq 1$ for all $x \in X$, $g(x) = 0$ on E, and $g(x) = 1$ on F.

(c) Let $0 \leq a < b < c < d$ be real numbers, let $f \in L$ be nonnegative, and define $E = f^{-1}([0, a])$, $F = f^{-1}([b, c])$, and $G = f^{-1}([d, \infty))$. Show that there exists an element g of L such that $0 \leq bg(x) \leq f(x)$ for all $x \in X$, $g(x) = 1$ on F, and $g(x) = 0$ on $E \cup G$.

(d) Let μ be a measure defined on a σ-algebra of subsets of the set X, and suppose $L = L^1(\mu)$ is the set of all (absolutely) integrable real-valued functions on X with respect to μ. Show that L is a vector lattice that satisfies Stone's axiom.

(e) Let μ and L be as in part d. Define $\phi : L \to \mathbb{R}$ by $\phi(f) = \int f \, d\mu$. Prove that ϕ is a positive linear functional on L, i.e., ϕ is a linear functional for which $\phi(f) \geq 0$ whenever $f(x) \geq 0$ for all $x \in X$.

We come now to our fundamental representation theorem for linear functionals.

THEOREM 1.1. *Let L be a vector lattice on a set X, and assume that L satisfies Stone's axiom, i.e., that if f is a nonnegative function in L, then $\min(f, 1) \in L$. Suppose I is a linear functional on the vector space L that satisfies:*

(1) *$I(f) \geq 0$ whenever $f(x) \geq 0$ for all $x \in X$. (I is a positive linear functional.)*
(2) *Suppose $\{f_n\}$ is a sequence of nonnegative elements of L, which increases pointwise to an element f of L, i.e., $f(x) = \lim f_n(x)$ for every x, and $f_n(x) \leq f_{n+1}(x)$ for every x and n. Then $I(f) = \lim I(f_n)$. (I satisfies the monotone convergence property.)*

Then there exists a (not necessarily finite) measure μ defined on a σ-algebra $\mathcal{M}$ of subsets of X such that every $f \in L$ is μ-measurable, μ-integrable, and

$$I(f) = \int f \, d\mu.$$

PROOF. We begin by defining an outer measure μ^* on all subsets of X. Thus, if $E \subseteq X$, put

$$\mu^*(E) = \inf \sum I(h_m),$$

where the infimum is taken over all sequences $\{h_m\}$ of nonnegative functions in L for which $\sum h_m(x) \geq 1$ for each $x \in E$. Note that if, for some set E, no such sequence $\{h_m\}$ exists, then $\mu^*(E) = \infty$, the infimum over an empty set being $+\infty$. In particular, if L does not contain the constant function 1, then $\mu^*(X)$ could be ∞, although not necessarily. See Exercise 1.2 below.

It follows routinely that μ^* is an outer measure. Again see Exercise 1.2 below.

We let μ be the measure generated by μ^*, i.e., μ is the restriction of μ^* to the σ-algebra $\mathcal{M}$ of all μ^*-measurable subsets of X. We wish to show that each $f \in L$ is μ-measurable, μ-integrable, and then that $I(f) = \int f \, d\mu$. Since L is a vector lattice, and both I and $\int \cdot \, d\mu$ are positive linear functionals on L, we need only verify the above three facts for nonnegative functions $f \in L$.

To prove that a nonnegative $f \in L$ is μ-measurable, it will suffice to show that each set $f^{-1}[a, \infty)$, for $a > 0$, is μ^*-measurable; i.e., we must show that for any $A \subseteq X$,

$$\mu^*(A) \geq \mu^*(A \cap f^{-1}[a, \infty)) + \mu^*(A \cap \widetilde{f^{-1}[a, \infty)}).$$

We first make the following observation.

Suppose $A \subseteq X$, $0 < a < b$, E is a subset of X for which $f(x) \leq a$ if $x \in E$, and F is a subset of X for which $f(x) \geq b$ if $x \in F$. Then

$$\mu^*(A \cap (E \cup F)) \geq \mu^*(A \cap E) + \mu^*(A \cap F).$$

Indeed, let g be the element of L defined by

$$g = \frac{\min(f, b) - \min(f, a)}{b - a}.$$

Then $g = 0$ on E, and $g = 1$ on F. If $\epsilon > 0$ is given, and $\{h_m\}$ is a sequence of nonnegative elements of L for which $\sum h_m(x) \geq 1$ on $A \cap (E \cup F)$, and $\sum I(h_m) < \mu^*(A \cap (E \cup F)) + \epsilon$, set $f_m = \min(h_m, g)$ and $g_m = h_m - \min(h_m, g)$. Then:

$$h_m = f_m + g_m$$

on X,

$$\sum f_m(x) \geq 1$$

for $x \in A \cap F$, and

$$\sum g_m(x) \geq 1$$

for $x \in A \cap E$. Therefore:

$$\begin{aligned}\mu^*(A \cap (E \cup F)) + \epsilon &\geq \sum I(h_m) \\ &= \sum I(f_m) + \sum I(g_m) \\ &\geq \mu^*(A \cap F) + \mu^*(A \cap E).\end{aligned}$$

It follows now by induction that if $\{I_1, ..., I_n\}$ is a finite collection of disjoint half-open intervals $(a_j, b_j]$, with $0 < b_1$ and $b_j < a_{j+1}$ for $1 \leq j < n$, and if $E_j = f^{-1}(I_j)$, then

$$\mu^*(A \cap (\cup E_j)) \geq \sum \mu^*(A \cap E_j)$$

for any subset A of X. In fact, using the monotonicity of the outer measure μ^*, the same assertion is true for any countable collection $\{I_j\}$ of such disjoint half-open intervals. See Exercise 1.3.

Now, Let A be an arbitrary subset of X, and let $a > 0$ be given. Write $E = f^{-1}[a, \infty)$. We must show that

$$\mu^*(A \cap E) + \mu^*(A \cap \tilde{E}) \leq \mu^*(A).$$

We may assume that $\mu^*(A)$ is finite, for otherwise the desired inequality is obvious. Let $\{c_1, c_2, \dots\}$ be a strictly increasing sequence of positive numbers that converges to a. We write the interval $(-\infty, a)$ as the countable union $\cup_{j=0}^{\infty} I_j$ of the disjoint half-open intervals $\{I_j\}$, where $I_0 = (-\infty, c_1]$, and for $j > 0$, $I_j = (c_j, c_{j+1}]$, whence

$$\tilde{E} = \cup_{j=0}^{\infty} E_j,$$

where $E_j = f^{-1}(I_j)$. Also, if we set $F_k = \cup_{j=0}^{k} E_j$, then $\tilde{E}$ is the increasing union of the F_k's. Then, using Exercise 1.3, we have:

$$\mu^*(A \cap (\cup_{j=0}^{\infty} E_{2j})) \geq \sum_{j=0}^{\infty} \mu^*(A \cap E_{2j}),$$

whence the infinite series on the right is summable.

Similarly, the infinite series $\sum_{j=0}^{\infty} \mu^*(A \cap E_{2j+1})$ is summable. Therefore,

$$\begin{aligned}\mu^*(A \cap F_k) &\leq \mu^*(A \cap \tilde{E}) \\ &= \mu^*((A \cap F_k) \cup (A \cap (\cup_{j=k+1}^{\infty} E_j))) \\ &\leq \mu^*(A \cap F_k) + \sum_{j=k+1}^{\infty} \mu^*(A \cap E_j),\end{aligned}$$

and this shows that $\mu^*(A \cap \tilde{E}) = \lim_k \mu^*(A \cap F_k)$.

So, recalling that $F_k = f^{-1}(-\infty, c_{k+1}]$, we have

$$\begin{aligned}\mu^*(A \cap E) + \mu^*(A \cap \tilde{E}) &= \mu^*(A \cap E) + \lim_k \mu^*(A \cap F_k) \\ &= \lim_k (\mu^*(A \cap E) + \mu^*(A \cap F_k)) \\ &= \lim_k \mu^*(A \cap (E \cup F_k)) \\ &\leq \mu^*(A),\end{aligned}$$

as desired. Therefore, f is μ-measurable for every $f \in L$.

It remains to prove that each $f \in L$ is μ-integrable, and that $I(f) = \int f\, d\mu$. It will suffice to show this for f's which are nonnegative, bounded, and 0 outside a set of finite μ measure. See Exercise 1.4. For such an f, let ϕ be a nonnegative measurable simple function, with $\phi(x) \geq f(x)$ for all x, and such that ϕ is 0 outside a set of finite measure. (We will use the fact from measure theory that there exists a sequence $\{\phi_n\}$ of such simple functions for which $\int f\, d\mu = \lim \int \phi_n\, d\mu$.) Write $\phi = \sum_{i=1}^{k} a_i \chi_{E_i}$, where each $a_i \geq 0$, and let $\epsilon > 0$ be given. For each i, let $\{h_{i,m}\}$ be a sequence of nonnegative elements in L for which $\sum_m h_{i,m}(x) \geq 1$ on E_i, and

$\sum_m I(h_{i,m}) < \mu^*(E_i) + \epsilon$. Then

$$\begin{aligned}\int \phi\, d\mu + \epsilon \sum_{i=1}^k a_i &= \sum_{i=1}^k a_i \mu(E_i) + \epsilon \sum_{i=1}^k a_i \\ &> \sum_{i=1}^k a_i \sum_m I(h_{i,m}) \\ &= \sum_m \sum_{i=1}^k a_i I(h_{i,m}) \\ &= \lim_M \sum_{m=1}^M \sum_{i=1}^k a_i I(h_{i,m}) \\ &= \lim_M I(h_M),\end{aligned}$$

where

$$h_M = \sum_{m=1}^M \sum_{i=1}^k a_i h_{i,m}.$$

Observe that $\{h_M\}$ is an increasing sequence of nonnegative elements of L, and that $\lim h_M(x) \geq \phi(x)$ for all $x \in X$, whence the sequence $\{\min(h_M, f)\}$ increases pointwise to f. Therefore, by the monotone convergence property of I, we have that

$$\begin{aligned}\int \phi\, d\mu + \epsilon \sum_{i=1}^k a_i &\geq \lim_M I(h_M) \\ &\geq \lim_M I(\min(h_M, f)) \\ &= I(f),\end{aligned}$$

showing that $\int \phi\, d\mu \geq I(f)$, for all such simple functions ϕ. It follows then that $\int f\, d\mu \geq I(f)$.

To show the reverse inequality, we may suppose that $0 \leq f(x) < 1$ for all x, since both I and $\int \cdot\, d\mu$ are linear. For each positive integer n and each $0 \leq i < 2^n$, define the set $E_{i,n}$ by

$$E_{i,n} = f^{-1}([i/2^n, (i+1)/2^n)),$$

and then a simple function ϕ_n by

$$\phi_n = \sum_{i=0}^{2^n-1} (i/2^n)\chi_{E_{i,n}}.$$

Using Exercise 1.1 part c, we choose, for each $0 \leq i < 2^n$ and each $m > 2^{n+1}$, a function $g_{i,m}$ satisfying:

(1) For $x \in f^{-1}([i/2^n, ((i+1)/2^n) - 1/m))$,
$$g_{i,m}(x) = i/2^n.$$

(2) For $x \in f^{-1}([0, (i/2^n) - 1/2m))$ and $x \in f^{-1}([((i+1)/2^n) - 1/2m, 1])$,
$$g_{i,m}(x) = 0.$$

(3) For all x,
$$0 \leq g_{i,m}(x) \leq f(x).$$

Then
$$\mu(E_{i,n}) = \lim_m \mu(f^{-1}([i/2^n \ , \ (i+1)/2^n - 1/m))).$$

And,
$$\begin{aligned}\sum_{i=0}^{2^n-1} (i/2^n)\mu(f^{-1}([i/2^n \ , \ (i+1)/2^n - 1/m))) &\leq \sum_{i=0}^{2^n-1} I(g_{i,m}) \\ &= I(h_m),\end{aligned}$$

where
$$h_m = \sum_{i=0}^{2^n-1} g_{i,m}.$$

Observe that $h_m(x) \leq f(x)$ for all x. It follows that
$$\begin{aligned}\int \phi_n \, d\mu &= \sum_{i=0}^{2^n-1} (i/2^n)\mu(E_{i,n}) \\ &= \lim_m \sum_{i=0}^{2^n-1} (i/2^n)\mu(f^{-1}([i/2^n \ , \ (i+1)/2^n - 1/m))) \\ &\leq \limsup_m I(h_m) \\ &\leq I(f),\end{aligned}$$

whence, by letting n tend to ∞, we see that $\int f \, d\mu \leq I(f)$.

The proof of the theorem is now complete.

EXERCISE 1.2. (a) Give an example of a vector lattice L of functions on a set X, such that the constant function 1 does not belong to L, but for which there exists a sequence $\{h_n\}$ of nonnegative elements of L satisfying $\sum h_n(x) \geq 1$ for all $x \in X$.

(b) Verify that the μ^* in the preceding proof is an outer measure on X by showing that:

(1) $\mu^*(\emptyset) = 0$.
(2) If E and F are subsets of X, with E contained in F, then $\mu^*(E) \leq \mu^*(F)$.
(3) μ^* is *countably subadditive*, i.e.,

$$\mu^*(\cup E_n) \leq \sum \mu^*(E_n)$$

for every sequence $\{E_n\}$ of subsets of X.

HINT: To prove the countable subadditivity, assume that each $\mu^*(E_n)$ is finite. Then, given any $\epsilon > 0$, let $\{h_{n,i}\}$ be a sequence of nonnegative functions in L for which $\sum_i h_{n,i}(x) \geq 1$ for all $x \in E_n$ and for which $\sum_i I(h_{n,i}) \leq \mu^*(E_n) + \epsilon/2^n$.

EXERCISE 1.3. Let $\{I_1, I_2, ...\}$ be a countable collection of half-open intervals $(a_j, b_j]$, with $0 < b_1$ and $b_j < a_{j+1}$ for all j. Let f be a nonnegative element of the lattice L of the preceding theorem, and set $E_j = f^{-1}(I_j)$. Show that for each $A \subseteq X$ we have

$$\mu^*(A \cap (\cup E_j)) = \sum \mu^*(A \cap E_j).$$

HINT: First show this, by induction, for a finite sequence $I_1, \ldots, I_n$, and then verify the general case by using the properties of the outer measure.

EXERCISE 1.4. Let L be the lattice of the preceding theorem.

(a) Show that there exist sets of finite μ-measure. In fact, if f is a nonnegative element of L, show that $f^{-1}([\epsilon, \infty))$ has finite measure for every positive ϵ.

(b) Let $f \in L$ be nonnegative. Show that there exists a sequence $\{f_n\}$ of bounded nonnegative elements of L, each of which is 0 outside some set of finite μ-measure, which increases to f. HINT: Use Stone's axiom.

(c) Conclude that, if $I(f) = \int f \, d\mu$ for every $f \in L$ that is bounded, nonnegative, and 0 outside a set of finite μ-measure, then $I(f) = \int f \, d\mu$ for every $f \in L$.

REMARK. One could imagine that all linear functionals defined on a vector lattice of functions on a set X are related somehow to integration over X. The following exercise shows that this is not the case; that is, some extra hypotheses on the functional I are needed.

EXERCISE 1.5. (a) Let X be the set of positive integers, and let L be the space of all functions f (sequences) on X for which $\lim_{n\to\infty} f(n)$ exists. Prove that L is a vector lattice that satisfies Stone's axiom.

(b) Let X and L be as in part a, and define $I : L \to \mathbb{R}$ by $I(f) = \lim_{n\to\infty} f(n)$. Prove that I is a positive linear functional.

(c) Let I be the positive linear functional from part b. Prove that there exists no measure μ on the set X for which $I(f) = \int f\,d\mu$ for all $f \in L$. HINT: If there were such a measure, there would have to exist a sequence $\{\mu_n\}$ such that $I(f) = \sum f(n)\mu_n$ for all $f \in L$. Show that each μ_n must be 0, and that this would lead to a contradiction.

(d) Let $X, L,$ and I be as in part b. Verify by giving an example that I fails to satisfy the monotone convergence property of Theorem 1.1.

DEFINITION. If Δ is a Hausdorff topological space, then the smallest σ-algebra $\mathcal{B}$ of subsets of Δ, which contains all the open subsets of Δ, is called the σ-algebra of *Borel* sets. A measure which is defined on this σ-algebra, is called a *Borel* measure. A function f from Δ into another topological space Δ' is called a *Borel function* if $f^{-1}(U)$ is a Borel subset of Δ whenever U is an open (Borel) subset of Δ'.

A real-valued (or complex-valued) function f on Δ is said to have *compact support* if the closure of the set of all $x \in \Delta$ for which $f(x) \neq 0$ is compact. The set of all continuous functions having compact support on Δ is denoted by $C_c(\Delta)$.

A real-valued (or complex-valued) function f on Δ is said to *vanish at infinity* if, for each $\epsilon > 0$, the set of all $x \in \Delta$ for which $|f(x)| \geq \epsilon$ is compact. The set of all continuous real-valued functions vanishing at infinity on Δ is denoted here by $C_0(\Delta)$. Sometimes, $C_0(\Delta)$ denotes the complex vector space of all continuous complex-valued functions on Δ that vanish at ∞. Hence, the context in which this symbol occurs dictates which meaning it has.

If Δ is itself compact, then every continuous function vanishes at infinity, and we write $C(\Delta)$ for the space of all continuous real-valued (complex-valued) functions on Δ. That is, if Δ is compact, then $C(\Delta) = C_0(\Delta)$.

EXERCISE 1.6. (a) Prove that a second countable locally compact Hausdorff space Δ is metrizable. (See Exercise 0.9.) Conclude that if

K is a compact subset of a second countable locally compact Hausdorff space Δ, then there exists an element $f \in C_c(\Delta)$ that is identically 1 on K.

(b) Let Δ be a locally compact Hausdorff space. Show that every element of $C_c(\Delta)$ is a Borel function, and hence is μ-measurable for every Borel measure μ on Δ.

(c) Show that, if Δ is second countable, Hausdorff, and locally compact, then the σ-algebra of Borel sets coincides with the smallest σ-algebra that contains all the compact subsets of Δ.

(d) If Δ is second countable, Hausdorff, and locally compact, show that the σ-algebra $\mathcal{B}$ of Borel sets coincides with the smallest σ-algebra $\mathcal{M}$ of subsets of Δ for which each $f \in C_c(\Delta)$ satisfies $f^{-1}(U) \in \mathcal{M}$ whenever U is open in $\mathbb{R}$.

(e) Suppose μ and ν are finite Borel measures on a second countable, locally compact, Hausdorff space Δ, and assume that $\int f\,d\mu = \int f\,d\nu$ for every $f \in C_c(\Delta)$. Prove that $\mu = \nu$. HINT: Show that μ and ν agree on compact sets, and hence on all Borel sets.

(f) Prove that a second countable locally compact Hausdorff space Δ is σ-compact. In fact, show that Δ is the increasing union $\cup K_n$ of a sequence of compact subsets $\{K_n\}$ of Δ such that K_n is contained in the interior of K_{n+1}. Note also that this implies that every closed subset F of Δ is the increasing union of a sequence of compact sets.

EXERCISE 1.7. Prove Dini's Theorem: If Δ is a compact topological space and $\{f_n\}$ is a sequence of continuous real-valued functions on Δ that increases monotonically to a continuous function f, then $\{f_n\}$ converges uniformly to f on Δ.

THEOREM 1.2. *Let Δ be a second countable locally compact Hausdorff space. Let I be a positive linear functional on $C_c(\Delta)$. Then there exists a unique Borel measure μ on Δ such that, for all $f \in C_c(\Delta)$, f is μ-integrable and $I(f) = \int f\,d\mu$.*

PROOF. Of course $C_c(\Delta)$ is a vector lattice that satisfies Stone's axiom. The given linear functional I is positive, so that this theorem will follow immediately from Theorem 1.1 and Exercise 1.6 if we show that I satisfies the monotone convergence property. Thus, let $\{f_n\}$ be a sequence of nonnegative functions in $C_c(\Delta)$ that increases monotonically to an element $f \in C_c(\Delta)$. If K denotes a compact set such that $f(x) = 0$ for $x \notin K$, Then $f_n(x) = 0$ for all $x \notin K$ and for all n. We let g be a nonnegative element of $C_c(\Delta)$ for which $g(x) = 1$ on K. On the compact set K, the sequence $\{f_n\}$ is converging monotonically to the continuous function f, whence, by Dini's Theorem, this convergence is uniform. Therefore, given an $\epsilon > 0$, there exists an N such that

$$f(x) - f_n(x) = |f(x) - f_n(x)| < \epsilon$$

for all x if $n \geq N$. Hence $f - f_n \leq \epsilon g$ everywhere on X, whence

$$|I(f - f_n)| = I(f - f_n) \leq I(\epsilon g) = \epsilon I(g).$$

Therefore $I(f) = \lim I(f_n)$, as desired.

DEFINITION. If f is a bounded real-valued function on a set X, we define the *supremum norm* or *uniform norm* of f, denoted by $\|f\|$, or $\|f\|_\infty$, by

$$\|f\| = \|f\|_\infty = \sup_{x \in X} |f(x)|.$$

A linear functional ϕ on a vector space E of bounded functions is called a *bounded linear functional* if there exists a positive constant M such that $|\phi(f)| \leq M\|f\|$ for all $f \in E$.

THEOREM 1.3. (Riesz Representation Theorem) *Suppose Δ is a second countable locally compact Hausdorff space and that I is a positive linear functional on $C_0(\Delta)$. Then there exists a unique finite Borel measure μ on Δ such that*

$$I(f) = \int f \, d\mu$$

for every $f \in C_0(\Delta)$. Further, I is a bounded linear functional on $C_0(\Delta)$. Indeed, $|I(f)| \leq \mu(\Delta)\|f\|_\infty$.

PROOF. First we show that the positive linear functional I on the vector lattice $C_0(\Delta)$ satisfies the monotone convergence property. Thus, let $\{f_n\}$ be a sequence of nonnegative functions in $C_0(\Delta)$, which increases to an element f, and let $\epsilon > 0$ be given. Choose a compact subset $K \subseteq \Delta$ such that $f(x) \leq \epsilon^2$ if $x \notin K$, and let $g \in C_0(\Delta)$ be nonnegative and such that $g = 1$ on K. Again, by Dini's Theorem, there exists an N such that

$$|f(x) - f_n(x)| \leq \epsilon/(1 + I(g))$$

for all $x \in K$ and all $n \geq N$. For $x \notin K$, we have:

$$\begin{aligned} |f(x) - f_n(x)| &= f(x) - f_n(x) \\ &\leq f(x) \\ &= (\sqrt{f(x)})^2 \\ &\leq \epsilon\sqrt{f(x)}, \end{aligned}$$

so that, for all $x \in \Delta$ and all $n \geq N$, we have

$$|f(x) - f_n(x)| \leq (\epsilon/(1 + I(g)))g(x) + \epsilon\sqrt{f(x)}.$$

Therefore,

$$|I(f) - I(f_n)| = I(f - f_n) \leq \epsilon(1 + I(\sqrt{f})).$$

This proves that $I(f) = \lim I(f_n)$, as desired.

Using Theorem 1.1 and Exercise 1.6, let μ be the unique Borel measure on Δ for which $I(f) = \int f\,d\mu$ for every $f \in C_0(\Delta)$. We show next that there exists a positive constant M such that $|I(f)| \leq M\|f\|_\infty$ for each $f \in C_0(\Delta)$; i.e., that I is a bounded linear functional on $C_0(\Delta)$. If there were no such M, there would exist a sequence $\{f_n\}$ of nonnegative elements of $C_0(\Delta)$ such that $\|f_n\|_\infty = 1$ and $I(f_n) \geq 2^n$ for all n. Then, defining $f_0 = \sum_n f_n/2^n$, we have that $f_0 \in C_0(\Delta)$. (Use the Weierstrass M-test.) On the other hand, since I is a positive linear functional, we see that

$$I(f_0) \geq \sum_{n=1}^{N} I(f_n/2^n) \geq N$$

for all N, which is a contradiction. Therefore, I is a bounded linear functional, and we let M be a fixed positive constant satisfying $|I(f)| \leq M\|f\|_\infty$ for all $f \in C_0(\Delta)$.

Observe next that if K is a compact subset of Δ, then there exists a nonnegative function $f \in C_0(\Delta)$ that is identically 1 on K and ≤ 1 everywhere on Δ. Therefore,

$$\mu(K) \leq \int f\,d\mu = I(f) \leq M\|f\|_\infty = M.$$

Because Δ is second countable and locally compact, it is σ-compact, i.e., the increasing union $\cup K_n$ of a sequence of compact sets $\{K_n\}$. Hence, $\mu(\Delta) = \lim \mu(K_n) \leq M$, showing that μ is a finite measure. Then, $|I(f)| = |\int f\,d\mu| \leq \mu(\Delta)\|f\|_\infty$, and this completes the proof.

THEOREM 1.4. *Let Δ be a second countable locally compact Hausdorff space, and let ϕ be a bounded linear functional on $C_0(\Delta)$. That is, suppose there exists a positive constant M for which $|\phi(f)| \leq M\|f\|_\infty$ for all $f \in C_0(\Delta)$. Then ϕ is the difference $\phi_1 - \phi_2$ of two positive linear functionals ϕ_1 and ϕ_2, whence there exists a unique finite signed Borel measure μ such that $\phi(f) = \int f\,d\mu$ for all $f \in C_0(\Delta)$.*

PROOF. For f a nonnegative element in $C_0(\Delta)$, define $\phi_1(f)$ by

$$\phi_1(f) = \sup_g \phi(g),$$

where the supremum is taken over all nonnegative functions $g \in C_0(\Delta)$ for which $0 \leq g(x) \leq f(x)$ for all x. Define $\phi_1(f)$, for an arbitrary element $f \in C_0(\Delta)$, by $\phi_1(f) = \phi_1(f_+) - \phi_1(f_-)$, where $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$. It follows from Exercise 1.8 below that ϕ_1 is well-defined and is a linear functional on $C_0(\Delta)$. Since the 0 function is one of the g's over which we take the supremum when evaluating $\phi_1(f)$ for f a nonnegative function, we see that ϕ_1 is a positive linear functional. We define ϕ_2 to be the difference $\phi_1 - \phi$. Clearly, since f itself is one of the g's over which we take the supremum when evaluating $\phi_1(f)$ for f a nonnegative function, we see that ϕ_2 also is a positive linear functional, so that the existence of a signed measure μ satisfying $\phi(f) = \phi_1(f) - \phi_2(f) = \int f\, d\mu$ follows from the Riesz Representation Theorem. The uniqueness of μ is a consequence of the Hahn decomposition theorem.

EXERCISE 1.8. Let L be a vector lattice of bounded functions on a set Δ, and let ϕ be a bounded linear functional on L. That is, suppose that M is a positive constant for which $|\phi(f)| \leq M\|f\|_\infty$ for all $f \in L$. For each nonnegative $f \in L$ define, in analogy with the preceding proof,

$$\phi_1(f) = \sup_g \phi(g),$$

where the supremum is taken over all $g \in L$ for which $0 \leq g(x) \leq f(x)$ for all $x \in \Delta$.

(a) If f is a nonnegative element of L, show that $\phi_1(f)$ is a finite real number.

(b) If f and f' are two nonnegative functions in L, show that $\phi_1(f + f') = \phi_1(f) + \phi_1(f')$.

(c) For each real-valued $f = f_+ - f_- \in L$, define $\phi_1(f) = \phi_1(f_+) - \phi_1(f_-)$. Suppose g and h are nonnegative elements of L and that $f = g - h$. Prove that $\phi_1(f) = \phi_1(g) - \phi_1(h)$. HINT: $f_+ + h = g + f_-$.

(d) Prove that ϕ_1, as defined in part c, is a positive linear functional on L.

EXERCISE 1.9. Let Δ be a locally compact, second countable, Hausdorff space, and let $U_1, U_2, \ldots$ be a countable basis for the topology on Δ for which the closure $\overline{U_n}$ of U_n is compact for every n. Let C be the set of all pairs (n, m) for which $\overline{U_n} \subseteq U_m$, and for each $(n, m) \in C$

let $f_{n,m}$ be a continuous function from Δ into $[0,1]$ that is 1 on $\overline{U_n}$ and 0 on the complement $\widetilde{U_m}$ of U_m.

(a) Show that each $f_{n,m}$ belongs to $C_0(\Delta)$ and that the set of $f_{n,m}$'s separate the points of Δ.

(b) Let A be the smallest algebra of functions containing all the $f_{n,m}$'s. Show that A is uniformly dense in $C_0(\Delta)$. HINT: Use the Stone-Weierstrass Theorem.

(c) Prove that there exists a countable subset D of $C_0(\Delta)$ such that every element of $C_0(\Delta)$ is the uniform limit of a sequence of elements of D. (That is, $C_0(\Delta)$ is a separable metric space with respect to the metric d given by $d(f,g) = \|f-g\|_\infty$.)

EXERCISE 1.10. (a) Define I on $C_c(\mathbb{R})$ by $I(f) = \int f(x)\,dx$. Show that I is a positive linear functional which is not a bounded linear functional.

(b) Show that there is no way to extend the positive linear functional I of part a to all of $C_0(\mathbb{R})$ so that the extension is still a positive linear functional.

EXERCISE 1.11. Let X be a complex vector space, and let f be a complex linear functional on X. Write $f(x) = u(x) + iv(x)$, where $u(x)$ and $v(x)$ are the real and imaginary parts of $f(x)$.

(a) Show that u and v are real linear functionals on the real vector space X.

(b) Show that $u(ix) = -v(x)$, and $v(ix) = u(x)$. Conclude that a complex linear functional is completely determined by its real part.

(c) Suppose a is a real linear functional on the complex vector space X. Define $g(x) = a(x) - ia(ix)$. Prove that g is a complex linear functional on X.

DEFINITION. Let S be a set and let $\mathcal{B}$ be a σ-algebra of subsets of S. By a *finite complex measure* on $\mathcal{B}$ we mean a mapping $\mu : \mathcal{B} \to \mathbb{C}$ that satisfies:

(1) $\mu(\emptyset) = 0$.
(2) If $\{E_n\}$ is a sequence of pairwise disjoint elements of $\mathcal{B}$, then the series $\sum \mu(E_n)$ is absolutely summable, and

$$\mu(\cup E_n) = \sum \mu(E_n).$$

EXERCISE 1.12. Let μ be a finite complex measure on a σ-algebra $\mathcal{B}$ of subsets of a set S.

(a) Show that there exists a constant M such that $|\mu(E)| \leq M$ for all $E \in \mathcal{B}$.

(b) Write the complex-valued function μ on $\mathcal{B}$ as $\mu_1 + i\mu_2$, where μ_1 and μ_2 are real-valued functions. Show that both μ_1 and μ_2 are finite signed measures on $\mathcal{B}$. Show also that $\bar{\mu} = \mu_1 - i\mu_2$ is a finite complex measure on $\mathcal{B}$.

(c) Let X denote the complex vector space of all bounded complex-valued $\mathcal{B}$-measurable functions on S. If $f \in X$, define

$$\int f\,d\mu = \int f\,d\mu_1 + i\int f\,d\mu_2.$$

Prove that the assignment $f \to \int f\,d\mu$ is a linear functional on X and that there exists a constant M such that

$$|\int f\,d\mu| \leq M\|f\|_\infty$$

for all $f \in X$.

(d) Show that

$$\int f\,d\bar{\mu} = \overline{\int \bar{f}\,d\mu}.$$

HINT: Write $\mu = \mu_1 + i\mu_2$ and $f = u + iv$.

THEOREM 1.5. (Riesz Representation Theorem, Complex Version) *Let Δ be a second countable locally compact Hausdorff space, and denote now by $C_0(\Delta)$ the complex vector space of all continuous complex-valued functions on Δ that vanish at infinity. Suppose ϕ is a linear functional on $C_0(\Delta)$ into the field $\mathbb{C}$, and assume that ϕ is a bounded linear functional, i.e., that there exists a positive constant M such that $|\phi(f)| \leq M\|f\|_\infty$ for all $f \in C_0(\Delta)$. Then there exists a unique finite complex Borel measure μ on Δ such that $\phi(f) = \int f\,d\mu$ for all $f \in C_0(\Delta)$. See the preceding exercise.*

EXERCISE 1.13. (a) Prove Theorem 1.5. HINT: Write ϕ in terms of its real and imaginary parts ψ and η. Show that each of these is a bounded real-valued linear functional on the real vector space $C_0(\Delta)$ of all real-valued continuous functions on Δ that vanish at infinity, and that

$$\psi(f) = \int f\,d\mu_1$$

and

$$\eta(f) = \int f \, d\mu_2$$

for all real-valued $f \in C_0(\Delta)$. Then show that

$$\phi(f) = \int f \, d\mu,$$

where $\mu = \mu_1 + i\mu_2$.

(b) Let Δ be a second countable, locally compact Hausdorff space, and let $C_0(\Delta)$ denote the space of continuous complex-valued functions on Δ that vanish at infinity. Prove that there is a 1-1 correspondence between the set of all finite complex Borel measures on Δ and the set of all bounded linear functionals on $C_0(\Delta)$.

REMARK. The hypothesis of second countability may be removed from the Riesz Representation Theorem. However, the notion of measurability must be reformulated. Indeed, the σ-algebra on which the measure is defined is, from Theorem 1.1, the smallest σ-algebra for which each element $f \in C_c(\Delta)$ is a measurable function. One can show that this σ-algebra is the smallest σ-algebra containing the compact G_δ sets. This σ-algebra is called the σ-algebra of *Baire* sets, and a measure defined on this σ-algebra is called a *Baire* measure. One can prove versions of Theorems 1.2-1.5, for an arbitrary locally compact Hausdorff space Δ, almost verbatim, only replacing the word "Borel" by the word "Baire."