CHAPTER IV

NORMED LINEAR SPACES AND BANACH SPACES

DEFINITION A *Banach space* is a real normed linear space that is a complete metric space in the metric defined by its norm. A *complex Banach space* is a complex normed linear space that is, as a real normed linear space, a Banach space. If X is a normed linear space, x is an element of X, and δ is a positive number, then $B_\delta(x)$ is called the *ball of radius* δ *around* x, and is defined by $B_\delta(x) = \{y \in X : \|y - x\| < \delta\}$. The *closed ball* $\overline{B}_\delta(x)$ of radius δ around x is defined by $\overline{B}_\delta(x) = \{y \in X : \|y - x\| \leq \delta\}$. By B_δ and $\overline{B}_\delta$ we shall mean the (open and closed) balls of radius δ around 0.

Two normed linear spaces X and Y are *isometrically isomorphic* if there exists a linear isomorphism $T : X \to Y$ which is an isometry of X onto Y. In this case, T is called an *isometric isomorphism.*

If $X_1, \ldots X_n$ are n normed linear spaces, we define a norm on the (algebraic) direct sum $X = \bigoplus_{i=1}^n X_i$ by

$$\|(x_1, \ldots , x_n)\| = \max_{i=1}^{n} \|x_i\|.$$

This is frequently called the *max norm.*

Our first order of business is to characterize those locally convex topological vector spaces whose topologies are determined by a norm, i.e., those locally convex topological vector spaces that are normable.

DEFINITION. Let X be a topological vector space. A subset $S \subseteq$

X is called *bounded* if for each neighborhood W of 0 there exists a positive scalar c such that $S \subseteq cW$.

THEOREM 4.1. (Characterization of Normable Spaces) *Let X be a locally convex topological vector space. Then X is a normable vector space if and only if there exists a bounded convex neighborhood of 0.*

PROOF. If X is a normable topological vector space, let $\|\cdot\|$ be a norm on X that determines the topology. Then B_1 is clearly a bounded convex neighborhood of 0.

Conversely, let U be a bounded convex neighborhood of 0 in X. We may assume that U is symmetric, since, in any event, $U \cap (-U)$ is also bounded and convex. Let ρ be the seminorm (Minkowski functional) on X associated to U as in Theorem 3.6. We show first that ρ is actually a norm.

Thus, let $x \neq 0$ be given, and choose a convex neighborhood V of 0 such that $x \notin V$. Note that, if $tx \in V$, then $|t| < 1$. Choose $c > 0$ so that $U \subseteq cV$, and note that if $tx \in U$, then $tx \in cV$, whence $|t| < c$. Therefore, recalling the definition of $\rho(x)$,

$$\rho(x) = \frac{1}{\sup_{t>0, tx\in U} t},$$

we see that $\rho(x) \geq 1/c > 0$, showing that ρ is a norm.

We must show finally that the given topology agrees with the one defined by the norm ρ. Since, by Theorem 3.6, ρ is continuous, it follows immediately that $B_\epsilon = \rho^{-1}(-\infty, \epsilon)$ is open in the given topology, showing that the topology defined by the norm is contained in the given topology. Conversely, if V is an open subset of the given topology and $x \in V$, let W be a neighborhood of 0 such that $x + W \subseteq V$. Choose $c > 0$ so that $U \subseteq cW$. Again using Theorem 3.6, we see that $B_1 = \rho^{-1}(-\infty, 1) \subseteq U \subseteq cW$, whence $B_{1/c} = \rho^{-1}(-\infty, (1/c)) \subseteq W$, and $x + B_{1/c} \subseteq V$. This shows that V is open in the topology defined by the norm. Q.E.D.

EXERCISE 4.1. (a) (Characterization of Banach Spaces) Let X be a normed linear space. Show that X is a Banach space if and only if every absolutely summable infinite series in X is summable in X. (An infinite series $\sum x_n$ is *absolutely summable* in X if $\sum \|x_n\| < \infty$.) HINT: If $\{y_n\}$ is a Cauchy sequence in X, choose a subsequence $\{y_{n_k}\}$ for which $\|y_{n_k} - y_{n_{k+1}}\| < 2^{-k}$.

(b) Use part a to verify that all the spaces $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, are Banach spaces, as is $C_0(\Delta)$.

(c) If c_0 is the set of all sequences $\{a_n\}$, $n = 0, 1, \dots$, satisfying $\lim a_n = 0$, and if we define $\|\{a_n\}\| = \max |a_n|$, show that c_0 is a Banach space.

(d) Let X be the set of all continuous functions on $[0, 1]$, which are differentiable on $(0, 1)$. Set $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Show that X is a normed linear space but is not a Banach space.

(e) If $X_1, \dots, X_n$ are normed linear spaces, show that the direct sum $\bigoplus_{i=1}^n X_i$, equipped with the max norm, is a normed linear space. If each X_i is a Banach space, show that $\bigoplus_{i=1}^n X_i$ is a Banach space.

(f) Let $X_1, \dots, X_n$ be normed linear spaces. Let $x = (x_1, \dots, x_n)$ be in $\bigoplus_{i=1}^n X_i$, and define $\|x\|_1$ and $\|x\|_2$ by

$$\|x\|_1 = \sum_{i=1}^n \|x_i\|,$$

and

$$\|x\|_2 = \sqrt{\sum_{i=1}^n \|x_i\|^2}.$$

Prove that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\bigoplus_{i=1}^n X_i$. Show further that

$$\|x\| \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|.$$

(g) Let $\{X_i\}$ be an infinite sequence of nontrivial normed linear spaces. Prove that the direct product $\prod X_i$ is a metrizable, locally convex, topological vector space, but that there is no definition of a norm on $\prod X_i$ that defines its topology. HINT: In a normed linear space, given any bounded set A and any neighborhood U of 0, there exists a number t such that $A \subseteq tU$.

EXERCISE 4.2. (Schwartz Space $\mathcal{S}$ is Not Normable) Let $\mathcal{S}$ denote Schwartz space, and let $\{\rho_n\}$ be the seminorms (norms) that define the topology on $\mathcal{S}$:

$$\rho_n(f) = \sup_x \max_{0\leq i,j\leq n} |x^j f^{(i)}(x)|.$$

(a) If V is a neighborhood of 0 in $\mathcal{S}$, show that there exists an integer n and an $\epsilon > 0$ such that $\rho_n^{-1}(-\infty, \epsilon) \subseteq V$; i.e., if $\rho_n(h) < \epsilon$, then $h \in V$.

(b) Given the nonnegative integer n from part a, show that there exists a C^∞ function g such that $g(x) = 1/x^{n+1/2}$ for $x \geq 2$. Note that

$$\sup_x \max_{0\leq i,j\leq n} |x^j g^{(i)}(x)| < \infty.$$

(Of course, g is not an element of $\mathcal{S}$.)

(c) Let n be the integer from part a and let f be a C^∞ function with compact support such that $|f(x)| \leq 1$ for all x and $f(0) = 1$. For each integer $M > 0$, define $g_M(x) = g(x)f(x - M)$, where g is the function from part b. Show that each $g_M \in \mathcal{S}$ and that there exists a positive constant c such that $\rho_n(g_M) < c$ for all M; i.e., $(\epsilon/c)g_M \in V$ for all M. Further, show that for each $M \geq 2$, $\rho_{n+1}(g_M) \geq \sqrt{M}$.

(d) Show that the neighborhood V of 0 from part a is not bounded in $\mathcal{S}$. HINT: Define W to be the neighborhood $\rho_{n+1}^{-1}(-\infty, 1)$, and show that no multiple of W contains V.

(e) Conclude that $\mathcal{S}$ is not normable.

THEOREM 4.2. (Subspaces and Quotient Spaces) *Let X be a Banach space and let M be a closed linear subspace.*

(1) *M is a Banach space with respect to the restriction to M of the norm on X.*
(2) *If $x + M$ is a coset of M, and if $\|x + M\|$ is defined by*

$$\|x + M\| = \inf_{y \in x+M} \|y\| = \inf_{m \in M} \|x + m\|,$$

then the quotient space X/M is a Banach space with respect to this definition of norm.
(3) *The quotient topology on X/M agrees with the topology determined by the norm on X/M defined in part 2.*

PROOF. M is certainly a normed linear space with respect to the restricted norm. Since it is a closed subspace of the complete metric space X, it is itself a complete metric space, and this proves part 1.

We leave it to the exercise that follows to show that the given definition of $\|x + M\|$ does make X/M a normed linear space. Let us show that this metric space is complete. Thus let $\{x_n + M\}$ be a Cauchy sequence in X/M. It will suffice to show that some subsequence has a limit in X/M. We may replace this Cauchy sequence by a subsequence for which

$$\|(x_{n+1} + M) - (x_n + M)\| = \|(x_{n+1} - x_n) + M\| < 2^{-(n+1)}.$$

Then, we may choose elements $\{y_n\}$ of X such that for each $n \geq 1$ we have

$$y_n \in (x_{n+1} - x_n) + M,$$

and $\|y_n\| < 2^{-(n+1)}$. We choose y_0 to be any element of $x_1 + M$. If $z_N = \sum_{n=0}^{N} y_n$, then it follows routinely that $\{z_N\}$ is a Cauchy sequence in X, whence has a limit z. We claim that $z + M$ is the limit of the sequence $\{x_N + M\}$. Indeed,

$$\begin{aligned}\|(z+M)-(x_N+M)\| &= \|(z-x_N)+M\| \\ &= \inf_{y\in(z-x_N)+M} \|y\|.\end{aligned}$$

Since $z = \sum_{n=0}^{\infty} y_n$, and since $\sum_{n=0}^{N-1} y_n \in x_N + M$, It follows that $\sum_{n=N}^{\infty} y_n \in (z - x_N) + M$. Therefore,

$$\begin{aligned}\|(z+M)-(x_N+M)\| &\leq \|\sum_{n=N}^{\infty} y_n\| \\ &\leq \sum_{n=N}^{\infty} 2^{-(n+1)} \\ &= 2^{-N},\end{aligned}$$

completing the proof of part 2.

We leave part 3 to the exercise that follows.

EXERCISE 4.3. Let X and M be as in the preceding theorem.

(a) Verify that the definition of $\|x + M\|$, given in the preceding theorem, makes X/M into a normed linear space.

(b) Prove that the quotient topology on X/M agrees with the topology determined by the norm on X/M.

(c) Suppose X is a vector space, ρ is a seminorm on X, and $M = \{x : \rho(x) = 0\}$. Prove that M is a subspace of X. Define p on X/M by

$$p(x+M) = \inf_{m\in M} \rho(x+m).$$

Show that p is a norm on the quotient space X/M.

EXERCISE 4.4. (a) Suppose X and Y are topologically isomorphic normed linear spaces, and let S denote a linear isomorphism of X onto Y that is a homeomorphism. Prove that there exist positive constants C_1 and C_2 such that

$$\|x\| \leq C_1\|S(x)\|$$

and

$$\|S(x)\| \leq C_2\|x\|$$

for all $x \in X$. Deduce that, if two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ determine identical topologies on a vector space X, then there exist constants C_1 and C_2 such that

$$\|x\|_1 \leq C_1\|x\|_2 \leq C_2\|x\|_1$$

for all $x \in X$.

(b) Suppose S is a linear transformation of a normed linear space X into a topological vector space Y. Assume that $S(\overline{B}_1)$ contains a neighborhood U of 0 in Y. Prove that S is an open map of X onto Y.

We come next to one of the important applications of the Baire category theorem in functional analysis.

THEOREM 4.3. (Isomorphism Theorem) *Suppose S is a continuous linear isomorphism of a Banach space X onto a Banach space Y. Then S^{-1} is continuous, and X and Y are topologically isomorphic.*

PROOF. For each positive integer n, let A_n be the closure in Y of $S(\overline{B}_n)$. Then, since S is onto, $Y = \cup A_n$. Because Y is a complete metric space, it follows from the Baire category theorem that some A_n, say A_N, must have nonempty interior. Therefore, let $y_0 \in Y$ and $\epsilon > 0$ be such that $B_\epsilon(y_0) \subset A_N$. Let $x_0 \in X$ be the unique element for which $S(x_0) = y_0$, and let k be an integer larger than $\|x_0\|$. Then A_{N+k} contains $A_N - y_0$, so that the closed set A_{N+k} contains $\overline{B}_\epsilon(0)$. This implies that if $w \in Y$ satisfies $\|w\| \leq \epsilon$, and if δ is any positive number, then there exists an $x \in X$ for which $\|S(x) - w\| < \delta$ and $\|x\| \leq N + k$. Write $M = (N + k)/\epsilon$. It follows then by scaling that, given any $w \in Y$ and any $\delta > 0$, there exists an $x \in X$ such that $\|S(x) - w\| < \delta$ and $\|x\| \leq M\|w\|$. We will use the existence of such an x recursively below.

We now complete the proof by showing that

$$\|S^{-1}(w)\| \leq 2M\|w\|$$

for all $w \in Y$, which will imply that S^{-1} is continuous. Thus, let $w \in Y$ be given. We construct sequences $\{x_n\}$, $\{w_n\}$ and $\{\delta_n\}$ as follows: Set $w_1 = w$, $\delta_1 = \|w\|/2$, and choose x_1 so that $\|w_1 - S(x_1)\| < \delta_1$ and $\|x_1\| \leq M\|w_1\|$. Next, set $w_2 = w_1 - S(x_1)$, $\delta_2 = \|w\|/4$, and choose x_2 such that $\|w_2 - S(x_2)\| < \delta_2$ and $\|x_2\| \leq M\|w_2\| \leq (M/2)\|w\|$. Continuing inductively, we construct the sequences $\{w_n\}$, $\{\delta_n\}$ and $\{x_n\}$ so that

$$w_n = w_{n-1} - S(x_{n-1}),$$

$$\delta_n = \|w\|/2^n,$$

and x_n so that

$$\|w_n - S(x_n)\| < \delta_n$$

and

$$\|x_n\| \leq M\|w_n\| < (M/2^{n-1})\|w\|.$$

It follows that the infinite series $\sum x_n$ converges in X, its sequence of partial sums being a Cauchy sequence, to an element x and that $\|x\| \leq 2M\|w\|$. Also, $w_n = w - \sum_{i=1}^{n-1} S(x_i)$. So, since S is continuous and $0 = \lim w_n$, we have that $S(x) = S(\sum_{n=1}^{\infty} x_n) = \sum_{n=1}^{\infty} S(x_n) = w$. Finally, for any $w \in Y$, we have that

$$\|S^{-1}(w)\| = \|x\| \leq 2M\|w\|,$$

and the proof is complete.

THEOREM 4.4. (Open Mapping Theorem) *Let T be a continuous linear transformation of a Banach space X onto a Banach space Y. Then T is an open map.*

PROOF. Since T is continuous, its kernel M is a closed linear subspace of X. Let S be the unique linear transformation of X/M onto Y satisfying $T = S \circ \pi$, where π denotes the natural map of X onto X/M. Then, by Theorems 3.4 and 4.2, S is a continuous isomorphism of the Banach space X/M onto the Banach space Y. Hence, S is an open map, whence T is an open map.

THEOREM 4.5. (Closed Graph Theorem) *Suppose T is a linear transformation of a Banach space X into a Banach space Y, and assume that the graph G of T is a closed subset of the product Banach space $X \times Y = X \oplus Y$. Then T is continuous.*

PROOF. Since the graph G is a closed linear subspace of the Banach space $X \oplus Y$, it is itself a Banach space in the restricted norm (max norm) from $X \oplus Y$. The map S from G to X, defined by $S(x, T(x)) = x$, is therefore a norm-decreasing isomorphism of G onto X. Hence S^{-1} is continuous by the Isomorphism Theorem. The linear transformation P of $X \oplus Y$ into Y, defined by $P(x, y) = y$, is norm-decreasing whence continuous. Finally, $T = P \circ S^{-1}$, and so is continuous.

EXERCISE 4.5. (a) Let X be the vector space of all continuous functions on $[0, 1]$ that have uniformly continuous derivatives on $(0, 1)$. Define a norm on X by $\|f\| = \sup_{0<x<1} |f(x)| + \sup_{0<x<1} |f'(x)|$. Let Y be the vector space of all uniformly continuous functions on $(0, 1)$, equipped with the norm $\|f\| = \sup_{0<x<1} |f(x)|$. Define $T : X \to Y$ by $T(f) = f'$. Prove that X and Y are Banach spaces and that T is a continuous linear transformation.

(b) Now let X be the vector space of all absolutely continuous functions f on $[0,1]$, for which $f(0) = 0$ and whose derivative f' is in L^p (for some fixed $1 \leq p \leq \infty$). Define a norm on X by $\|f\| = \|f\|_p$. Let $Y = L^p$, and define $T : X \to Y$ by $T(f) = f'$. Prove that T is not continuous, but that the graph of T is closed in $X \times Y$. How does this example relate to the preceding theorem?

(c) Prove analogous results to Theorems 4.3, 4.4, and 4.5 for locally convex, Fréchet spaces.

DEFINITION. Let X and Y be normed linear spaces. By $L(X,Y)$ we shall mean the set of all continuous linear transformations from X into Y. We refer to elements of $L(X,Y)$ as *operators* from X to Y.

If $T \in L(X,Y)$, we define the *norm* of T, denoted by $\|T\|$, by

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|.$$

EXERCISE 4.6. Let X and Y be normed linear spaces.

(a) Let T be a linear transformation of X into Y. Verify that $T \in L(X,Y)$ if and only if

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\| < \infty.$$

(b) Let T be in $L(X,Y)$. Show that the norm of T is the infimum of all numbers M for which $\|T(x)\| \leq M\|x\|$ for all $x \in X$.

(c) For each $x \in X$ and $T \in L(X,Y)$, show that $\|T(x)\| \leq \|T\|\|x\|$.

THEOREM 4.6. *Let X and Y be normed linear spaces.*

1. *The set $L(X,Y)$ is a vector space with respect to pointwise addition and scalar multiplication. If X and Y are complex normed linear spaces, then $L(X,Y)$ is a complex vector space.*
2. *$L(X,Y)$, equipped with the norm defined above, is a normed linear space.*
3. *If Y is a Banach space, then $L(X,Y)$ is a Banach space.*

PROOF. We prove part 3 and leave parts 1 and 2 to the exercises. Thus, suppose Y is a Banach space, and let $\{T_n\}$ be a Cauchy sequence in $L(X,Y)$. Then the sequence $\{\|T_n\|\}$ is bounded, and we let M be a number for which $\|T_n\| \leq M$ for all n. For each $x \in X$, we have that $\|(T_n(x) - T_m(x))\| \leq \|T_n - T_m\|\|x\|$, whence the sequence $\{T_n(x)\}$ is a Cauchy sequence in the complete metric space Y. Hence there exists an element $T(x) \in Y$ such that $T(x) = \lim T_n(x)$. This mapping T, being the pointwise limit of linear transformations, is a linear transformation, and it is continuous, since $\|T(x)\| = \lim \|T_n(x)\| \leq M\|x\|$. Consequently, T is an element of $L(X, Y)$.

We must show finally that T is the limit in $L(X, Y)$ of the sequence $\{T_n\}$. To do this, let $\epsilon > 0$ be given, and choose an N such that $\|T_n - T_m\| < \epsilon/2$ if $n, m \geq N$. If $x \in X$ and $\|x\| \leq 1$, then

$$\begin{aligned}\|T(x) - T_n(x)\| &\leq \limsup_m \|T(x) - T_m(x)\| + \limsup_m \|T_m(x) - T_n(x)\| \\ &\leq 0 + \limsup_m \|T_m - T_n\|\|x\| \\ &\leq \epsilon/2,\end{aligned}$$

whenever $n \geq N$. Since this is true for an arbitrary x for which $\|x\| \leq 1$, it follows that

$$\|T - T_n\| \leq \epsilon/2 < \epsilon$$

whenever $n \geq N$, as desired.

EXERCISE 4.7. Prove parts 1 and 2 of Theorem 4.6.

The next theorem gives another application to functional analysis of the Baire category theorem.

THEOREM 4.7. (Uniform Boundedness Principle) *Let X be a Banach space, let Y be a normed linear space, and suppose $\{T_n\}$ is a sequence of elements in $L(X, Y)$. Assume that, for each $x \in X$, the sequence $\{T_n(x)\}$ is bounded in Y. (That is, the sequence $\{T_n\}$ is pointwise bounded.) Then there exists a positive constant M such that $\|T_n\| \leq M$ for all n. (That is, the sequence $\{T_n\}$ is uniformly bounded.)*

PROOF. For each positive integer j, let A_j be the set of all $x \in X$ such that $\|T_n(x)\| \leq j$ for all n. Then each A_j is closed ($A_j = \cap_n T_n^{-1}(\overline{B}_j)$), and $X = \cup A_j$. By the Baire category theorem, some A_j, say A_J, has nonempty interior. Let $\epsilon > 0$ and $x_0 \in X$ be such that A_J contains $B_\epsilon(x_0)$. It follows immediately that $A_J - x_0 \subseteq A_{2J}$, from which it follows that A_{2J} contains B_ϵ. Hence, if $\|z\| < \epsilon$, then $\|T_n(z)\| \leq 2J$ for all n.

Now, given a nonzero $x \in X$, we write $z = (\epsilon/2\|x\|)x$. So, for any n,

$$\begin{aligned}\|T_n(x)\| &= (2\|x\|/\epsilon)\|T_n(z)\| \\ &\leq (2\|x\|/\epsilon)(2J) \\ &= M\|x\|,\end{aligned}$$

where $M = 4J/\epsilon$. It follows then that $\|T_n\| \leq M$ for all n, as desired.

THEOREM 4.8. *Let X be a Banach space, let Y be a normed linear space, let $\{T_n\}$ be a sequence of elements of $L(X,Y)$, and suppose that $\{T_n\}$ converges pointwise to a function $T : X \to Y$. Then T is a continuous linear transformation of X into Y; i.e., the pointwise limit of a sequence of continuous linear transformations from a Banach space into a normed linear space is continuous and linear.*

PROOF. It is immediate that the pointwise limit (when it exists) of a sequence of linear transformations is again linear. Since any convergent sequence in Y, e.g., $\{T_n(x)\}$, is bounded, it follows from the preceding theorem that there exists an M so that $\|T_n\| \leq M$ for all n, whence $\|T_n(x)\| \leq M\|x\|$ for all n and all $x \in X$. Therefore, $\|T(x)\| \leq M\|x\|$ for all x, and this implies that T is continuous.

EXERCISE 4.8. (a) Extend the Uniform Boundedness Principle from a sequence to a set S of elements of $L(X,Y)$.

(b) Restate the Uniform Boundedness Principle for a sequence $\{f_n\}$ of continuous linear functionals, i.e., for a sequence in $L(X,\mathbb{R})$ or $L(X,\mathbb{C})$.

(c) Let c_c denote the vector space of all sequences $\{a_j\},\ j = 1, 2, \ldots$ that are eventually 0, and define a norm on c_c by

$$\|\{a_j\}\| = \max |a_j|.$$

Define a linear functional f_n on c_c by $f_n(\{a_j\}) = na_n$. Prove that the sequence $\{f_n\}$ is a sequence of continuous linear functionals that is pointwise bounded but not uniformly bounded in norm. Why doesn't this contradict the Uniform Boundedness Principle?

(d) Let c_c be as in part c. Define a sequence $\{f_n\}$ of linear functionals on c_c by $f_n(\{a_j\}) = \sum_{j=1}^n a_j$. Show that $\{f_n\}$ is a sequence of continuous linear functionals that converges pointwise to a discontinuous linear functional. Why doesn't this contradict Theorem 4.8?

(e) Let c_0 denote the Banach space of sequences $a_0, a_1, \ldots$ for which $\lim a_n = 0$, where the norm on c_0 is given by

$$\|\{a_n\}\| = \max |a_n|.$$

If $\alpha = \{n_1 < n_2 < \ldots < n_k\}$ is a finite set of positive integers, define f_α on c_0 by

$$f_\alpha(\{a_j\}) = f_{n_1,\ldots,n_k}(\{a_j\}) = n_1 a_{n_k}.$$

Show that each f_α is a continuous linear functional on c_0.

(f) Let D denote the set consisting of all the finite sets $\alpha = \{n_1 < n_2 < \ldots < n_k\}$ of positive integers. Using inclusion as the partial ordering on D, show that D is a directed set, and let $\{f_\alpha\}$ be the corresponding net of linear functionals, as defined in part e, on c_0. Show that $\lim_\alpha f_\alpha = 0$. Show also that the net $\{f_\alpha\}$ is not uniformly bounded in norm. Explain why this does not contradict part a of this exercise.

DEFINITION. A *Banach algebra* is a Banach space A on which there is also defined a binary operation $\times$ of multiplication that is associative, (left and right) distributive over addition, satisfies

$$\lambda(x \times y) = (\lambda x) \times y = x \times (\lambda y)$$

for all scalars λ and all $x, y \in A$, and for which $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

EXERCISE 4.9. Let X be a Banach space. Using composition of transformations as a multiplication, show that $L(X, X)$ is a Banach algebra.

EXERCISE 4.10. Let X be the Banach space $\mathbb{R}^2$ with respect to the usual norm

$$\|x\| = \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2},$$

and let $(1, 0)$ and $(0, 1)$ be the standard basis for X. Let T be an element of $L(X, X)$, and represent T by a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Compute the norm of T in terms of a, b, c, d. Can you do the same for $X = \mathbb{R}^3$?

EXERCISE 4.11. Let X be a normed linear space, let Y be a dense subspace of X, and let Z be a Banach space.

(a) If $T \in L(Y, Z)$, show that there exists a unique element $T' \in L(X, Z)$ such that the restriction of T' to Y is T. That is, T has a unique continuous extension to all of X.

(b) Show that the map $T \to T'$, of part a, is an isometric isomorphism of $L(Y, Z)$ onto $L(X, Z)$.

(c) Suppose $\{T_n\}$ is a uniformly bounded sequence of elements of $L(X, Z)$. Suppose that the sequence $\{T_n(y)\}$ converges for every $y \in Y$. Show that the sequence $\{T_n(x)\}$ converges for every $x \in X$.

EXERCISE 4.12. Let X be a normed linear space, and let $\overline{X}$ denote the completion of the metric space X (e.g., the space of equivalence classes of Cauchy sequences in X). Show that $\overline{X}$ is in a natural way a Banach space with X isometrically imbedded as a dense subspace.

THEOREM 4.9. (Hahn-Banach Theorem, Normed Linear Space Version) *Let Y be a subspace of a normed linear space X. Suppose f is a continuous linear functional on Y; i.e., $f \in L(Y, \mathbb{R})$. Then there exists a continuous linear functional g on X, i.e., an element of $L(X, \mathbb{R})$, such that*

(1) *g is an extension of f.*
(2) $\|g\| = \|f\|$.

PROOF. If ρ is defined on X by $\rho(x) = \|f\|\|x\|$, then ρ is a seminorm on X. Clearly,

$$f(y) \leq |f(y)| \leq \|f\|\|y\| = \rho(y)$$

for all $y \in Y$. By the seminorm version of the Hahn-Banach Theorem, there exists a linear functional g on X, which is an extension of f, such that $g(x) \leq \rho(x) = \|f\|\|x\|$, for all $x \in X$, and this implies that g is continuous, and $\|g\| \leq \|f\|$. Obviously $\|g\| \geq \|f\|$ since g is an extension of f.

EXERCISE 4.13. (a) Let X be a normed linear space and let $x \in X$. Show that $\|x\| = \sup_f f(x)$, where the supremum is taken over all continuous linear functionals f for which $\|f\| \leq 1$. Show, in fact, that this supremum is actually attained.

(b) Use part a to derive the integral form of Minkowski's inequality. That is, if (X, μ) is a σ-finite measure space, and $F(x, y)$ is a $\mu \times \mu$-measurable function on $X \times X$, then

$$(\int |\int F(x,y)\,dy|^p\,dx)^{1/p} \leq \int (\int |F(x,y)|^p\,dx)^{1/p}\,dy,$$

where $1 \leq p < \infty$.

(c) Let $1 \leq p < \infty$, and let X be the complex Banach space $L^p(\mathbb{R})$. Let p' be such that $1/p + 1/p' = 1$, and let D be a dense subspace of $L^{p'}(\mathbb{R})$. If $f \in X$, show that

$$\|f\|_p = \sup_{\|g\|_{p'}=1} |\int f(x)g(x)\,dx|.$$

EXERCISE 4.14. Let X and Y be normed linear spaces, and let $T \in L(X,Y)$. Prove that the norm of T is given by

$$\|T\| = \sup_x \sup_f |f(T(x))|,$$

where the supremum is taken over all $x \in X$, $\|x\| \leq 1$ and all $f \in L(Y,\mathbb{R})$ for which $\|f\| \leq 1$.

We close this chapter with a theorem from classical analysis.

THEOREM 4.10. (Riesz Interpolation Theorem) *Let D be the linear space of all complex-valued measurable simple functions on $\mathbb{R}$ that have compact support, and let T be a linear transformation of D into the linear space M of all complex-valued measurable functions on $\mathbb{R}$. Let $1 \leq p_0 < p_1 < \infty$ be given, and suppose that:*

(1) *There exist numbers q_0 and m_0, with $1 < q_0 \leq \infty$, such that $\|T(f)\|_{q_0} \leq m_0\|f\|_{p_0}$ for all $f \in D$; i.e., T has a unique extension to a bounded operator T_0 from L^{p_0} into L^{q_0}, and $\|T_0\| \leq m_0$.*
(2) *There exist numbers q_1 and m_1, with $1 < q_1 \leq \infty$, such that $\|T(f)\|_{q_1} \leq m_1\|f\|_{p_1}$ for all $f \in D$; i.e., T has a unique extension to a bounded operator T_1 from L^{p_1} into L^{q_1}, and $\|T_1\| \leq m_1$.*

Let p satisfy $p_0 < p < p_1$, and define $t \in (0,1)$ by

$$1/p = (1-t)/p_0 + t/p_1;$$

i.e.,

$$t = \frac{1/p - 1/p_0}{1/p_1 - 1/p_0}.$$

Now define q by

$$1/q = (1-t)/q_0 + t/q_1.$$

Then

$$\|T(f)\|_q \leq m_p\|f\|_p,$$

for all $f \in D$, where

$$m_p = m_0^{1-t} m_1^t.$$

Hence, T has a unique extension to a bounded operator T_p from L^p into L^q, and $\|T_p\| \leq m_p$.

PROOF. For any $1 < r < \infty$, we write r' for the conjugate number defined by $1/r + 1/r' = 1$. Let $f \in D$ be given, and suppose that

$\|f\|_p = 1$. If the theorem holds for all such f, it will hold for all $f \in D$. (Why?) Because $T(f)$ belongs to L^{q_0} and to L^{q_1} by hypothesis, it follows that $T(f) \in L^q$, so that it is only the inequality on the norms that we must verify. We will show that $|\int [T(f)](y)g(y)\,dy| \leq m_p$, whenever $g \in D \cap L^{q'}$ with $\|g\|_{q'} = 1$. This will complete the proof (see Exercise 4.13). Thus, let g be such a function. Write $f = \sum_{j=1}^n a_j \chi_{A_j}$ and $g = \sum_{k=1}^m b_k \chi_{B_k}$, for $\{A_j\}$ and $\{B_k\}$ disjoint bounded measurable sets and a_j and b_k nonzero complex numbers.

For each $z \in \mathbb{C}$, define

$$\alpha(z) = (1-z)/p_0 + z/p_1$$

and

$$\beta(z) = (1-z)/q_0' + z/q_1'.$$

Note that $\alpha(t) = 1/p$ and $\beta(t) = 1/q'$.

We extend the definition of the signum function to the complex plane as follows: If λ is a nonzero complex number, define $\operatorname{sgn}(\lambda)$ to be $\lambda/|\lambda|$. For each complex z, define the simple functions

$$f_z = \sum_{j=1}^n \operatorname{sgn}(a_j)|a_j|^{\alpha(z)/\alpha(t)} \chi_{A_j}$$

and

$$g_z = \sum_{k=1}^m \operatorname{sgn}(b_k)|b_k|^{\beta(z)/\beta(t)} \chi_{B_k},$$

and finally put

$$\begin{aligned} F(z) &= \int [T(f_z)](y) g_z(y)\,dy \\ &= \sum_{j=1}^n \sum_{k=1}^m \operatorname{sgn}(a_j)\operatorname{sgn}(b_k)|a_j|^{p\alpha(z)}|b_k|^{q'\beta(z)} \int [T(\chi_{A_j})](y)\chi_{B_k}(y)\,dy \\ &= \sum_{j=1}^n \sum_{k=1}^m c_{jk} e^{d_{jk} z}, \end{aligned}$$

where the c_{jk}'s are complex numbers and the d_{jk}'s are real numbers.

Observe that F is an entire function of the complex variable z, and that it is bounded on the closed strip $0 \leq \Re z \leq 1$. Note also that

$\int [T(f)](y)g(y)\,dy$, the quantity we wish to estimate, is precisely $F(t)$. Observe next that

$$\begin{aligned}
\sup_{s\in\mathbb{R}} |F(is)| &= \sup_s |\int [T(f_{is})](y) g_{is}(y)\,dy| \\
&\leq \sup_s (\int |[T(f_{is})](y)|^{q_0}\,dy)^{1/q_0} (\int |g_{is}(y)|^{q_0'}\,dy)^{1/q_0'} \\
&\leq \sup_s m_0 \|f_{is}\|_{p_0} \|g_{is}\|_{q_0'} \\
&= \sup_s m_0 (\int \sum_j |(|a_j|^{p_0\alpha(is)/\alpha(t)})| \chi_{A_j}(y)\,dy)^{1/p_0} \\
&\qquad \times (\int \sum_k |(|b_k|^{q_0'\beta(is)/\beta(t)})| \chi_{B_k}(y)\,dy)^{1/q_0'} \\
&= m_0 \sup_s \int \sum_j |a_j|^p \chi_{A_j}(y)\,dy)^{1/p_0} \\
&\qquad \times (\int \sum_k |b_k|^{q'} \chi_{B_k}(y)\,dy)^{1/q_0'} \\
&= m_0 \|f\|_p^{p/p_0} \|g\|_{q'}^{q'/q_0'} \\
&= m_0.
\end{aligned}$$

By a similar calculation, we see that

$$\sup_{s\in\mathbb{R}} |F(1+is)| \leq m_1.$$

The proof of the theorem is then completed by appealing to the lemma from complex variables that follows.

LEMMA. *Suppose F is a complex-valued function that is bounded and continuous on the closed strip $0 \leq \Re z \leq 1$ and analytic on the open strip $0 < \Re z < 1$. Assume that m_0 and m_1 are real numbers satisfying*

$$m_0 \geq \sup_{s\in\mathbb{R}} |F(is)|$$

and

$$m_1 \geq \sup_{s\in\mathbb{R}} |F(1+is)|.$$

Then

$$\sup_{s\in\mathbb{R}} |F(t+is)| \leq m_0^{1-t} m_1^t$$

for all $0 \leq t \leq 1$.

PROOF. We may assume that m_0 and m_1 are positive. Define a function G on the strip $0 \leq \Re z \leq 1$ by

$$G(z) = F(z)/m_0^{1-z} m_1^z.$$

Then G is continuous and bounded on this strip and is analytic on the open strip $0 < \Re z < 1$. It will suffice to prove that

$$\sup_{s \in \mathbb{R}} |G(t + is)| \leq 1.$$

For each positive integer n, define $G_n(z) = G(z)e^{z^2/n}$. Then each function G_n is continuous and bounded on the strip $0 \leq \Re z \leq 1$ and analytic on the open strip $0 < \Re z < 1$. Also, $G(z) = \lim G_n(z)$ for all z in the strip. It will suffice then to show that $\lim |G_n(z)| \leq 1$ for each z for which $0 < \Re z < 1$. Fix $z_0 = x_0 + iy_0$ in the open strip, and choose a $Y > |y_0|$ such that $|G_n(z)| = |G_n(x + iy)| = |G(z)|e^{(x^2 - y^2)/n} < 1$ whenever $|y| \geq Y$. Let Γ be the rectangular contour determined by the four points $(0, -Y),\ (1, -Y),\ (1, Y)$, and $(0, Y)$. Then, by the Maximum Modulus Theorem, we have

$$\begin{aligned} |G_n(z_0)| &\leq \max_{z \in \Gamma} |G_n(z)| \\ &\leq \max(1, \sup_{s \in \mathbb{R}} |G_n(1 + is)|\ ,\ 1, \sup_{s \in \mathbb{R}} |G_n(is)|) \\ &= e^{1/n}, \end{aligned}$$

proving that $\lim |G_n(z_0)| \leq 1$, and this completes the proof of the lemma.

EXERCISE 4.15. Verify that the Riesz Interpolation Theorem holds with $\mathbb{R}$ replaced by any regular σ-finite measure space.