CHAPTER IX

PROJECTION-VALUED MEASURES

DEFINITION Let S be a set and let $\mathcal{B}$ be a σ-algebra of subsets of S. We refer to the elements of $\mathcal{B}$ as *Borel* subsets of S and we call the pair $(S, \mathcal{B})$ a *Borel space.*

If H is a separable (complex) Hilbert space, we say that a mapping $E \to p_E$, of $\mathcal{B}$ into the set $\mathcal{P}$ of projections on H, is a *projection-valued measure* (or an *H-projection-valued measure*) on $(S, \mathcal{B})$ if:

(1) $p_S = I$, and $p_\emptyset = 0$.
(2) If $\{E_i\}$ is a countable collection of pairwise disjoint elements of $\mathcal{B}$, then $\{p_{E_i}\}$ is a pairwise orthogonal collection of projections, and $p_{\cup E_i} = \sum p_{E_i}$.

If p $(E \to p_E)$ is an H-projection-valued measure and M is a closed subspace of H, for which $p_E(M) \subseteq M$ for all $E \in \mathcal{B}$, then M is called an *invariant subspace* for p or simply a *p-invariant* subspace. The assignment $E \to (p_E)|_M$ is called the *restriction* of p to M. See Exercise 9.1.

Two functions f and g on S are said to *agree a.e.p* if the set E of all x for which $f(x) \neq g(x)$ satisfies $p_E = 0$.

A function $f : S \to \mathbb{C}$ is called a *Borel function* or *$\mathcal{B}$-measurable* if $f^{-1}(U) \in \mathcal{B}$ whenever U is an open subset of $\mathbb{C}$. A complex-valued $\mathcal{B}$-measurable function f is said to belong to $L^\infty(p)$ if there exists a positive real number M such that

$$p_{|f|^{-1}(M,\infty)} = p_{\{x:|f(x)|>M\}} = 0,$$

and the L^∞ *norm* (really only a seminorm) of such a function f is defined to be the infimum of all such numbers M. By $L^\infty(p)$, we mean the vector space (or algebra) of all L^∞ functions f equipped with the ∞-norm. See Exercise 9.1.

If H and H' are two separable Hilbert spaces, and if $E \to p_E$ is an H-projection-valued measure and $E \to p'_E$ is an H'-projection-valued measure, both defined on the same Borel space $(S, \mathcal{B})$, we say that p and p' are *unitarily equivalent* if there exists a unitary transformation $U : H \to H'$ such that

$$U \circ p_E \circ U^{-1} = p'_E$$

for every $E \in \mathcal{B}$.

If we are thinking of the set $\mathcal{P}$ as a model for the set Q of all questions (see Chapter VII), and the Borel space S is the real line $\mathbb{R}$, then the set of projection-valued measures will correspond to the set O of all observables.

EXERCISE 9.1. Let $E \to p_E$ be a projection-valued measure on $(S, \mathcal{B})$.

(a) If $E \in \mathcal{B}$, show that $p_{\tilde{E}} = I - p_E$.

(b) If $E, F \in \mathcal{B}$, show that $p_{E\cap F} = p_E p_F$. HINT: Show first that if $E \cap F = \emptyset$, then p_E and p_F are orthogonal, i.e., that $p_E p_F = p_F p_E = 0$.

(c) If S is the increasing union $\cup E_n$ of elements of $\mathcal{B}$, show that the union of the ranges of the projections p_{E_n} is dense in H. HINT: Write $F_1 = E_1$, and for $n > 1$ define $F_n = E_n - E_{n-1}$. Note that $S = \cup F_n$, whence $x = \sum p_{F_n}(x)$ for each $x \in H$.

(d) Suppose $\{E_n\}$ is a sequence of elements of $\mathcal{B}$ for which $p_{E_n} = 0$ for all n. Prove that $p_{\cup E_n} = 0$.

(e) Show that $\|f\|_\infty$ is a seminorm on $L^\infty(p)$. Show further that $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$ for all $f, g \in L^\infty(p)$. If M denotes the subset of $L^\infty(p)$ consisting of the functions f for which $\|f\|_\infty = 0$, i.e., the functions that are 0 a.e.p, prove that $L^\infty(p)/M$ is a Banach space (even a Banach algebra). See part c of Exercise 4.3. Sometimes the notation $L^\infty(p)$ stands for this Banach space $L^\infty(p)/M$.

(f) Suppose M is a closed invariant subspace for p. Show that the assignment $E \to (p_E)|_M$ is an M-projection-valued measure.

(g) Let $\{H_i\}$ be a sequence of separable Hilbert spaces, and for each i let $E \to p^i_E$ be an H_i-projection-valued measure on the Borel space $(S, \mathcal{B})$. Let $H = \bigoplus H_i$ be the Hilbert space direct sum of the H_i's, and define a map $E \to p_E$ of $\mathcal{B}$ into the set of projections on H by

$$p_E = \sum_i p_E^i.$$

Prove that $E \to p_E$ is a projection-valued measure. This projection-valued measure is called the *direct sum* of the projection-valued measures $\{p^i\}$.

THEOREM 9.1. *Let $(S, \mathcal{B})$ be a Borel space, let H be a separable Hilbert space, and let $E \to p_E$ be an H-projection-valued measure on $(S, \mathcal{B})$. If $x \in H$, define μ_x on $\mathcal{B}$ by*

$$\mu_x(E) = (p_E(x), x).$$

Then μ_x is a finite positive measure on the σ-algebra $\mathcal{B}$ and $\mu_x(S) = \|x\|^2$.

EXERCISE 9.2. (a) Prove Theorem 9.1.

(b) Show that each measure μ_x, as defined in the preceding theorem, is absolutely continuous with respect to p. That is, show that if $p_E = 0$ then $\mu_x(E) = 0$.

(c) Let $S, \mathcal{B}, H$ and p be as in the preceding theorem. If x and y are vectors in H, and if $\mu_{x,y}$ is defined on $\mathcal{B}$ by

$$\mu_{x,y}(E) = (p_E(x), y),$$

show that $\mu_{x,y}$ is a finite complex measure on $\mathcal{B}$. Show also that

$$\|\mu_{x,y}\| \leq \|x\|\|y\|.$$

See Exercise 5.12.

(d) Let $S, \mathcal{B}, H, p$, and μ_x be as in the preceding theorem. Suppose p' is any H-projection-valued measure on $\mathcal{B}$ for which $\mu_x(E) = (p'_E(x), x)$ for all $x \in H$. Show that $p' = p$. That is, the measures $\{\mu_x\}$ uniquely determine the projection-valued measure p.

(e) Let ϕ be a $\mathcal{B}$-measurable simple function on S, and suppose

$$\phi = \sum_{i=1}^n a_i \chi_{E_i}$$

and

$$\phi = \sum_{j=1}^m b_j \chi_{F_j}$$

are two different representations of ϕ as finite linear combinations of characteristic functions of elements of $\mathcal{B}$. Prove that for each $x \in H$, we have

$$\sum_{i=1}^{n} a_i p_{E_i}(x) = \sum_{j=1}^{m} b_j p_{F_j}(x).$$

HINT: Show this by taking inner products.

THEOREM 9.2. *Let (S, μ) be a σ-finite measure space, let $\mathcal{B}$ be the σ-algebra of μ-measurable subsets of S, and let $H = L^2(\mu)$. For each measurable set $E \subseteq S$, define p_E to be the projection in $B(H)$ given by $p_E = m_{\chi_E}$. That is,*

$$p_E(f) = \chi_E f.$$

Then $E \to p_E$ is a projection-valued measure on H.

DEFINITION. The projection-valued measure of the preceding theorem is called the *canonical projection-valued measure* on $L^2(\mu)$.

EXERCISE 9.3. (a) Prove Theorem 9.2.

(b) Let U denote the L^2 Fourier transform on $L^2(\mathbb{R})$, and, for each Borel subset $E \subseteq \mathbb{R}$, define an operator p_E on $L^2(\mathbb{R})$ by

$$p_E(f) = U^{-1}(\chi_E U(f)).$$

Show that each operator p_E is a projection on $L^2(\mathbb{R})$ and that $E \to p_E$ is a projection-valued measure. Note that this projection-valued measure is unitarily equivalent to the canonical one on $L^2(\mathbb{R})$. Show that $p_{[-1,1]}$ can be expressed as a convolution operator:

$$p_{[-1,1]}f(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds,$$

where k is a certain L^2 function.

(c) Let $(S, \mathcal{B})$ and $(S', \mathcal{B}')$ be two Borel spaces, and let h be a map of S into S' for which $h^{-1}(E') \in \mathcal{B}$ whenever $E' \in \mathcal{B}'$. Such a map h is called a *Borel* map of S into S'. Suppose $E \to p_E$ is an H-projection-valued measure on $(S, \mathcal{B})$, and define a map $E' \to q_{E'}$ on $\mathcal{B}'$ by

$$q_{E'} = p_{h^{-1}(E')}.$$

Prove that $E' \to q_{E'}$ is an H-projection-valued measure on $(S', \mathcal{B}')$. This projection-valued measure q is frequently denoted by $h_*(p)$.

EXERCISE 9.4. Let (S, μ) be a σ-finite measure space, and let $E \to p_E$ be the canonical projection-valued measure on $L^2(\mu)$. Prove that there exists a vector f in $L^2(\mu)$ such that the linear span of the vectors $p_E(f)$, for E running over the μ-measurable subsets of S, is dense in $L^2(\mu)$. HINT: Do this first for a finite measure μ.

DEFINITION. Let $(S, \mathcal{B})$ be a Borel space, let H be a separable Hilbert space, and let $E \to p_E$ be an H-projection-valued measure on $(S, \mathcal{B})$. A vector $x \in H$ is called a *cyclic vector* for p if the linear span of the vectors $p_E(x)$, for $E \in \mathcal{B}$, is dense in H.

A vector x is a *separating vector* for p if: $p_E = 0$ if and only if $p_E(x) = 0$.

A vector x is a *supporting vector* for p if the measure μ_x of Theorem 9.1 satisfies: $\mu_x(E) = 0$ if and only if $p_E = 0$.

EXERCISE 9.5. (a) Show that a canonical projection-valued measure has a cyclic vector. (See Exercise 9.4.)

(b) Show that every cyclic vector for a projection-valued measure is a separating vector.

(c) Show that a vector x is a separating vector for a projection-valued measure if and only if it is a supporting vector.

(d) Give an example to show that not every separating vector need be cyclic. HINT: Use a one-point set S and a 2 dimensional Hilbert space.

THEOREM 9.3. *An H-projection-valued measure $E \to p_E$ on a Borel space $(S, \mathcal{B})$ has a cyclic vector if and only if there exists a finite measure μ on $(S, \mathcal{B})$ such that p is unitarily equivalent to the canonical projection-valued measure on $L^2(\mu)$.*

PROOF. The "if" part follows from part a of Exercise 9.5. Conversely, let x be a cyclic vector for p and write μ for the (finite) measure μ_x of Theorem 9.1 on $\mathcal{B}$. For each $\mathcal{B}$-measurable simple function $\phi = \sum a_i \chi_{E_i}$ on S, define $U(\phi) \in H$ by

$$U(\phi) = \sum a_i p_{E_i}(x).$$

Then $U(\phi)$ is well-defined by part e of Exercise 9.2, and the range of U is dense in H because x is a cyclic vector. It follows directly that U is a well-defined linear transformation of the complex vector space X of all simple $\mathcal{B}$-measurable functions on S into H. Furthermore, writing

$\phi = \sum a_i \chi_{E_i}$, where $E_i \cap E_j = \emptyset$ for $i \neq j$, then

$$\begin{aligned}
\|U(\phi)\|^2 &= (\sum_i a_i p_{E_i}(x)\,,\, \sum_j a_j p_{E_j}(x)) \\
&= \sum \sum a_i \overline{a_j} (p_{E_i}(x), p_{E_j}(x)) \\
&= \sum \sum a_i \overline{a_j} (p_{E_j \cap E_i}(x), x) \\
&= \sum |a_i|^2 (p_{E_i}(x), x) \\
&= \sum |a_i|^2 \mu(E_i) \\
&= \int |\phi|^2 \, d\mu \\
&= \|\phi\|_2^2,
\end{aligned}$$

showing that U is an isometry of X onto a dense subspace of H.

Therefore, U has a unique extension from the dense subspace X to a unitary operator from all of $L^2(\mu)$ onto all of H.

Finally, if p' denotes the canonical projection-valued measure on $L^2(\mu)$, $\phi = \sum a_i \chi_{E_i}$ is an element of X, and $y = U(\phi)$ is the corresponding element in the range of U on X, we have

$$\begin{aligned}
(U \circ p'_E \circ U^{-1})(y) &= (U \circ p'_E)(\phi) \\
&= U(\chi_E \phi) \\
&= U(\chi_E \sum a_i \chi_{E_i}) \\
&= U(\sum a_i \chi_{E \cap E_i}) \\
&= \sum a_i p_{E \cap E_i}(x) \\
&= \sum a_i p_E(p_{E_i}(x)) \\
&= p_E(\sum a_i p_{E_i}(x)) \\
&= p_E(U(\phi)) \\
&= p_E(y),
\end{aligned}$$

which shows that $U \circ p'_E \circ U^{-1}$ and p_E agree on a dense subspace of H, whence are equal everywhere. This completes the proof.

EXERCISE 9.6. Let $E \to p_E$ be an H-projection-valued measure.

(a) Let x be an element of H, and let M be the closed linear span of the vectors $p_E(x)$ for $E \in \mathcal{B}$. Prove that M is invariant under p, and that the restriction of p to M has a cyclic vector.

(b) Use the Hausdorff Maximality Principle to prove that there exists a sequence $\{M_i\}$ of pairwise orthogonal closed p-invariant subspaces of H, such that $E \to (p_E)|_{M_i}$ has a cyclic vector for each i, and such that H is the Hilbert space direct sum $\bigoplus M_i$.

We next take up the notion of integrals with respect to a projection-valued measure.

THEOREM 9.4. *Let p be an H-projection-valued measure on a Borel space $(S, \mathcal{B})$. Let ϕ be a $\mathcal{B}$-measurable simple function, and suppose that*

$$\phi = \sum a_i \chi_{E_i} = \sum b_j \chi_{F_j},$$

where each E_i and F_j are elements of $\mathcal{B}$ and each a_i and b_j are complex numbers. Then

$$\sum a_i p_{E_i} = \sum b_j p_{F_j}.$$

EXERCISE 9.7. Prove Theorem 9.4.

DEFINITION. If p is an H-projection-valued measure on a Borel space $(S, \mathcal{B})$, and ϕ is a $\mathcal{B}$-measurable simple function on S, we define an operator, which we denote by $\int \phi \, dp$, on H by

$$\int \phi \, dp = \sum a_i p_{E_i},$$

where $\phi = \sum a_i \chi_{E_i}$. This operator is well-defined in view of the preceding theorem.

THEOREM 9.5. *Let p be an H-projection-valued measure on a Borel space $(S, \mathcal{B})$, and let X denote the space of all $\mathcal{B}$-measurable simple functions on S. Then the map L that sends ϕ to $\int \phi \, dp$ has the following properties:*

(1) *$L(\phi) = \int \phi \, dp$ is a bounded operator on H, and*

$$\|L(\phi)\| = \|\int \phi \, dp\| = \|\phi\|_\infty.$$

(2) *L is linear; i.e.,*

$$\int (\phi + \psi) \, dp = \int \phi \, dp + \int \psi \, dp$$

and

$$\int \lambda\phi\, dp = \lambda \int \phi\, dp$$

for all complex numbers λ *and all* $\phi, \psi \in X$.

(3) L *is multiplicative; i.e.,*

$$\int (\phi\psi)\, dp = \int \phi\, dp \circ \int \psi\, dp$$

for all $\phi, \psi \in X$.

(4) L *is essentially 1-1, i.e.;* $\int \phi\, dp = \int \psi\, dp$ *if and only if* $\phi = \psi$ *a.e.p.*

(5) *For each* $\phi \in X$, *we have*

$$(\int \phi\, dp)^* = \int \overline{\phi}\, dp,$$

whence $\int \phi\, dp$ *is selfadjoint if and only if* ϕ *is real-valued a.e.p.*

(6) $\int \phi\, dp$ *is a positive operator if and only if* ϕ *is nonnegative a.e.p.*

(7) $\int \phi\, dp$ *is unitary if and only if* $|\phi| = 1$ *a.e.p.*

(8) $\int \phi\, dp$ *is a projection if and only if* $\phi^2 = \phi$ *a.e.p; i.e., if and only if* ϕ *agrees with a characteristic function a.e.p.*

PROOF. Let x and y be unit vectors in H, and let $\mu_{x,y}$ be the complex measure on S defined in part c of Exercise 9.2. Then

$$\begin{aligned} |([\int \phi\, dp](x), y)| &= |(\sum a_i p_{E_i}(x), y)| \\ &= |\sum a_i \mu_{x,y}(E_i)| \\ &= |\int \phi\, d\mu_{x,y}| \\ &\leq \|\phi\|_\infty \|\mu_{x,y}\| \\ &\leq \|\phi\|_\infty, \end{aligned}$$

showing that $\int \phi\, dp$ is a bounded operator and that $\|\int \phi\, dp\| \leq \|\phi\|_\infty$. See part c of Exercise 9.2 and part c of Exercise 5.12. On the other hand, we may assume that the sets $\{E_i\}$ are pairwise disjoint, that $p_{E_1} \neq 0$, and that $|a_1| = \|\phi\|_\infty$. Choosing x to be any unit vector in the range of

p_{E_1}, we see that

$$\begin{aligned}[\int \phi\, dp](x) &= \sum a_i p_{E_i}(p_{E_1}(x)) \\ &= \sum a_i p_{E_i \cap E_1}(x) \\ &= a_1 p_{E_1}(x) \\ &= a_1 x,\end{aligned}$$

showing that $\|[\int \phi\, dp](x)\| = \|\phi\|_\infty$, and this finishes the proof of part 1.

Part 2 is left to the exercises.

To see part 3, write $\phi = \sum_{i=1}^n a_i \chi_{E_i}$, and $\psi = \sum_{j=1}^m b_j \chi_{F_j}$. Then

$$\begin{aligned}\int \phi\psi\, dp &= \int (\sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{E_i} \chi_{F_j})\, dp \\ &= \int (\sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{E_i \cap F_j})\, dp \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j p_{E_i \cap F_j} \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j p_{E_i} p_{F_j} \\ &= \sum_{i=1}^n a_i p_{E_i} \circ \sum_{j=1}^m b_j p_{F_j} \\ &= \int \phi\, dp \circ \int \psi\, dp,\end{aligned}$$

proving part 3.

We have next that $\int \phi\, dp = \int \psi\, dp$ if and only if

$$([\int \phi\, dp](x), x) = ([\int \psi\, dp](x), x)$$

for every $x \in H$. Therefore $\int \phi\, dp = \int \psi\, dp$ if and only if $\int \phi\, d\mu_x = \int \psi\, d\mu_x$ for every $x \in H$. If $\phi = \psi$ a.e.p, then $\phi = \psi$ a.e.μ_x for every $x \in H$, whence $\int \phi\, d\mu_x = \int \psi\, d\mu_x$ for all x, and $\int \phi\, dp = \int \psi\, dp$. Conversely, if ϕ and ψ are not equal a.e.p, then, without loss of generality, we may assume that there exists a set $E \subseteq S$ and a $\delta > 0$ such that $\phi(s) - \psi(s) > \delta$ for all $s \in E$ and $p_E \neq 0$. Letting x be a unit vector in the range of the projection p_E, we have that

$$\begin{aligned}
([\int \phi\,dp](x), x) - ([\int \psi\,dp](x), x) &= ([\int (\phi - \psi)\,dp](x), x) \\
&= ([\int (\phi - \psi)\,dp](p_E(x)), x) \\
&= ([\int (\phi - \psi)\,dp][\int \chi_E\,dp](x), x) \\
&= ([\int (\phi - \psi)\chi_E\,dp](x), x) \\
&= \int (\phi - \psi)\chi_E\,d\mu_x \\
&\geq \int \delta\chi_E\,d\mu_x \\
&= \delta \int \chi_E\,d\mu_x \\
&= \delta(p_E(x), x) \\
&= \delta(x, x) \\
&> 0,
\end{aligned}$$

proving that $\int \phi\,dp \neq \int \psi\,dp$, which gives part 4.

To see part 5, let x and y be arbitrary vectors in H. Then

$$\begin{aligned}
([\int \phi\,dp]^*(x), y) &= (x, [\int \phi\,dp](y)) \\
&= (x, (\sum a_i p_{E_i}(y))) \\
&= \sum \overline{a_i}(x, p_{E_i}(y)) \\
&= \sum \overline{a_i}(p_{E_i}(x), y) \\
&= ((\sum \overline{a_i} p_{E_i})(x), y) \\
&= ([\int \overline{\phi}\,dp](x), y).
\end{aligned}$$

Parts 6, 7, and 8 now follow from parts 4 and 5, and we leave the details to the exercises.

EXERCISE 9.8. Prove parts 2,6,7, and 8 of Theorem 9.5.

THEOREM 9.6. *Let p be an H-projection-valued measure on a Borel space $(S,\mathcal{B})$. Then the map $\phi \to L(\phi) = \int \phi\, dp$, of the space X of all $\mathcal{B}$-measurable simple functions on S into $B(H)$, extends uniquely to a map (also called L) of $L^\infty(p)$ into $B(H)$ that satisfies:*

(1) *L is linear.*
(2) *L is multiplicative; i.e., $L(fg) = L(f)L(g)$ for all $f, g \in L^\infty(p)$.*
(3) *$\|L(f)\| = \|f\|_\infty$ for all $f \in L^\infty(p)$.*

EXERCISE 9.9. (a) Prove Theorem 9.6.

(b) If M denotes the subspace of $L^\infty(p)$ consisting of the functions f for which $f = 0$ a.e.p, show that the map L of Theorem 9.6 induces an isometric isomorphism of the Banach algebra $L^\infty(p)/M$. See part e of Exercise 9.1.

DEFINITION. If $f \in L^\infty(p)$, for p an H-projection-valued measure on $(S,\mathcal{B})$, we denote the bounded operator that is the image of f under the isometry L of the preceding theorem by $\int f\, dp$ or $\int f(s)\, dp(s)$, and we call it the *integral* of f with respect to the projection-valued measure p.

EXERCISE 9.10. Verify the following properties of the integral with respect to a projection-valued measure p.

(a) Suppose $f \in L^\infty(p)$ and $x, y \in H$. Then the matrix coefficient $([\int f\, dp](x), y)$ is given by

$$([\int f\, dp](x), y) = \int f\, d\mu_{x,y},$$

where $\mu_{x,y}$ is the complex measure defined in part c of Exercise 9.2.

(b) $[\int f\, dp]^* = \int \overline{f}\, dp$, whence $\int f\, dp$ is selfadjoint if and only if f is real-valued a.e.p.

(c) $\int f\, dp$ is a unitary operator if and only if $|f| = 1$ a.e.p.

(d) $\int f\, dp$ is a positive operator if and only if f is nonnegative a.e.p.

(e) We say that an element f in $L^\infty(p)$ is *essentially* bounded away from 0 if and only if there exists a $\delta > 0$ such that

$$p_{f^{-1}(B_\delta(0))} = 0.$$

Show that $\int f\, dp$ is invertible in $B(H)$ if and only if f is essentially bounded away from zero. HINT: If f is not essentially bounded away from 0, let $\{x_n\}$ be a sequence of unit vectors for which x_n belongs to the range of the projection $p_{f^{-1}(B_{1/n}(0))}$. Show that

$$\|[\int f\,dp](x_n)\| \leq 1/n,$$

so that no inverse of $\int f\,dp$ could be bounded.

EXERCISE 9.11. Let p be a projection-valued measure on the Borel space $(S, \mathcal{B})$.

(a) Suppose there exists a point $s \in S$ for which $p_{\{s\}} \neq 0$. Show that, for each $f \in L^\infty(p)$, the operator $\int f\,dp$ has an eigenvector belonging to the eigenvalue $\lambda = f(s)$. Indeed, any nonzero vector in the range of $p_{\{s\}}$ will suffice.

(b) Let f be an element of $L^\infty(p)$, let λ_0 be a complex number, let $\epsilon > 0$ be given, and write $B_\epsilon(\lambda_0)$ for the open ball of radius ϵ around λ_0. Define $E = f^{-1}(B_\epsilon(\lambda_0))$, and let x be a vector in H. Prove that x belongs to the range of p_E if and only if

$$\lim_{n\to\infty} \frac{1}{\epsilon^n} \|(\int f\,dp - \lambda_0 I)^n(x)\| = 0.$$

If x is in the range of p_E, show that

$$(|\lambda_0| - \epsilon)\|x\| \leq \|[\int f\,dp](x)\| \leq (|\lambda_0| + \epsilon)\|x\|.$$

More particularly, suppose f is real-valued, that $0 < a < b \leq \infty$, and let $E = f^{-1}(a, b)$. If x is in the range of p_E, show that

$$a\|x\| \leq \|[\int f\,dp](x)\| \leq b\|x\|.$$

(c) Suppose $f \in L^\infty(p)$ is such that the operator $T = \int f\,dp$ has an eigenvector with eigenvalue λ. Define $E = f^{-1}(\{\lambda\})$. Prove that $p_E \neq 0$, and show further that $x \in H$ is an eigenvector for T belonging to the eigenvalue λ if and only if x belongs to the range of p_E.

EXERCISE 9.12. Let $E \to p_E$ be the canonical projection-valued measure on $L^2(\mu)$. Verify that $\int f\,dp$ is the multiplication operator m_f for every $f \in L^\infty(p)$. HINT: Do this first for characteristic functions χ_E.

EXERCISE 9.13. (Change of Variables) Let $(S, \mathcal{B})$ and $(S', \mathcal{B}')$ be two Borel spaces, and let h be a Borel map from S into S'; i.e., h

maps S into S' and $h^{-1}(E') \in \mathcal{B}$ whenever $E' \in \mathcal{B}'$. Suppose p is a projection-valued measure on $(S, \mathcal{B})$, and as in part c of Exercise 9.3 define a projection-valued measure $q = h_*(p)$ on $(S', \mathcal{B}')$ by

$$q_E = p_{h^{-1}(E)}.$$

If f is any bounded $\mathcal{B}'$-measurable function on S', show that

$$\int f\, dq = \int (f \circ h)\, dp.$$

HINT: Check this equality for characteristic functions, then simple functions, and finally bounded functions.

THEOREM 9.7. (A "Riesz" Representation Theorem) *Let Δ be a second countable compact Hausdorff space, let H be a separable Hilbert space, and let T be a linear transformation from the complex normed linear space $C(\Delta)$ of all continuous complex-valued functions on Δ into $B(H)$. Assume that T satisfies*

(1) *$T(fg) = T(f) \circ T(g)$ for all $f, g \in C(\Delta)$.*
(2) *$T(\overline{f}) = [T(f)]^*$ for all $f \in C(\Delta)$.*
(3) *$T(1) = I$, where 1 denotes the identically 1 function and I denotes the identity operator on H.*

Then there exists a unique projection-valued measure $E \to p_E$ from the σ-algebra $\mathcal{B}$ of Borel subsets of Δ such that

$$T(f) = \int f\, dp$$

for every $f \in C(\Delta)$.

PROOF. Note first that assumptions 1 and 2 imply that $T(f)$ is a positive operator if $f \geq 0$. Consequently, since $|f(s)|^2 \leq \|f\|_\infty^2$, we have that $\|f\|_\infty^2 I - T(\bar{f}) \circ T(f)$ is a positive operator. Hence,

$$\|f\|_\infty^2 \|x\|^2 \geq ([T(\bar{f}) \circ T(f)](x), x) = \|[T(f)](x)\|^2,$$

showing that $\|T(f)\| \leq \|f\|_\infty$ for all $f \in C(\Delta)$. That is, T is a bounded linear transformation of norm ≤ 1.

Next, for each pair (x, y) of vectors in H, define $\phi_{x,y}$ on $C(\Delta)$ by

$$\phi_{x,y}(f) = (T(f)(x), y).$$

Then $\phi_{x,y}$ is a bounded linear functional on $C(\Delta)$, and we write $\nu_{x,y}$ for the unique finite complex Borel measure on Δ for which

$$\phi_{x,y}(f) = \int f\, d\nu_{x,y}$$

for all $f \in C(\Delta)$. See Theorem 1.5 and Exercise 1.12. We see immediately that

(1) The linear functional $\phi_{x,x}$ is a positive linear functional, whence the measure $\nu_{x,x}$ is a positive measure.
(2) For each fixed $y \in H$, the map $x \to \nu_{x,y}$ is a linear transformation of H into the vector space $M(\Delta)$ of all finite complex Borel measures on Δ.
(3) $\nu_{x,y} = \overline{\nu_{y,x}}$ for all $x, y \in H$.
(4) $\|\nu_{x,y}\| = \|\phi_{x,y}\| \leq \|x\|\|y\|$.

For each bounded, real-valued, Borel function h on Δ, consider the map $L_h : H \times H \to \mathbb{C}$ given by

$$L_h(x, y) = \int h\, d\nu_{x,y}.$$

It follows from the results above that for each fixed $y \in H$ the map $x \to L_h(x, y)$ is linear. Also,

$$\begin{aligned} L_h(y, x) &= \int h\, d\nu_{y,x} \\ &= \int h\, d\overline{\overline{\nu}}_{y,x} \\ &= \overline{\int \bar{h}\, d\bar{\nu}_{y,x}} \\ &= \overline{\int h\, d\nu_{x,y}} \\ &= \overline{L_h(x, y)} \end{aligned}$$

for all $x, y \in H$. Furthermore, using Exercise 5.12 we have that

$$\begin{aligned} |L_h(x, y)| &= |\int h\, d\nu_{x,y}| \\ &\leq \|h\|_\infty \|\nu_{x,y}\| \\ &\leq \|h\|_\infty \|x\|\|y\|. \end{aligned}$$

Now, using Theorem 8.5, let $T(h)$ be the unique bounded operator on H for which

$$L_h(x,y) = (T(h)(x),y)$$

for all $x, y \in H$. Note that since the measures $\nu_{x,x}$ are positive measures, it follows that the matrix coefficients

$$(T(h)(x),x) = L_h(x,x) = \int h\,d\nu_{x,x}$$

are all real, implying that the operator $T(h)$ is selfadjoint.

If E is a Borel subset of Δ, set $p_E = T(\chi_E)$. We will eventually see that the assignment $E \to p_E$ is a projection-valued measure on $(\Delta, \mathcal{B})$.

Fix $g \in C(\Delta)$ and $x, y \in H$. Note that the two bounded linear functionals

$$f \to \int fg\,d\nu_{x,y} = \phi_{x,y}(fg) = (T(fg)(x),y)$$

and

$$f \to \int f\,d\nu_{T(g)(x),y} = \phi_{T(g)(x),y}(f) = (T(f)(T(g)(x)),y)$$

agree on $C(\Delta)$. Since they are both represented by integrals (Theorem 1.5), it follows that

$$\int hg\,d\nu_{x,y} = \int h\,d\nu_{T(g)(x),y}$$

for every bounded Borel function h. Now, for each fixed bounded, real-valued, Borel function h and each pair $x, y \in H$, the two bounded linear functionals

$$g \to \int gh\,d\nu_{x,y} = \int hg\,d\nu_{x,y}$$

and

$$\begin{aligned} g \to \int h\,d\nu_{T(g)(x),y} &= (T(h)(T(g)(x)),y) \\ &= (T(g)(x),T(h)(y)) \\ &= \int g\,d\nu_{x,T(h)(y)} \end{aligned}$$

agree on $C(\Delta)$. Again, since both functionals can be represented as integrals, it follows that

$$\int hk\,d\nu_{x,y} = \int k\,d\nu_{x,T(h)(y)}$$

for all bounded, real-valued, Borel functions h and k. Therefore,

$$\begin{aligned}(T(hk)(x), y) &= L_{hk}(x, y) \\ &= \int hk\, d\nu_{x,y} \\ &= \int k\, d\nu_{x,T(h)(y)} \\ &= L_k(x, T(h)(y)) \\ &= (T(k)(x), T(h)(y)) \\ &= (T(h)(T(k)(x)), y),\end{aligned}$$

showing that $T(hk) = T(h)T(k)$ for all bounded, real-valued, Borel functions h and k.

We see directly from the preceding calculation that each $p_E = T(\chi_E)$ is a projection. Clearly $p_\Delta = T(1) = I$ and $p_\emptyset = T(0) = 0$, so that to see that $E \to p_E$ is a projection-valued measure we must only check the countable additivity condition. Thus, let $\{E_n\}$ be a sequence of pairwise disjoint Borel subsets of Δ, and write $E = \cup E_n$. For any vectors $x, y \in H$, we have

$$\begin{aligned}(p_E(x), y) &= (T(\chi_E)(x), y) \\ &= L_{\chi_E}(x, y) \\ &= \int \chi_E\, d\nu_{x,y} \\ &= \nu_{x,y}(E) \\ &= \sum \nu_{x,y}(E_n) \\ &= \sum (p_{E_n}(x), y) \\ &= ([\sum p_{E_n}](x), y),\end{aligned}$$

as desired.

Finally, let us show that $T(f) = \int f\, dp$ for every $f \in C(\Delta)$. Note that, for vectors $x, y \in H$, we have that the measure $\nu_{x,y}$ agrees with the measure $\mu_{x,y}$, where $\mu_{x,y}$ is the measure defined in part c of Exercise 9.2 by

$$\mu_{x,y}(E) = (p_E(x), y).$$

We then have

$$\begin{aligned}(T(f)(x),y) &= \phi_{x,y}(f)\\ &= \int f\,d\nu_{x,y}\\ &= \int f\,d\mu_{x,y}\\ &= ([\int f\,dp](x),y),\end{aligned}$$

by part a of Exercise 9.10. This shows the desired equality of $T(f)$ and $\int f\,dp$.

The uniqueness of the projection-valued measure p, satisfying $T(f) = \int f\,dp$ for all $f \in C(\Delta)$, follows from part d of Exercise 9.2 and part a of Exercise 9.10.

We close this chapter by attempting to extend the definition of integral with respect to a projection-valued measure to unbounded measurable functions. For simplicity, we will restrict our attention to real-valued functions.

DEFINITION. Let p be an H-projection-valued measure on the Borel space $(S,\mathcal{B})$, and let f be a real-valued, $\mathcal{B}$-measurable function on S. For each integer n, define $E_n = f^{-1}(-n,n)$, and write T_n for the bounded selfadjoint operator on H given by $T_n = \int f\chi_{E_n}\,dp$. We define $D(f)$ to be the set of all $x \in H$ for which $\lim_n T_n(x)$ exists, and we define $T_f : D(f) \to H$ by $T_f(x) = \lim T_n(x)$.

EXERCISE 9.14. Using the notation of the preceding definition, show that

(a) If x is in the range of p_{E_n}, then $x \in D(f)$, and $T_f(x) = T_n(x)$.

(b) $x \in D(f)$ if and only if the sequence $\{T_n(x)\}$ is bounded. HINT: $x = p_{E_n}(x) + p_{\tilde{E}_n}(x)$. Show further that the sequence $\{\|T_n(x)\|\}$ is nondecreasing.

(c) $D(f)$ is a subspace of H and T_f is a linear transformation of $D(f)$ into H.

THEOREM 9.8. *Let the notation be as in the preceding definition.*

(1) *$D(f)$ is a dense subspace of H.*

(2) *T_f is symmetric on $D(f)$; i.e.,*

$$(T_f(x),y) = (x,T_f(y))$$

for all $x,y \in D(f)$.

(3) *The graph of* T_f *is a closed subspace in* $H \times H$.

(4) *The following are equivalent: i)* $D(f) = H$; *ii)* T_f *is continuous from* $D(f)$ *into* H; *iii)* $f \in L^\infty(p)$.

(5) *The linear transformations* $I \pm iT_f$ *are both 1-1 and onto from* $D(f)$ *to* H.

(6) *The linear transformation* $U_f = (I - iT_f)(I + iT_f)^{-1}$ *is 1-1 and onto from* H *to* H *and is in fact a unitary operator for which -1 is not an eigenvalue. (This operator* U_f *is called the* Cayley transform *of* T_f.*)*

(7) *The range of* $I + U_f$ *equals* $D(f)$, *and*

$$T_f = -i(I - U_f)(I + U_f)^{-1}.$$

PROOF. That $D(f)$ is dense in H follows from part a of Exercise 9.14 and part c of Exercise 9.1.

Each operator T_n is selfadjoint. So, if $x, y \in D(f)$, then

$$(T_f(x), y) = \lim(T_n(x), y) = \lim(x, T_n(y)) = (x, T_f(y)),$$

showing that T_f is symmetric on its domain $D(f)$.

The graph of T_f, like the graph of any linear transformation of H into itself, is clearly a subspace of $H \times H$. To see that the graph of T_f is closed, let (x, y) be in the closure of the graph, i.e., $x = \lim x_j$ and $y = \lim T_f(x_j)$, where each $x_j \in D(f)$. We must show that $x \in D(f)$ and then that $y = T_f(x)$. Now the sequence $\{T_f(x_j)\}$ is bounded in norm, and for each n we have from the preceding exercise that $\|T_n(x_j)\| \leq \|T_f(x_j)\|$. Hence, there exists a constant M such that $\|T_n(x_j)\| \leq M$ for all n and j. Writing $T_n(x) = T_n(x - x_j) + T_n(x_j)$, we have that

$$\|T_n(x)\| \leq \lim_j \|T_n(x - x_j)\| + M = M$$

for all n, whence $x \in D(f)$ by Exercise 9.14. Now, for any $z \in D(f)$ we have

$$\begin{aligned}(y, z) &= \lim(T_f(x_j), z) \\ &= \lim(x_j, T_f(z)) \\ &= (x, T_f(z)) \\ &= (T_f(x), z),\end{aligned}$$

proving that $y = T_f(x)$ since $D(f)$ is dense in H.

We prove part 4 by showing that i) implies ii), ii) implies iii), and iii) implies i). First, if $D(f) = H$, then by the Closed Graph Theorem we have that T_f is continuous. Next, if f is not an element of $L^\infty(p)$, then there exists an increasing sequence $\{n_k\}$ of positive integers for which either

$$p_{f^{-1}(n_k, n_{k+1})} \neq 0$$

for all k, or

$$p_{f^{-1}(-n_{k+1}, -n_k)} \neq 0$$

for all k. Without loss of generality, suppose that

$$p_{f^{-1}(n_k, n_{k+1})} \neq 0$$

for all k. Write $F_k = f^{-1}(n_k, n_{k+1})$, and note that $F_k \subseteq E_{n_{k+1}}$. Now, for each k, let x_k be a unit vector in the range of p_{F_k}. Then each $x_k \in D(f)$, and

$$\begin{aligned}
(T_f(x_k), x_k) &= (T_{n_{k+1}}(x_k), x_k) \\
&= ((T_{n_{k+1}} \circ p_{F_k})(x_k), x_k) \\
&= \int f\chi_{F_k}\, d\mu_{x_k} \\
&\geq n_k \|x_k\|^2 \\
&= n_k,
\end{aligned}$$

proving that $\|T_f(x_k)\| \geq n_k$, whence T_f is not continuous. Finally, if $f \in L^\infty(p)$, then clearly $T_f = T_n$ for any $n \geq \|f\|_\infty$, and $D(f) = H$. This proves part 4.

We show part 5 for $I + iT_f$. An analogous argument works for $I - iT_f$. Observe that, for $x \in D(f)$, we have

$$\|(I + iT_f)(x)\|^2 = ((I + iT_f)(x), (I + iT_f)(x)) = \|x\|^2 + \|T_f(x)\|^2.$$

Therefore, $I + iT_f$ is norm-increasing, whence is 1-1. Now, if $\{(I + iT_f)(x_j)\}$ is a sequence of elements in the range of $I + iT_f$ that converges to a point $y \in H$, then the sequence $\{(I + iT_f)(x_j)\}$ is a Cauchy sequence and therefore, since $I + iT_f$ is norm-increasing, the sequence $\{x_j\}$ is a Cauchy sequence as well. Let $x = \lim_j x_j$. It follows that $y = x + iz$, where $z = \lim_j T_f(x_j)$. Since the graph of T_f is closed, we must have that $x \in D(f)$ and $z = T_f(x)$. Hence, $y = (I + iT_f)(x)$ belongs to the range of $I + iT_f$, showing that this range is closed. We complete the proof then of part 5 by showing that the range of $I + iT_f$ is dense in H.

Thus, if $y \in H$ is orthogonal to every element of the range of $I + iT_f$, then for each n we have

$$\begin{aligned} 0 &= ((I + iT_f)(p_{E_n}(y)), y) \\ &= ((I + iT_f)(p^2_{E_n}(y)), y) \\ &= ((I + iT_n)(p_{E_n}(y)), y) \\ &= (p_{E_n}(I + iT_n)p_{E_n}(y), y) \\ &= ((I + iT_n)p_{E_n}(y), p_{E_n}(y)) \\ &= \|p_{E_n}(y)\|^2 + i(T_n(p_{E_n}(y)), p_{E_n}(y)) \\ &= \|p_{E_n}(y)\|^2 + i(T_n(y), y). \end{aligned}$$

But then $\|p_{E_n}(y)\|^2 = -i(T_n(p_{E_n}(y)), p_{E_n}(y))$, which, since T_n is self-adjoint, can happen only if $p_{E_n}(y) = 0$. But then $y = \lim_n p_{E_n}(y)$ must be 0. Therefore, the range of $I + iT_f$ is dense, whence is all of H.

Next, since $I + iT_f$ and $I - iT_f$ are both 1-1 from $D(f)$ onto H, it follows that $U_f = (I - iT_f)(I + iT_f)^{-1}$ is 1-1 from H onto itself. Further, writing $y \in D(f)$ as $(I + iT_f)^{-1}(x)$, we have

$$\begin{aligned} \|U_f(x)\|^2 &= \|(I - iT_f)((I + iT_f)^{-1}(x))\|^2 \\ &= \|(I - iT_f)(y)\|^2 \\ &= \|y\|^2 + \|T_f(y)\|^2 \\ &= \|(I + iT_f)(y)\|^2 \\ &= \|x\|^2, \end{aligned}$$

proving that U_f is unitary. Writing the identity operator I as $(I + iT_f)(I + iT_f)^{-1}$, we have that $I + U_f = 2(I + iT_f)^{-1}$, which is 1-1. Consequently, -1 is not an eigenvalue for U_f.

We leave the verification of part 7 to the exercises. This completes the proof.

DEFINITION. We call the operator $T_f : D(f) \to H$ of the preceding theorem the *integral* of f with respect to p, and we denote it by $\int f\, dp$ or $\int f(s)\, dp(s)$. It is not in general an element of $B(H)$. Indeed, as we have seen in the preceding theorem, $\int f\, dp$ is in $B(H)$ if and only if f is in $L^\infty(p)$.

EXERCISE 9.15. (a) Prove part 7 of Theorem 9.8.

(b) Suppose (S, μ) is a σ-finite measure space, that p is the canonical projection-valued measure on $L^2(\mu)$, and that f is a real-valued measurable function on S. Verify that $D(f)$ is the set of all L^2functions g for which $fg \in L^2(\mu)$, and that $[\int f\, dp](g) = fg$ for all $g \in D(f)$.

(c) Suppose $(S,\mathcal{B})$ and p are as in the preceding theorem. Suppose g is an everywhere nonzero, bounded, real-valued, measurable function on S, and write T for the bounded operator $\int g\,dp$. Prove that the operator $\int(1/g)\,dp$ is a left inverse for the operator T.

(d) Let $(S,\mathcal{B})$ and $(S',\mathcal{B}')$ be two Borel spaces, and let h be a Borel map from S into S'. Suppose p is a projection-valued measure on $(S,\mathcal{B})$, and as in part c of Exercise 9.3 define a projection-valued measure $q = h_*(p)$ on $(S',\mathcal{B}')$ by

$$q_E = p_{h^{-1}(E)}.$$

If f is any (possibly unbounded) real-valued $\mathcal{B}'$-measurable function on S', show that

$$\int f\,dq = \int (f \circ h)\,dp.$$

EXERCISE 9.16. Let p be the projection-valued measure on the Borel space $(\mathbb{R},\mathcal{B})$ of part b of Exercise 9.3.

(a) Show that

$$\int f\,dp = U^{-1} \circ m_f \circ U$$

for every $f \in L^\infty(p)$.

(b) If $f(x) = x$, and $T_f = \int f\,dp$, show that $D(f)$ consists of all the L^2 functions g for which $x[U(g)](x) \in L^2(\mathbb{R})$, and then show that every such g is absolutely continuous and has an L^2 derivative.

(c) Conclude that the operator $\int f\,dp$ of part b has for its domain the set of all L^2 absolutely continuous functions having L^2derivatives, and that $[\int f\,dp](g) = (1/2\pi i)g'$.