# Single particle Green's functions and interacting topological insulators

Victor Gurarie


Nordita, Jan 2011

# Topological insulators are free fermion systems characterized by topological invariants.

## In this talk

1. All the invariants can be constructed out of single particle Green's functions of these insulators

2. It is generally believed that at the boundaries of topological insulators there must be zero energy "edge states". The Green's functions provide a very simple proof of this statement.

3. In the presence of interactions, edge states can disappear and get replaced by the "zeroes" of the Green's functions.

VG, arxiv:1011.2273
A. Essin, VG, work in progress
Discussions with A.W.W. Ludwig

# Noninteracting topological insulators

# Topological insulators

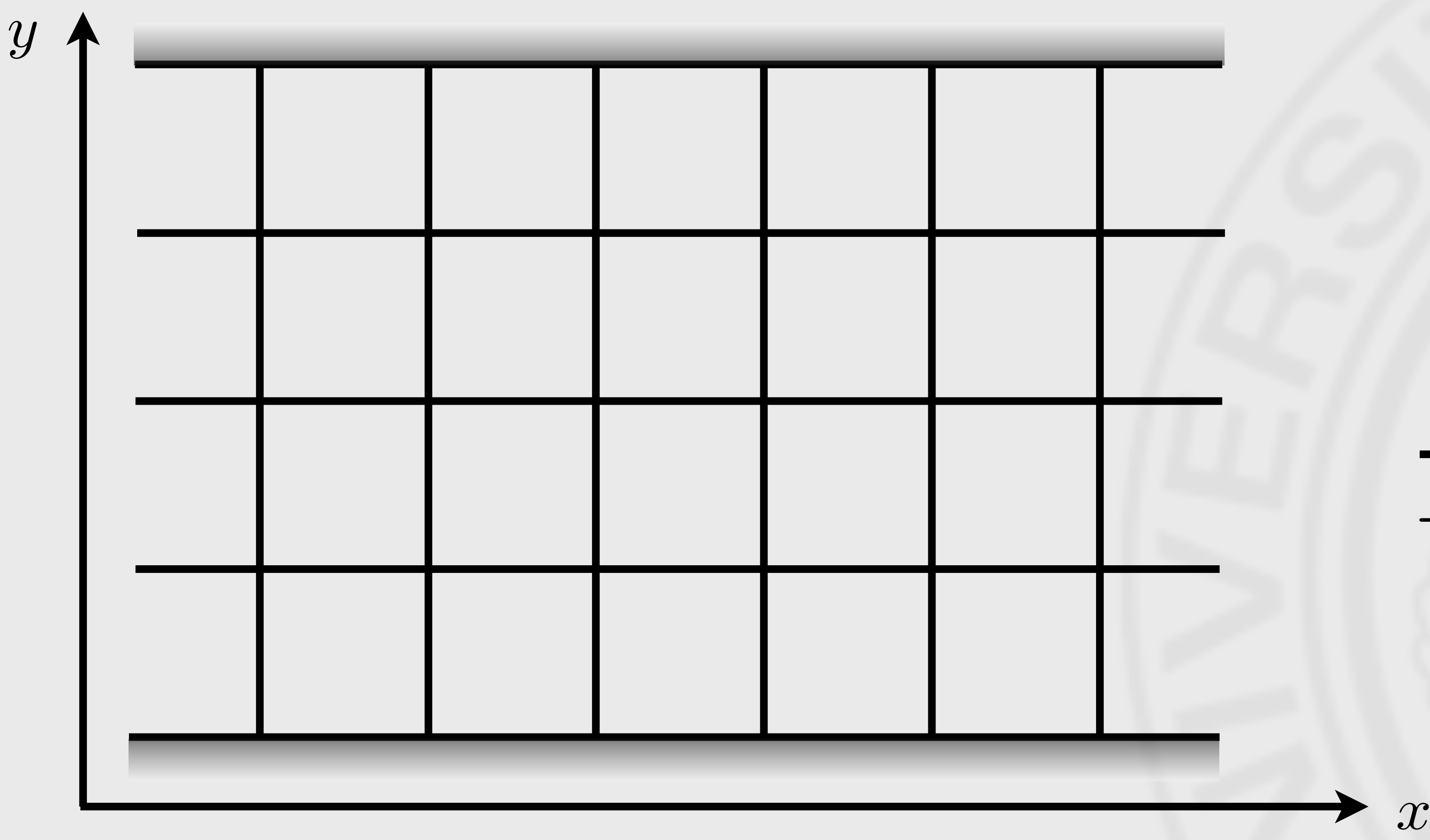
Topological insulators are free fermion systems

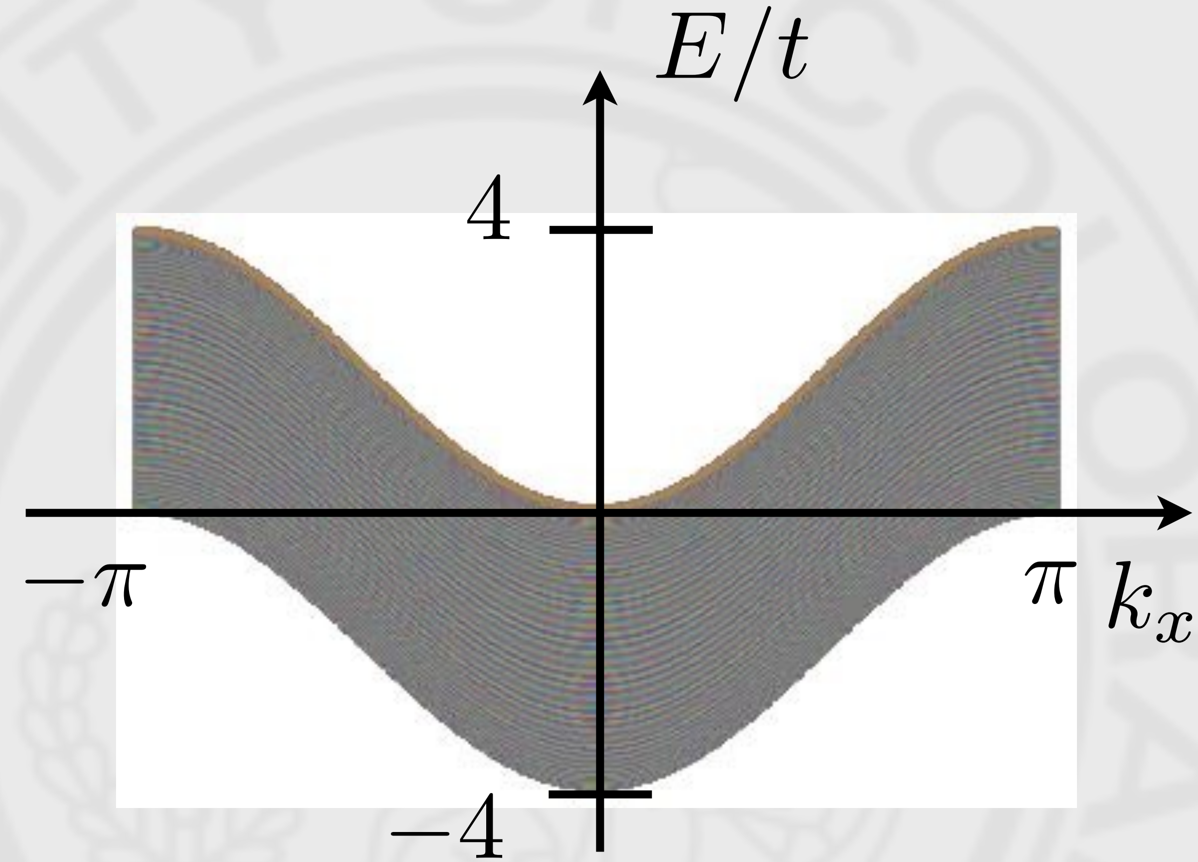
$$\hat{H} = \sum_{ij} \mathcal{H}_{ij} \hat{a}_i^\dagger \hat{a}_j$$

fermionic creation and annihilation operators

which happen to be band insulators of a special type

# Band insulators



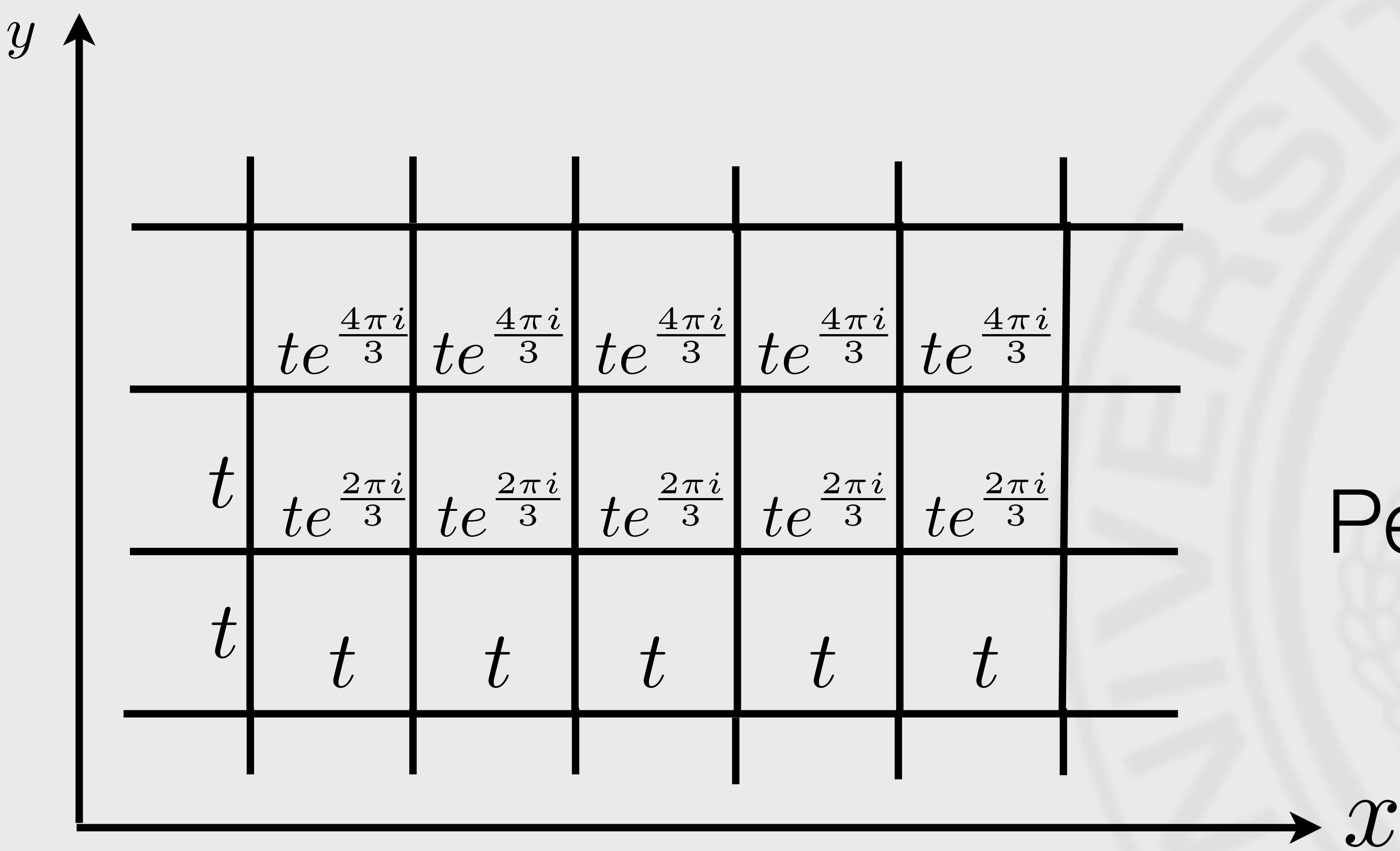




$$\hat{H} = t\sum_{xy}\left[\hat{a}^{\dagger}_{x+1,y}\hat{a}_{x,y} + \hat{a}^{\dagger}_{x,y+1}\hat{a}_{x,y}\right] + \text{h. c.}$$

$$E(k_x, k_y) = -2t\cos\left(k_x\right) - 2t\cos\left(k_y\right)$$

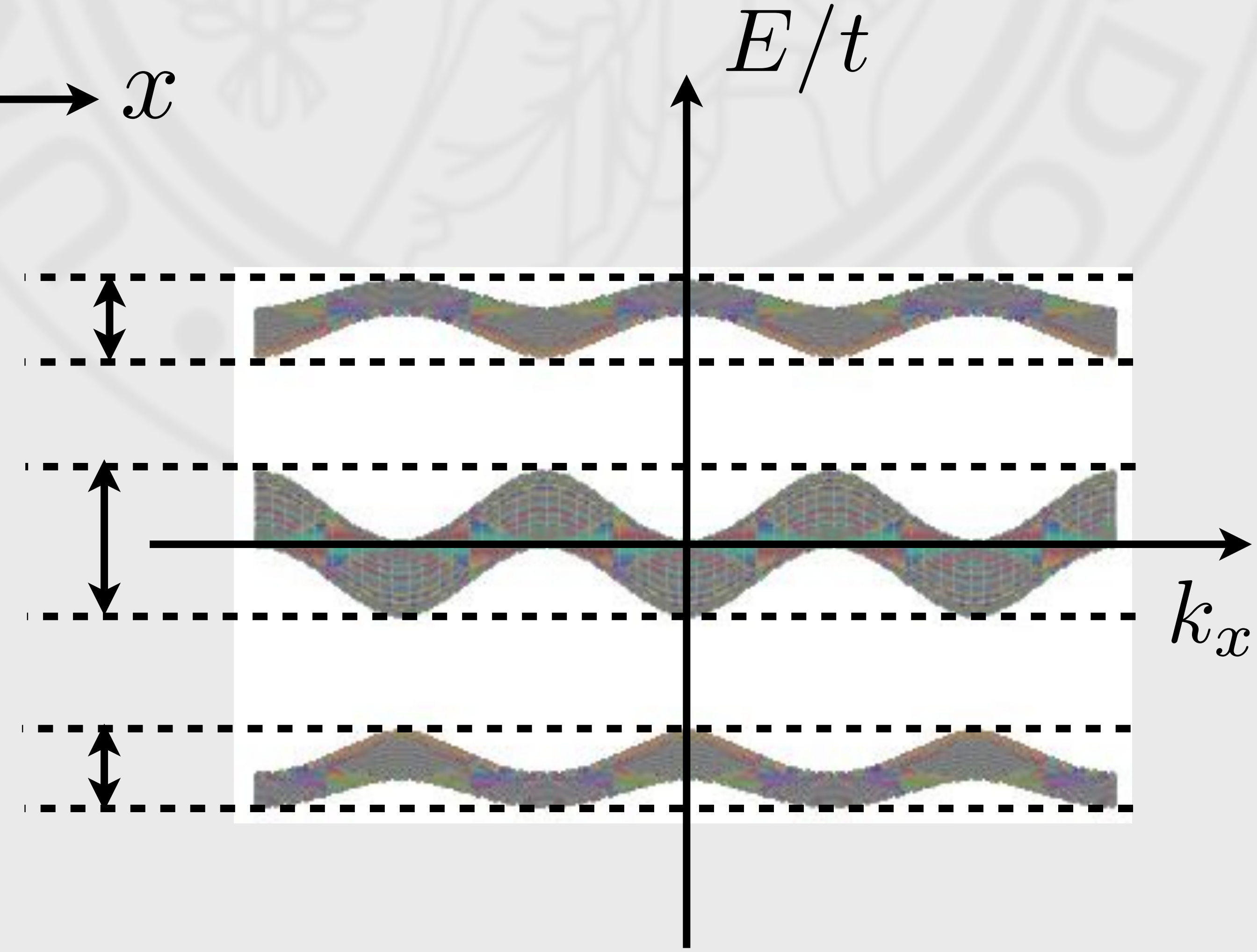
Spectrum is essentially the same regardless of whether the boundary conditions in the y-direction periodic or hard wall.

# Integer quantum Hall effect as a topogical insulator




Periodic boundary conditions in the y-direction

Same tight binding model but with 2π/3 magnetic flux through each plaquette



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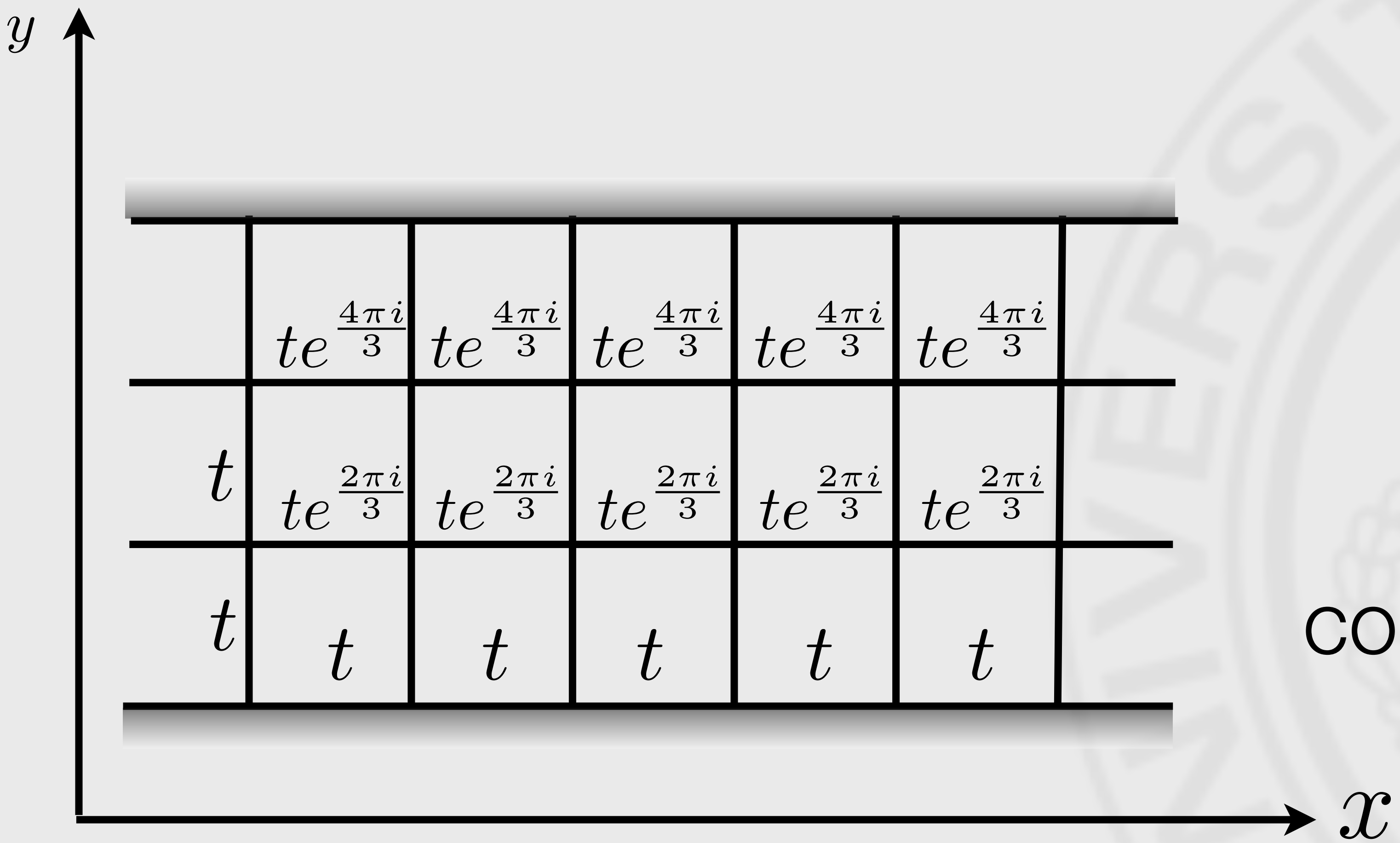
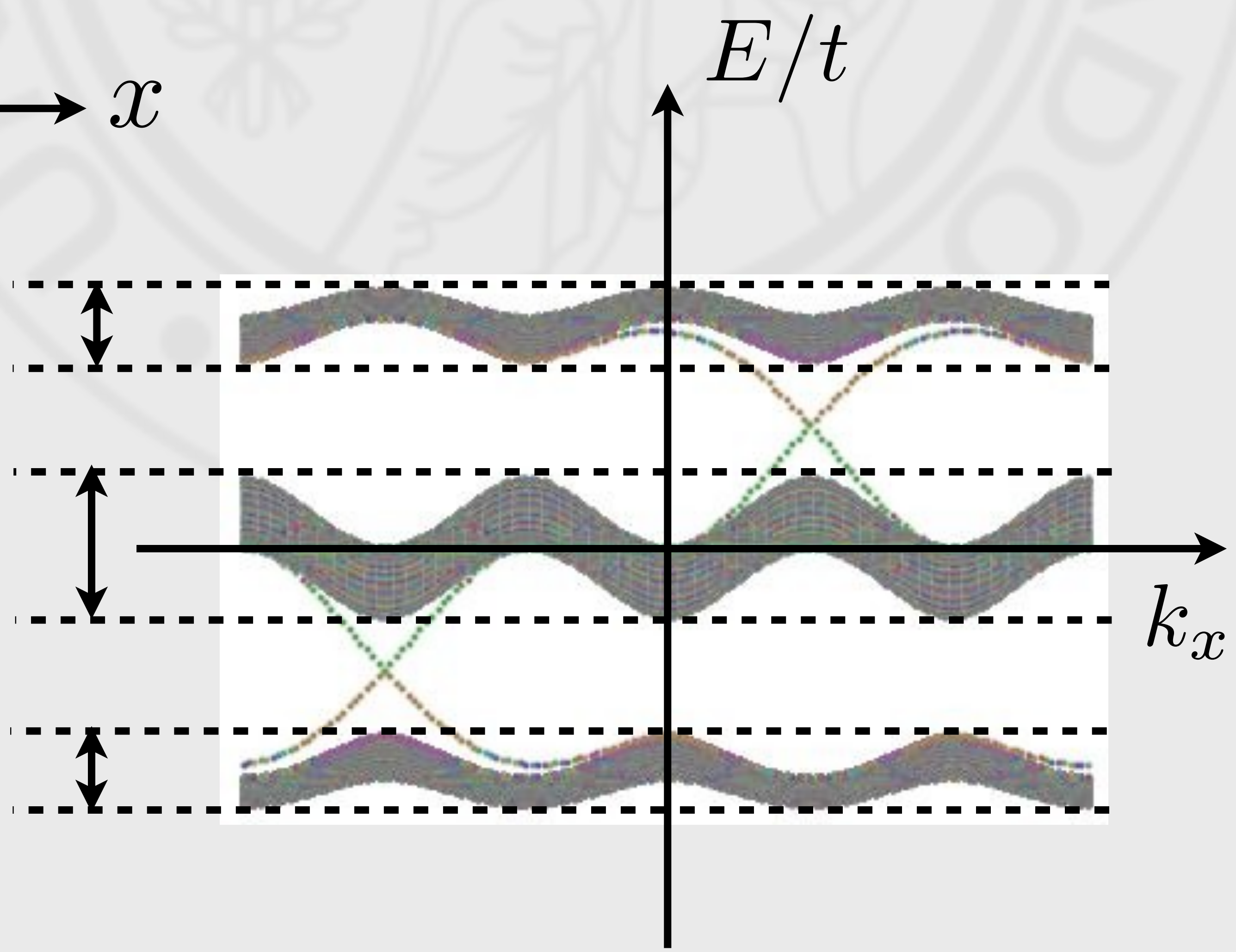



Hard wall boundary conditions in the y-direction

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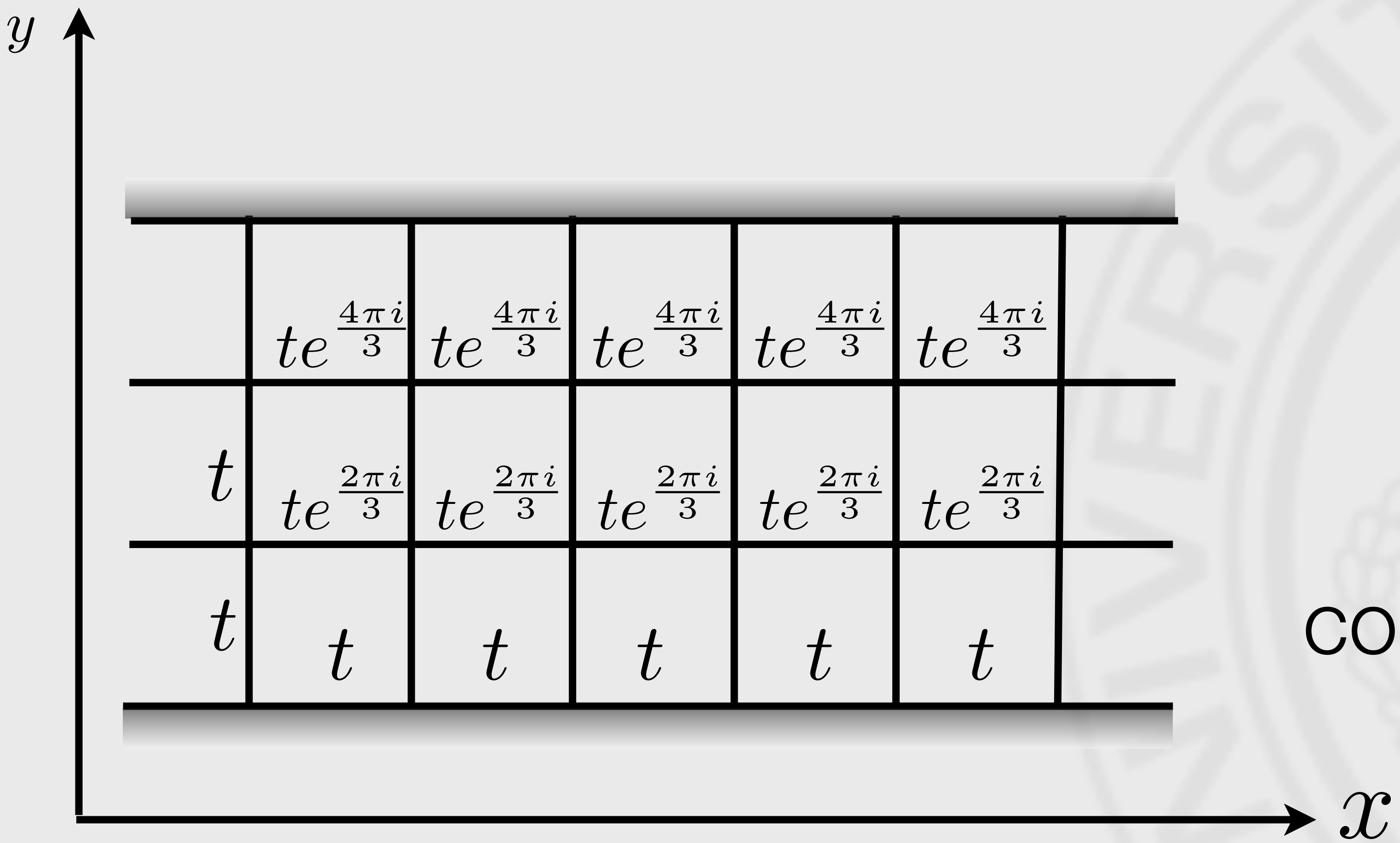


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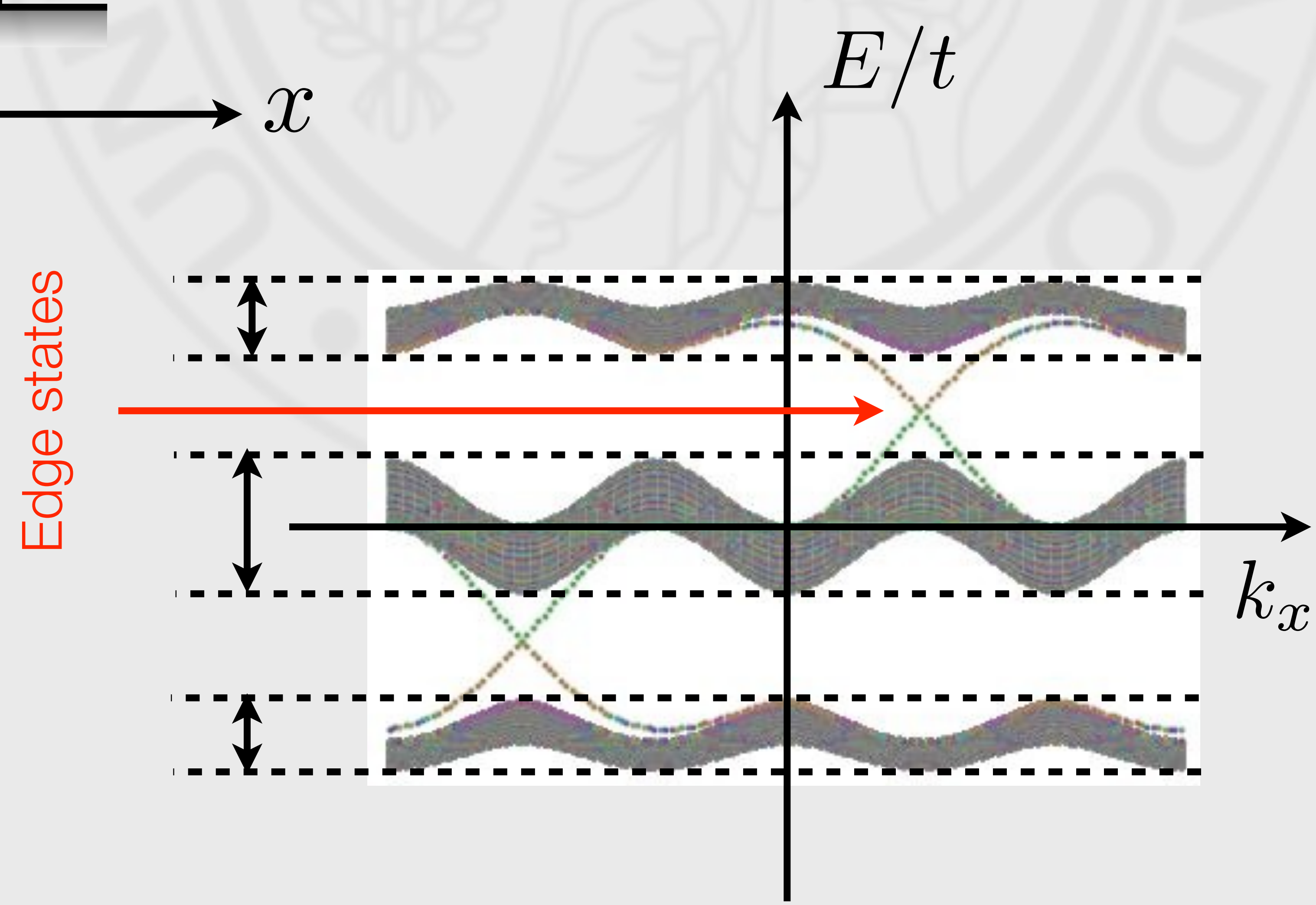
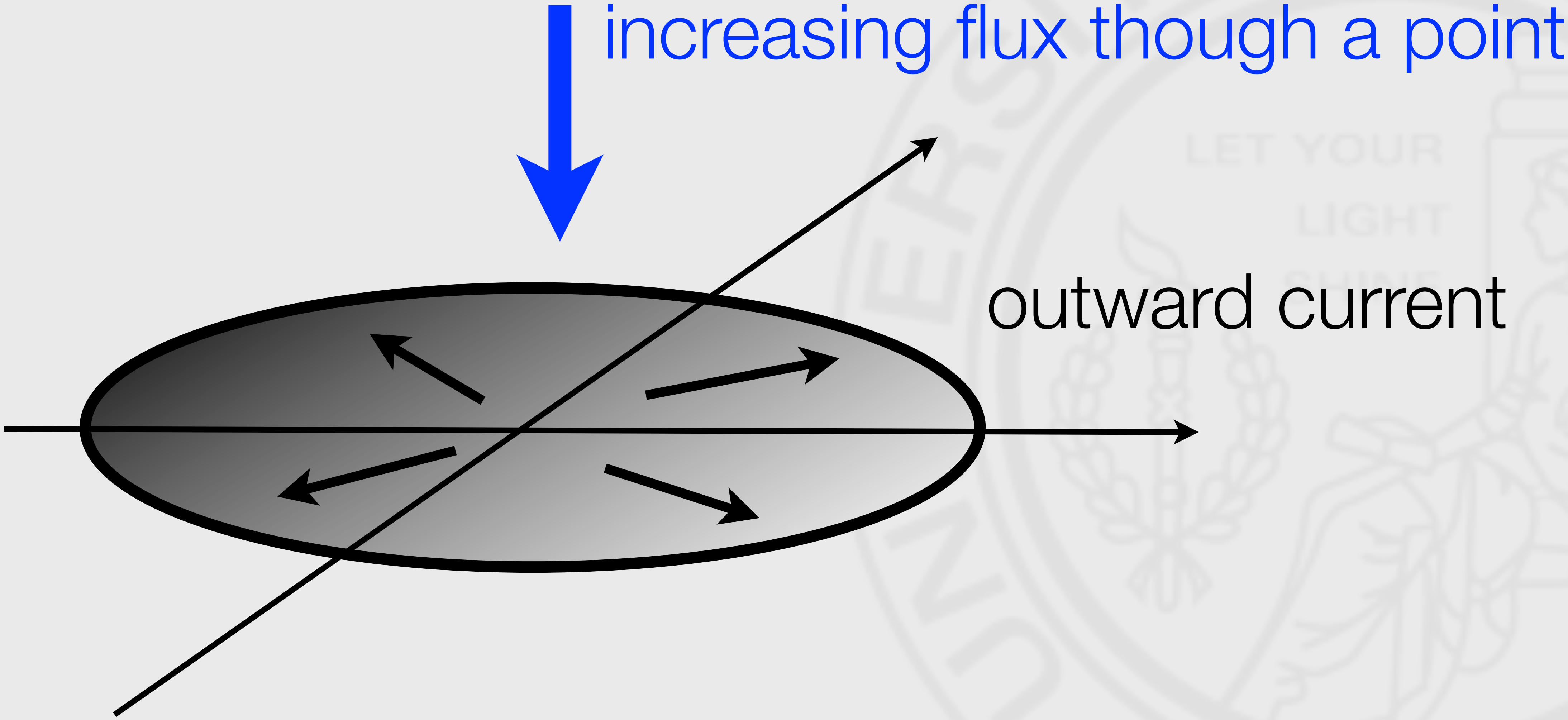


# Laughlin's argument




Outward current deposits charge somewhere.

There must be zero energy edge states to absorb the charge

# TKNN invariant

Thouless, Kohmoto, Nightingale, Den Nijs, 1982

$$\sigma_{xy} = \frac{ie^2}{2\pi h} \int d^2k \int d^2r \left( \frac{\partial u^*}{\partial k_x} \frac{\partial u}{\partial k_y} - \frac{\partial u^*}{\partial k_y} \frac{\partial u}{\partial k_x} \right)$$

This is a topological invariant (always integer times 2πi)

$u(k_x, k_y; \vec{r})$ Bloch waves

# TKNN invariant

Band structure topological invariant →

quantized Hall conductance →

Laughlin argument →

edge states

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4. Modern 2D and 3D topological insulators.
Kane and Mele (2005); Zhang, Hughes, Bernevig, (2006); Moore, Balents, (2007); Fu, Kane, Mele (2007)

# Altland-Zirnbauer's tenfold way

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| AI (orthogonal) | +1 | 0 | 0 |
| AII (symplectic) | −1 | 0 | 0 |
| AIII (ch. unit.) | 0 | 0 | 1 |
| BDI (ch. orth.) | +1 | +1 | 1 |
| CII (ch. sympl.) | −1 | −1 | 1 |
| D (BdG) | 0 | +1 | 0 |
| C (BdG) | 0 | −1 | 0 |
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From: Ryu, Schnyder, Furusaki, Ludwig, 2010

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IQHE → A (unitary)

spin-orbit coupling coupling: modern top. ins. → AII (symplectic)

Su-Schrieffer-Heeger solitons → BDI (ch. orth.)

p-wave spin-polarized superconductors → D (BdG)

$^3$He, phase B → DIII (BdG)

s-wave superconductors → CI (BdG)

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symmetry classes

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IQHE

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| DIII | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | ... |
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| CII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | ... |
| C | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | ... |
| CI | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | ... |

IQHE

Su, Schrieffer, Heeger

2D p-wave superconductor

symmetry classes

$^3$He, phase B

New Kane-Mele topological insulators

Kitaev, 2009;
Ludwig, Ryu, Schnyder, Furusaki, 2009.

# Chiral symmetry



$$\mathcal{H} = -\Sigma^{\dagger}\mathcal{H}\Sigma$$

$$\hat{H} = \sum_{ij} \mathcal{H}_{ij} \hat{a}_i^{\dagger} \hat{a}_j$$

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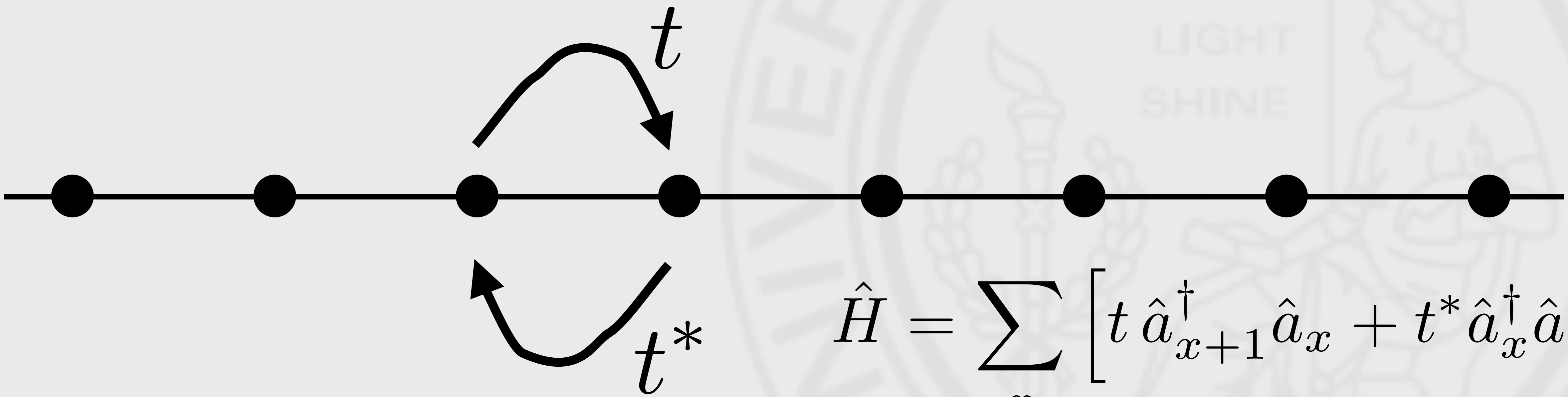


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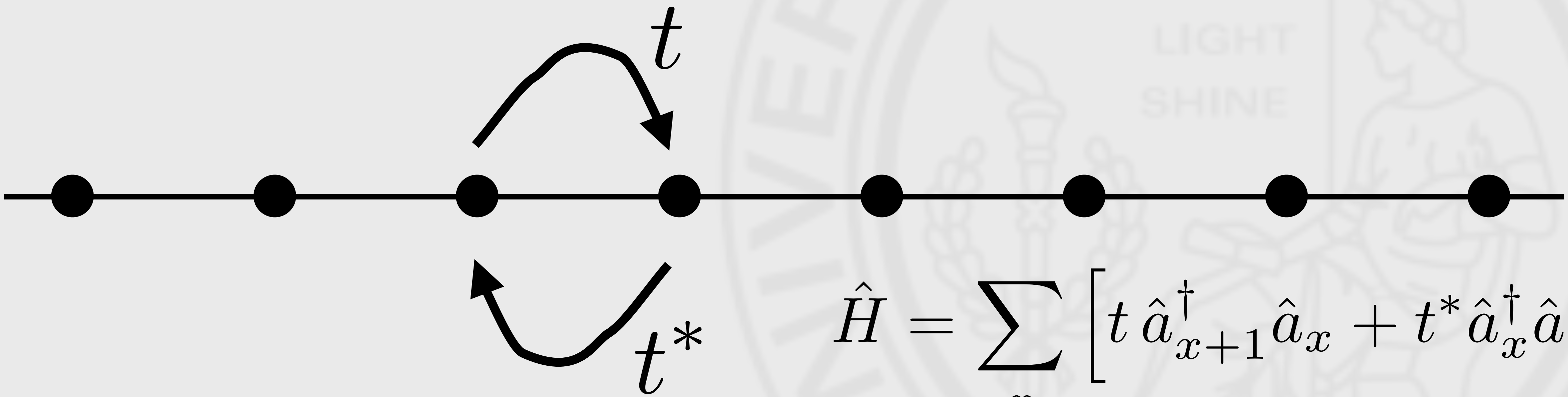


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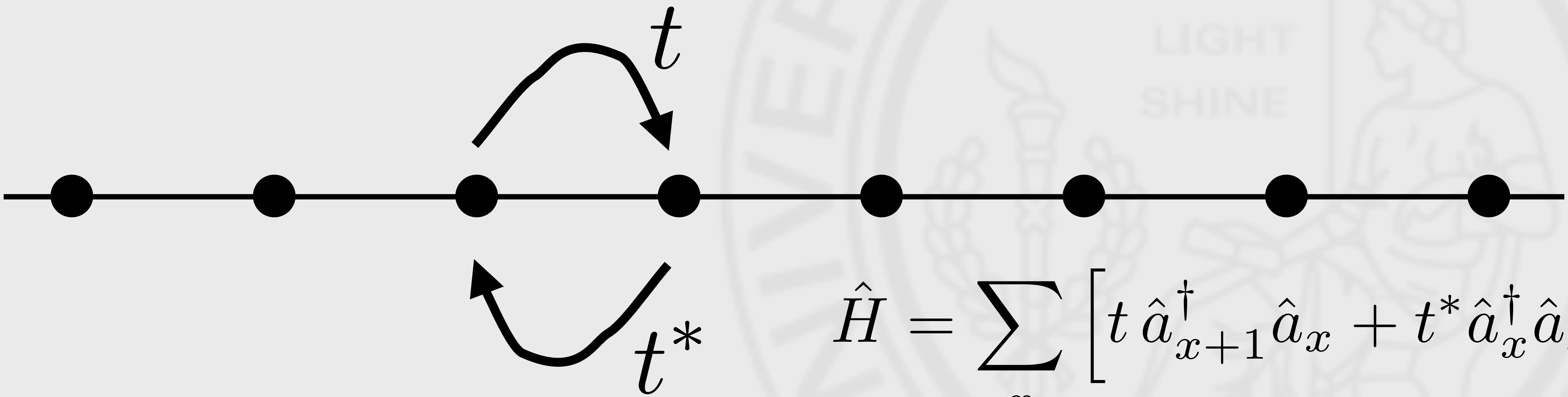


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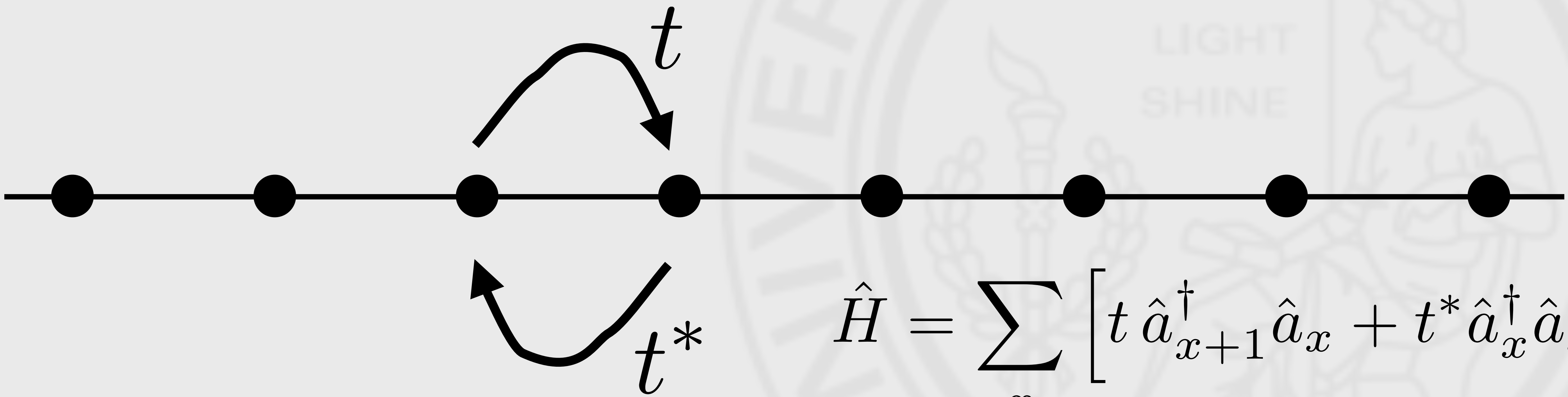


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$\#_R$-$\#_L$ is a topological invariant (index theorem)

# Chiral vs nonchiral systems

| Non-chiral systems | Chiral systems |
| --- | --- |
| can be characterized by an integer topological invariant in even spacial dimensions only | can be characterized by an integer topological invariant in odd spacial dimensions only |

# Chiral vs nonchiral systems

| Cartan | | | | | | $d$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | ... |
| *Complex case:* | | | | | | | | | | | | | |
| A | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | ... |
| AIII | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | ... |
| *Real case:* | | | | | | | | | | | | | |
| AI | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | ... |
| BDI | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | ... |
| D | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | ... |
| DIII | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | ... |
| AII | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | ... |
| CII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | ... |
| C | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | ... |
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Chiral (AIII, BDI, DIII, CII, CI)

Nonchiral (A, AI, D, AII, C)

# Topological invariants via single particle Green's functions

# Topological invariants for nonchiral insulators

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$$d \text{ even; } D = d + 1 \text{ odd}$$

space-time → $D$

space → $d$

# Topological invariants for nonchiral insulators



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space-time → $D$ ; space → $d$

$$G_{ij}(\omega) = [i\omega - \mathcal{H}_{ij}]^{-1}$$

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$$N_D \sim \sum_{\alpha_1, \alpha_2, \ldots, \alpha_D} \epsilon_{\alpha_1, \alpha_2 \ldots, \alpha_D} \operatorname{tr} \int \frac{d\omega d^d k}{(2\pi)^D} \left[G^{-1} \partial_{k_{\alpha_1}} G\right] \left[G^{-1} \partial_{k_{\alpha_2}} G\right] \ldots \left[G^{-1} \partial_{k_{\alpha_D}} G\right]$$

$$k_0 \equiv \omega$$

Notes:

1. *d* must be even. If *d=2* this coincides with the TKNN invariant
Niu, Thouless, Wu (1985)

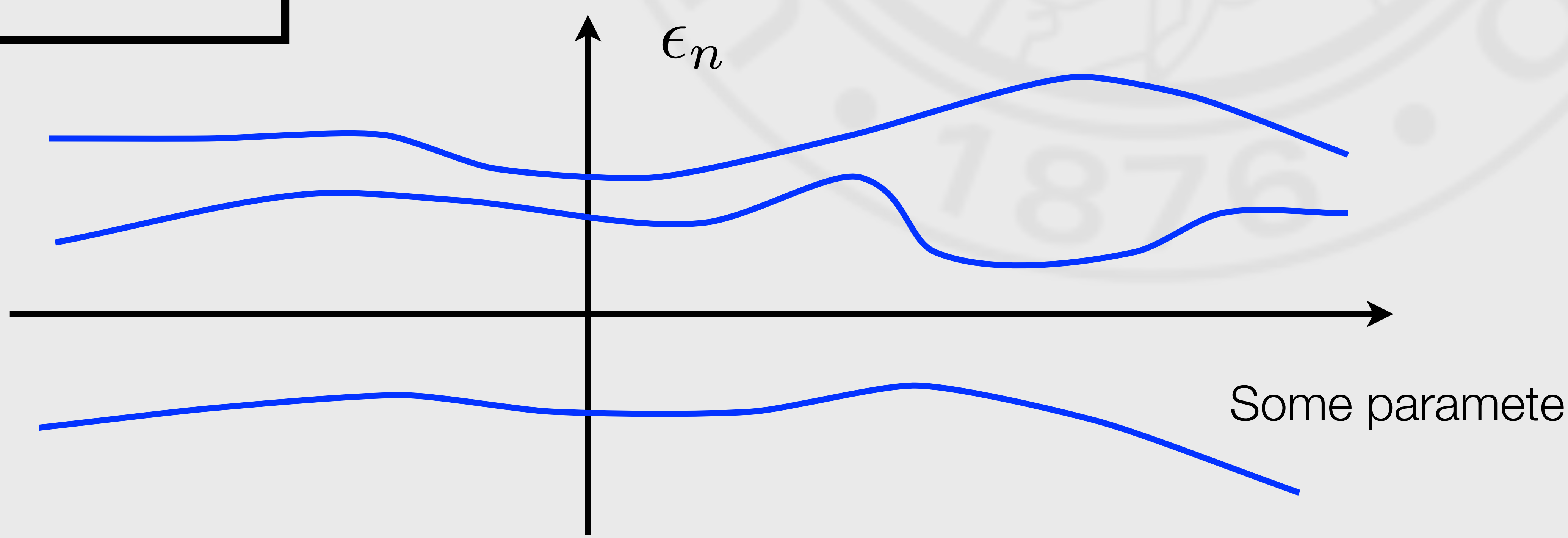
2. Subsequently used by Volovik in a variety of contexts (80's and 90's)

# The meaning of the invariant at *d*=0, *D*=1

$$N_1 = \mathrm{tr} \int_{-\infty}^{\infty} \frac{d\omega}{\pi i} G^{-1} \partial_\omega G \; = \int_{-\infty}^{\infty} \frac{d\omega}{\pi i} \, \partial_\omega \ln \det G$$

$$G_{ij}(\omega) = \left[ i\omega - \mathcal{H}_{ij} \right]^{-1} \qquad \det G = \prod_n \frac{1}{i\omega - \epsilon_n}$$

$$\boxed{N_1 = \sum_n \mathrm{sign}\, \epsilon_n}$$




As long as the system remains gapful, $N_1$ is an invariant

# Topological invariants for chiral insulators

$$d \text{ odd; } D = d + 1 \text{ even}$$

space-time → $D$; space → $d$

$$\mathcal{H} = -\Sigma^\dagger \mathcal{H} \Sigma$$

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VG, 2010

# Topological invariants for chiral insulators

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Translational invariance $\quad G_{ij}(\omega) \to G_{ab}(\omega, \mathbf{k})$

$$Q(\omega, \mathbf{k}) = G^{-1}(\omega, \mathbf{k}) \, \Sigma \, G(\Omega, \mathbf{k}) \qquad Q^2 = 1$$

$$I_D \sim \sum_{\alpha_1, \alpha_2, \ldots, \alpha_D} \epsilon_{\alpha_1, \alpha_2 \ldots, \alpha_D} \, \mathrm{tr} \int_0^\infty \frac{d\omega}{2\pi} \int \frac{d^d k}{(2\pi)^d} \, Q \, \partial_{k_{\alpha_1}} Q \, \partial_{k_{\alpha_2}} Q \ldots \partial_{k_{\alpha_D}} Q$$

VG, 2010

# Topological invariants for chiral insulators

$d$ odd: $D = d + 1$ even

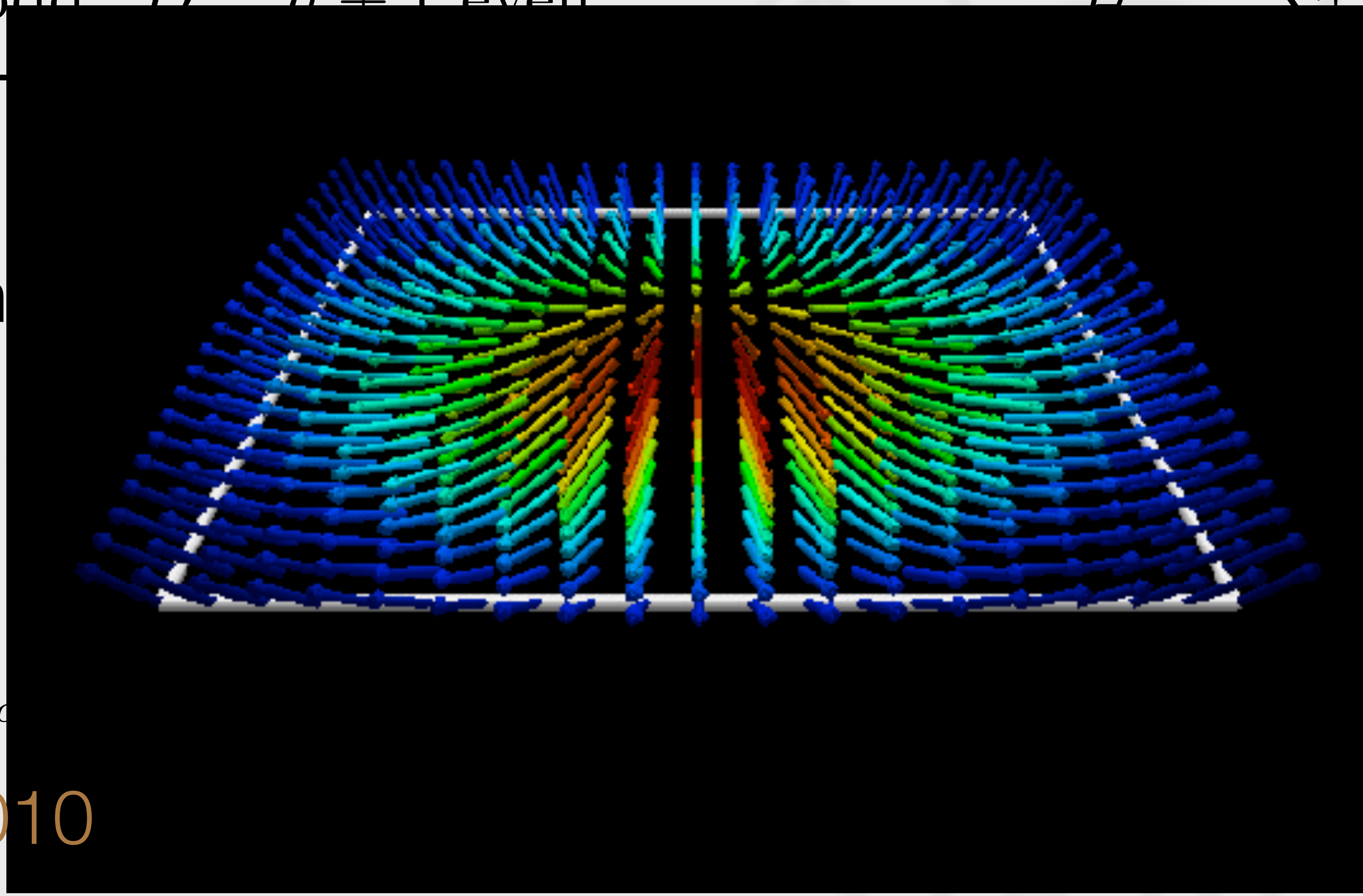
space-time

$\mathcal{H} \to \Sigma^\dagger \mathcal{H} \Sigma$

Tran

$Q(\omega$

$I_D \sim$ … $\alpha_1,$ … $\ldots \partial_{k_{\alpha_D}} Q$



VG, 2010

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$$I_2 \sim \int dx dy \, \vec{n} \cdot \nabla_\alpha \vec{n} \times \nabla_\beta \vec{n}$$

$\vec{n}^2 = 1$ Skyrmion number

pic. by N. Cooper

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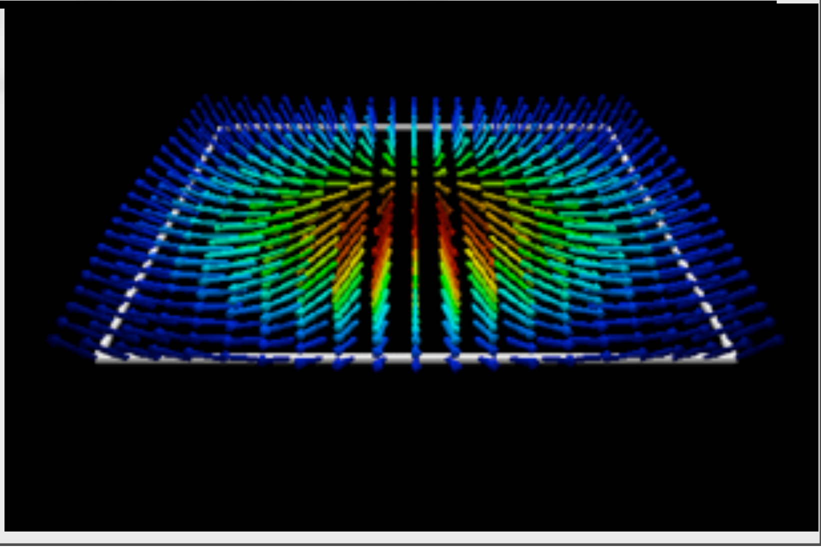


pic. by N. Cooper

# The meaning of the invariant at $D$=0



$$I_0 = \mathrm{tr}\, Q = \mathrm{tr}\, \Sigma = \#_R - \#_L$$

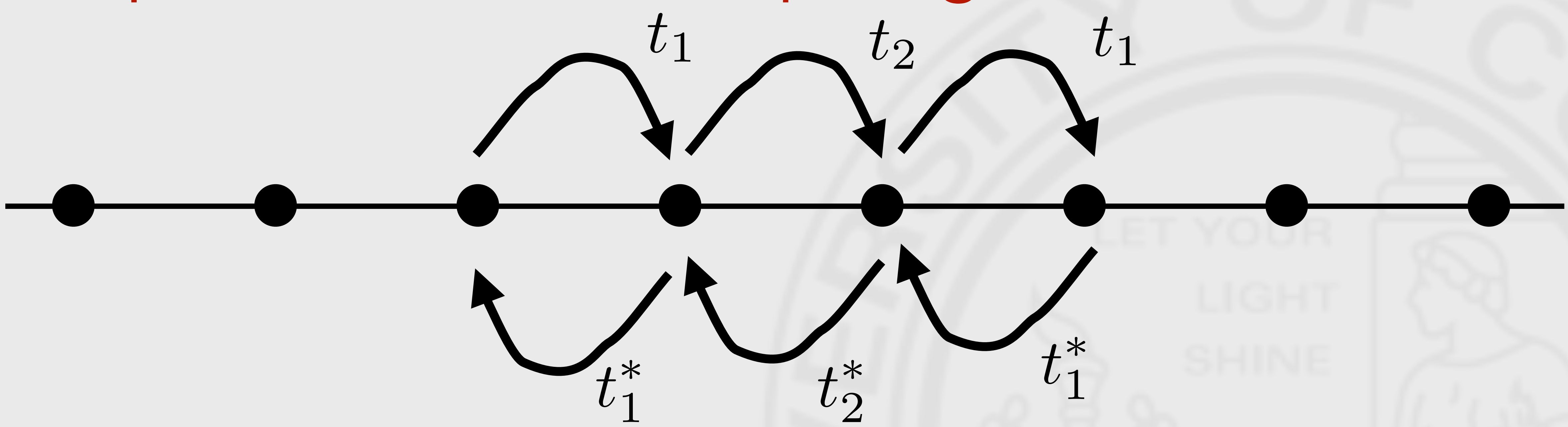
Properties of chiral systems

$\mathcal{H}\psi = E\psi \;\rightarrow\; \mathcal{H}\Sigma\psi = -E\Sigma\psi$ All levels come in pairs $\pm E$

$$\text{If } \mathcal{H}\psi = 0 \text{ then } \begin{cases} \Sigma\psi = \psi & \text{right zero modes} \\ \Sigma\psi = -\psi & \text{left zero modes} \end{cases}$$

$\#_R$-$\#_L$ is a topological invariant (index theorem)

# Simplest d=1 chiral topological insulator




$$\hat{H} = \sum_{x \text{ even}} \left[ t_1 \hat{a}^\dagger_{x+1} \hat{a}_x + t_2 \, \hat{a}^\dagger_{x+2} \hat{a}_{x+1} \right] + h.c.$$

Su, Schrieffer, Heeger (1978)

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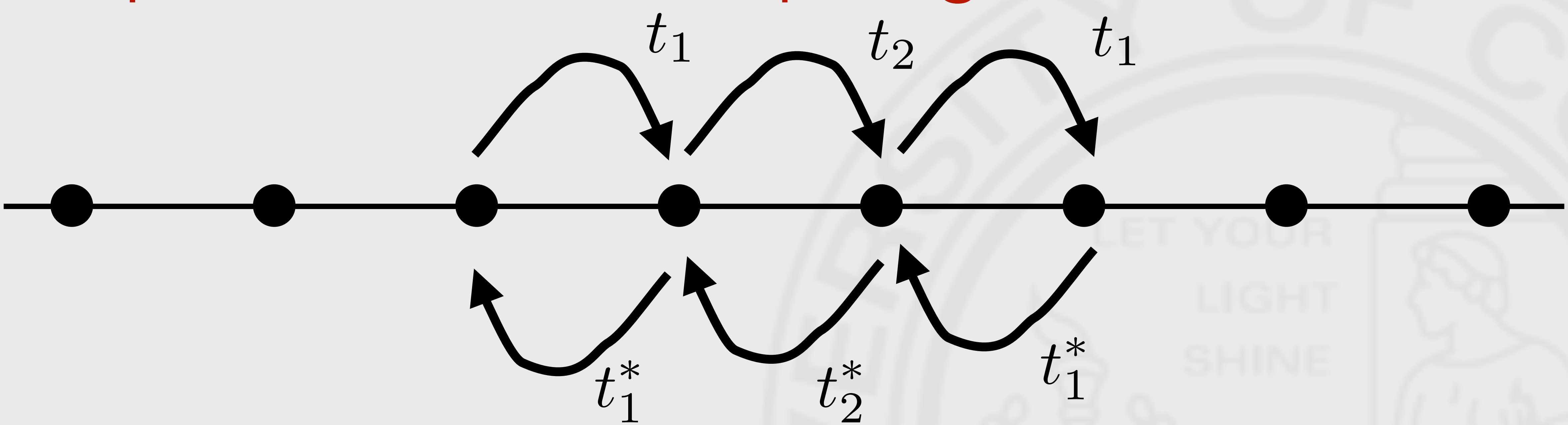


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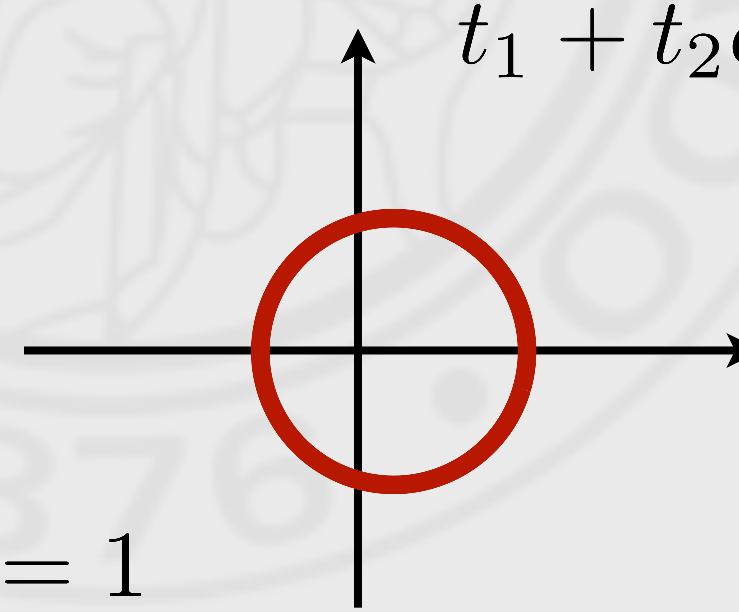
$$I_2 \sim \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \, \partial_k \ln \left( t_1 + t_2 e^{ik} \right)$$

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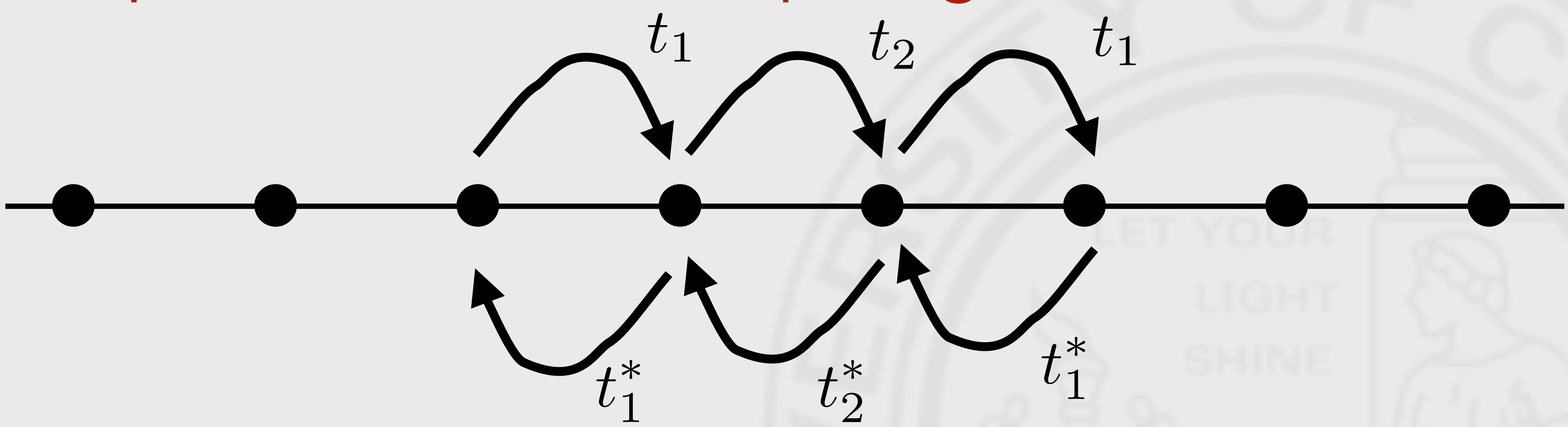


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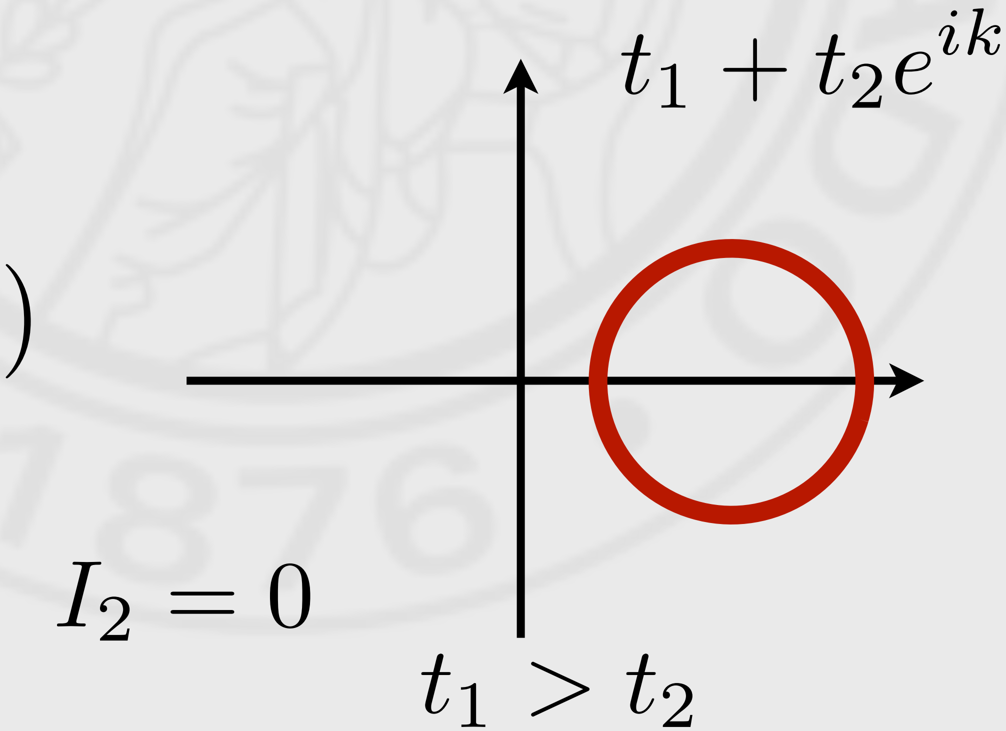


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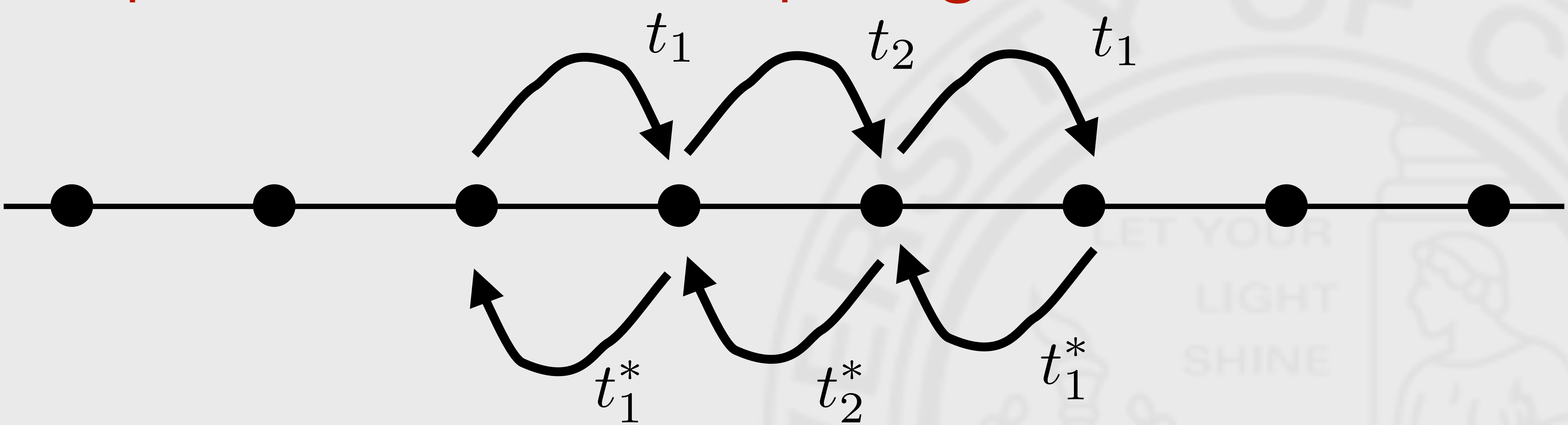


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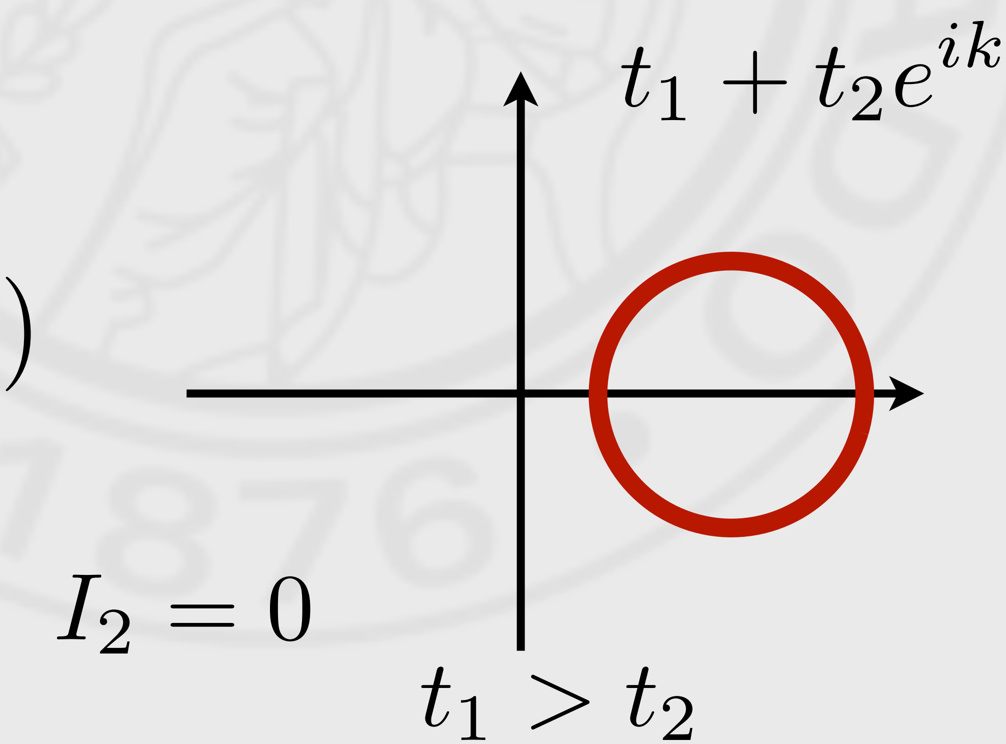



Zero mode (edge state) satisfy

$$t_1 \psi_x + t_2 \psi_{x+2} = 0$$

$$\psi_x = \left( -\frac{t_1}{t_2} \right)^{\frac{x}{2}}$$

If $x>0$, exist only if $t_1<t_2$.

Su, Schrieffer, Heeger (1978)

# Topological invariants and the edge states

# d=2; Classes A, D, C

G. Volovik, 1980s

| Cartan | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | … |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Complex case:* | | | | | | | | | | | | | |
| A IQHE | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | … |
| AIII | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | … |
| *Real case:* | | | | | | | | | | | | | |
| AI | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | … |
| BDI | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | … |
| D p-wave s.c. | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | … |
| DIII $^3$He B | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | … |
| AII | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | … |
| CII | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | … |
| C | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | … |
| CI singlet s.c. | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | … |

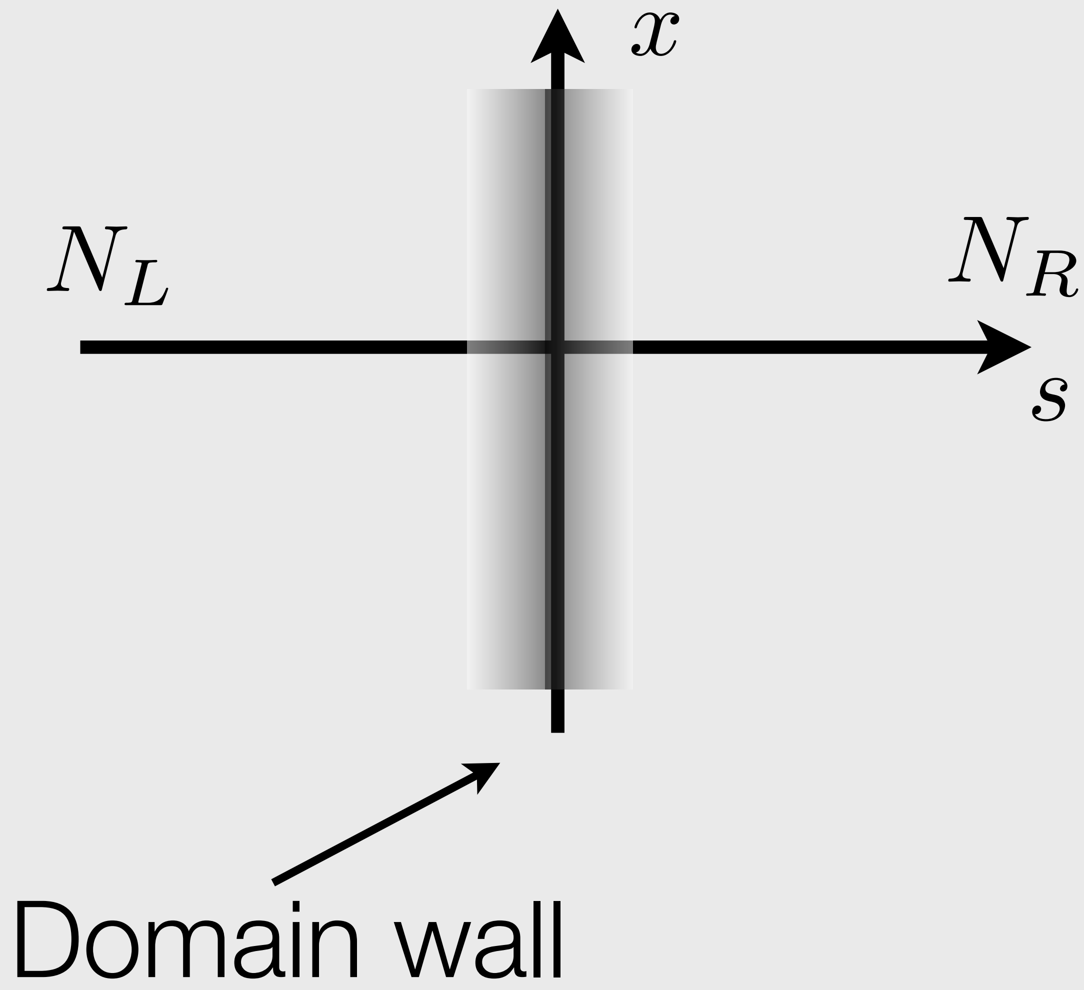
Column heading over 0–11: *d*

Chiral

Nonchiral

# d=2; Classes A, D, C

G. Volovik, 1980s



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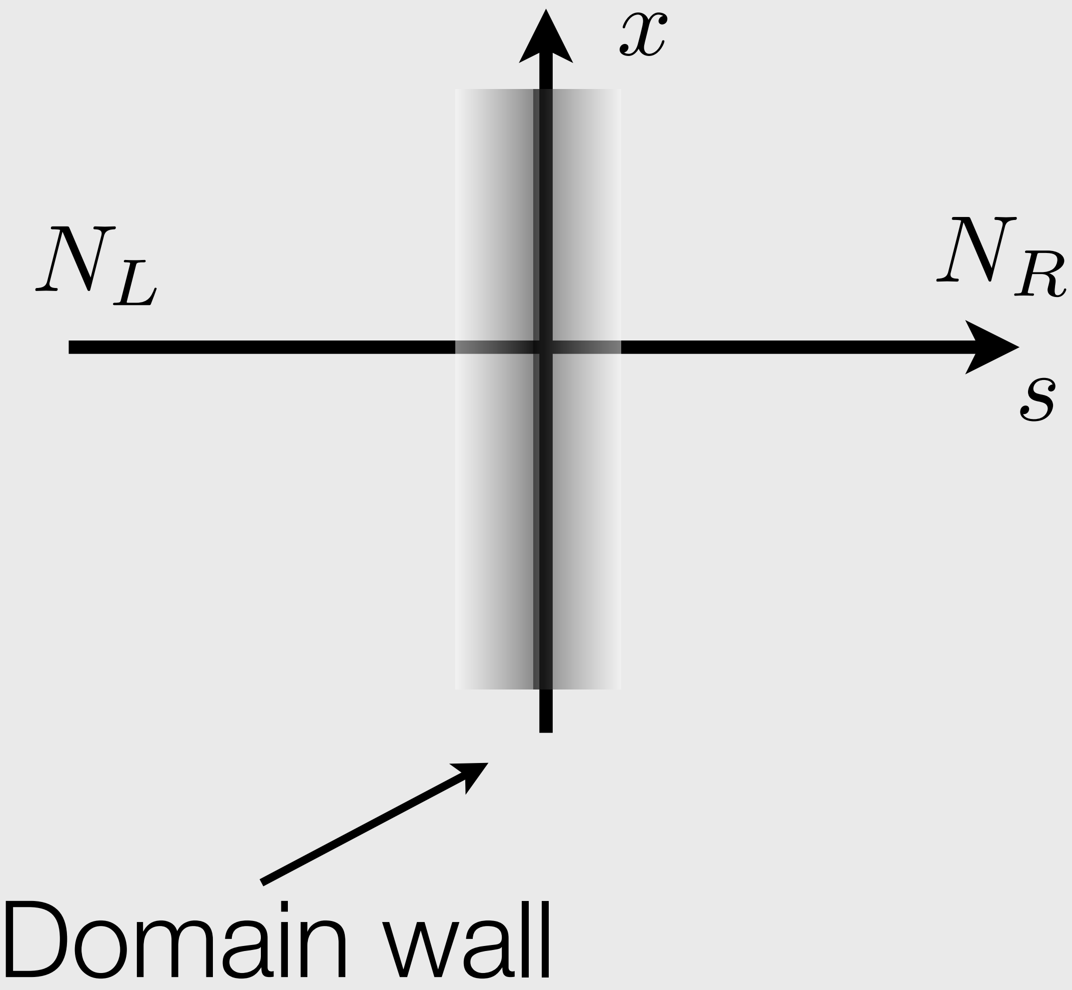
G. Volovik, 1980s

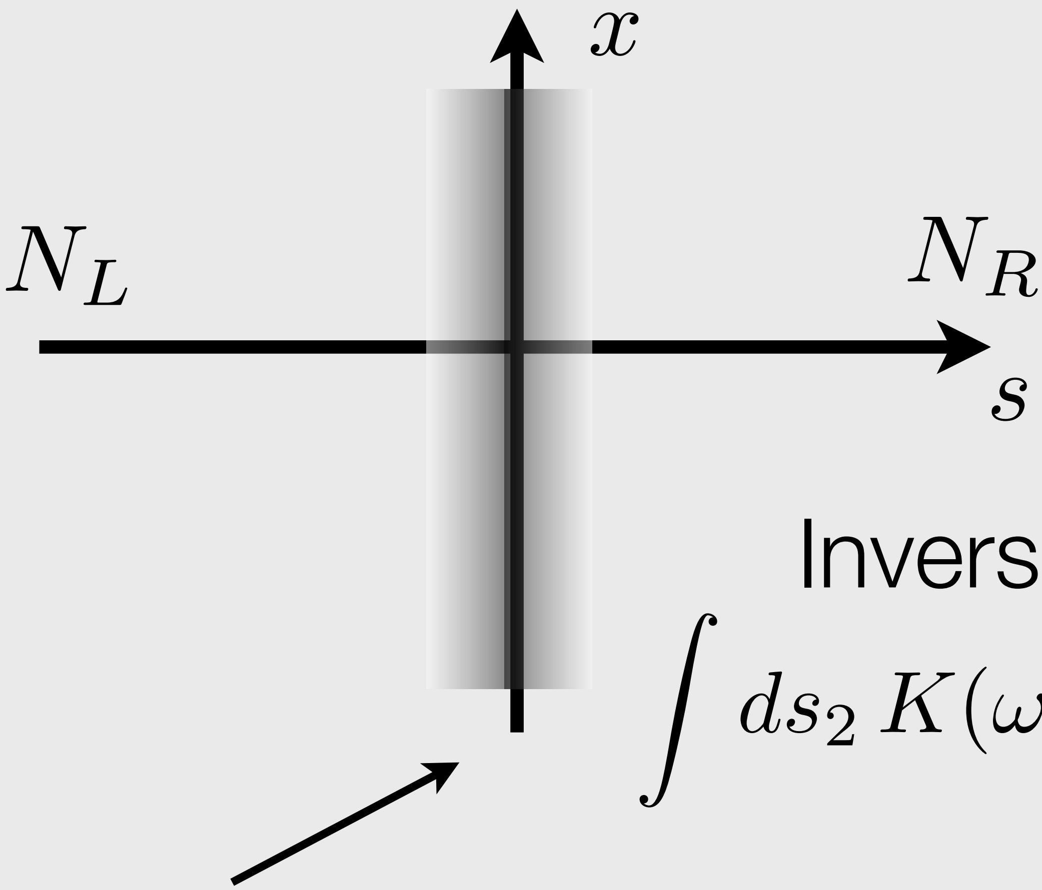



Green's function

$G(\omega; p_x; s_1, s_2)$

# d=2; Classes A, D, C

G. Volovik, 1980s




Domain wall

Green's function

$$G(\omega; p_x; s_1, s_2)$$

Inverse green's function

$$\int ds_2\, K(\omega; p_x; s_1, s_2)\, G(\omega; p_x; s_2, s_3) = \delta(s_1 - s_3)$$

# d=2; Classes A, D, C

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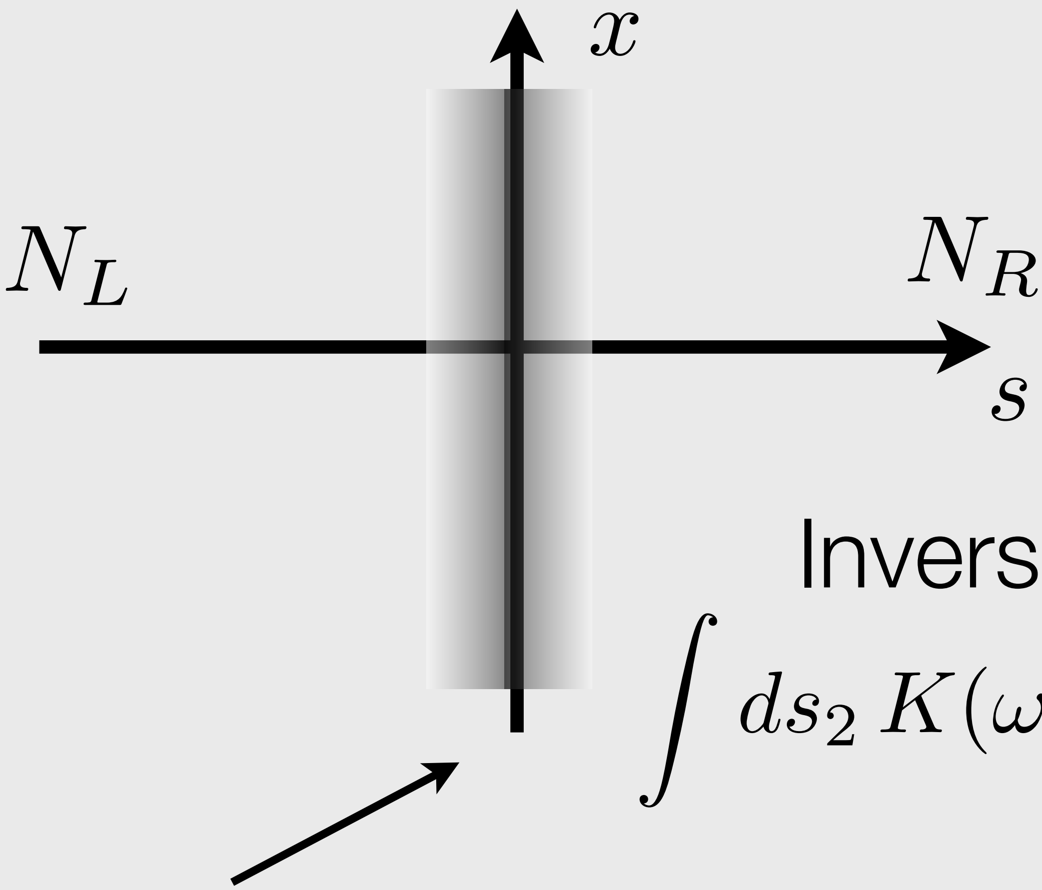


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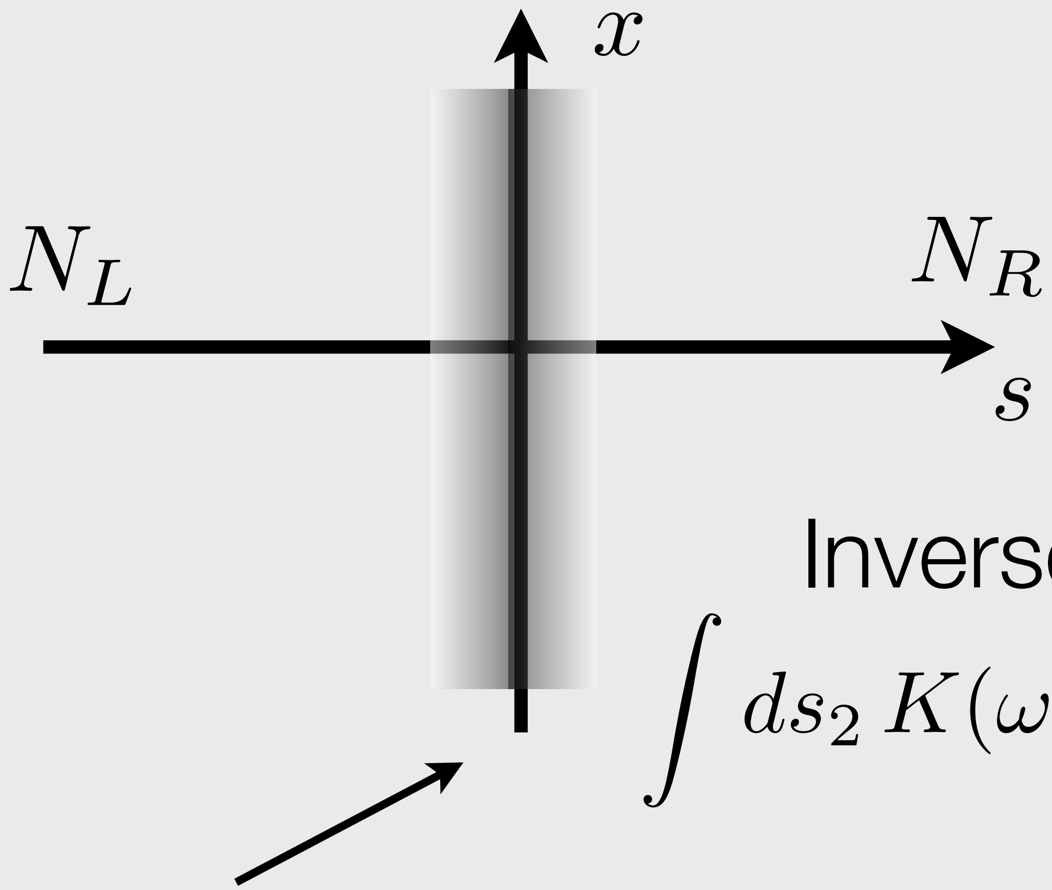
Construct the simplest topological invariant

$$N_1(p_x) = \int \frac{ds_1 ds_2 d\omega}{\pi i} K(\omega; p_x; s_1, s_2) \, \partial_\omega G(\omega; p_x; s_2, s_1) = \sum_n \text{sign } \epsilon_n|_{p_x}$$

# d=2; Classes A, D, C

G. Volovik, 1980s

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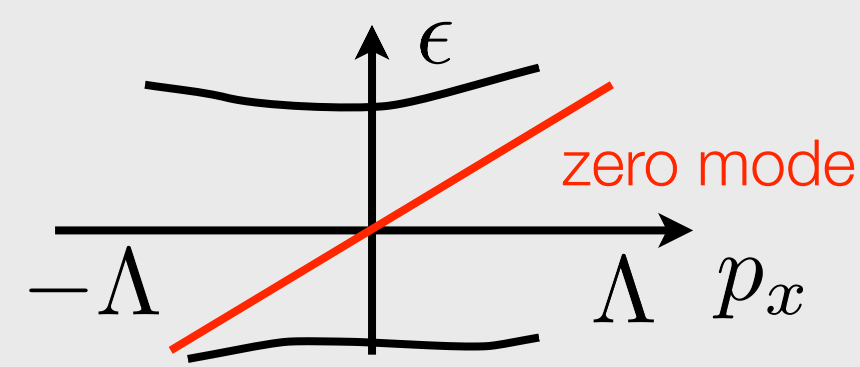
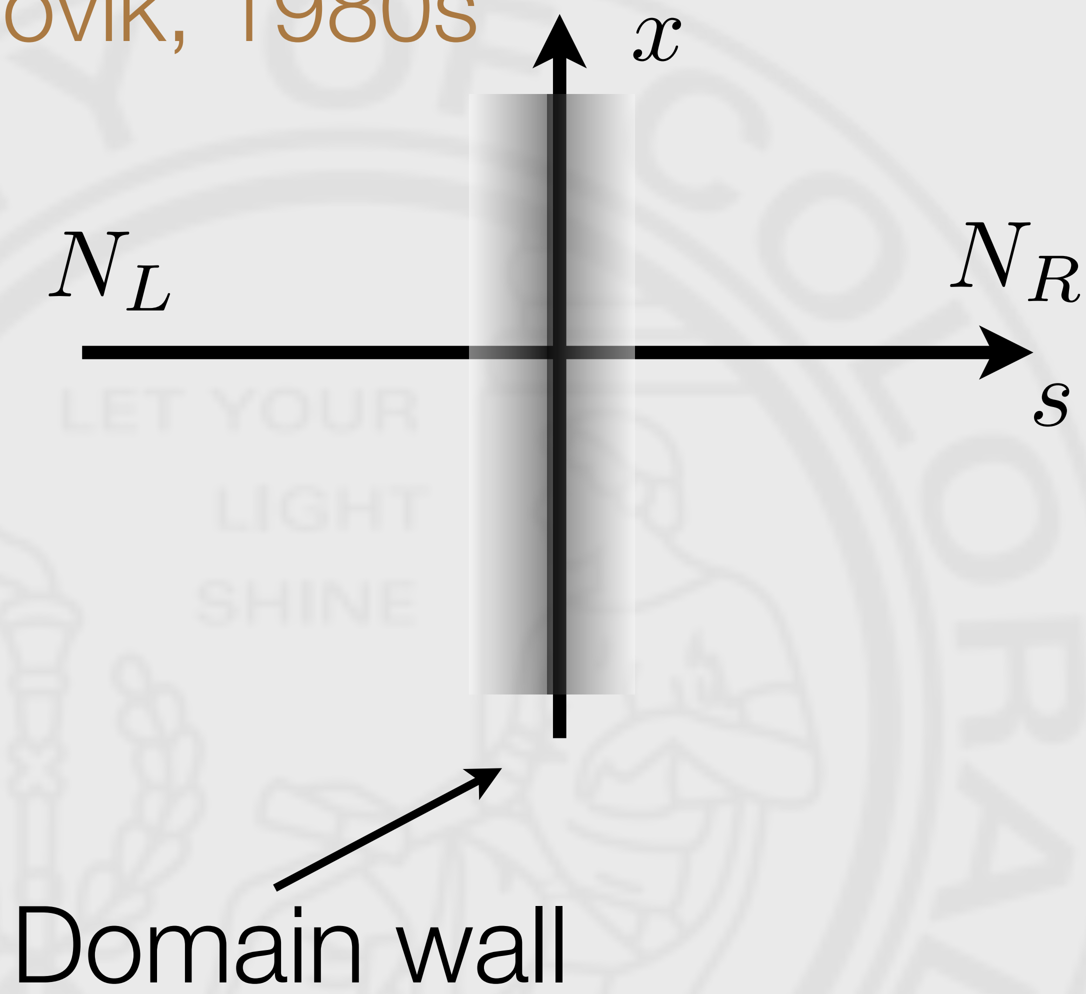
$$\#(\text{zero modes}) = \frac{1}{2}\left[N_1(\Lambda) - N_1(-\Lambda)\right]$$



# d=2; Classes A, D, C

G. Volovik, 1980s

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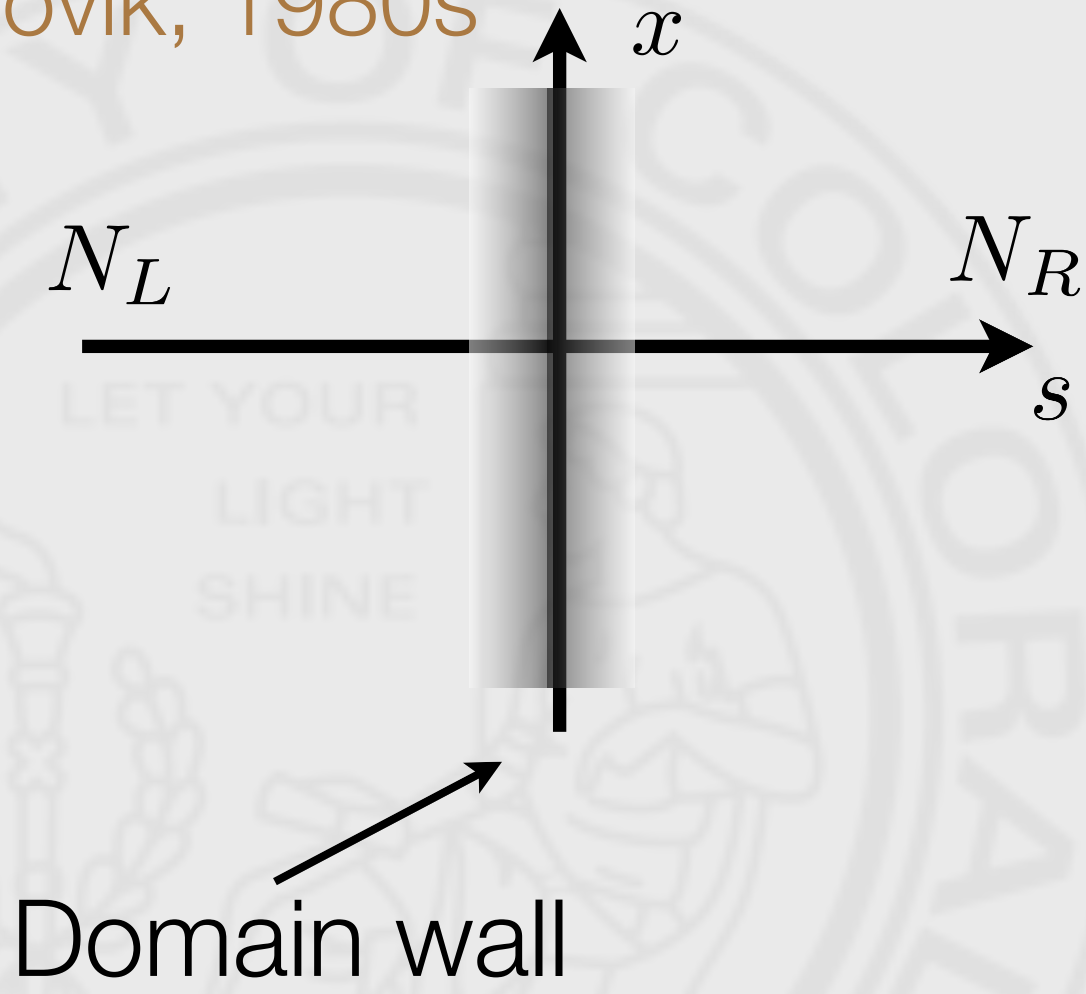
Wigner transformed Green's function

$$G(\omega; p_x; p_s, s) = \int dr\, e^{-ip_s r}\, G\left(\omega; p_x; s + \frac{r}{2}, s - \frac{r}{2}\right)$$

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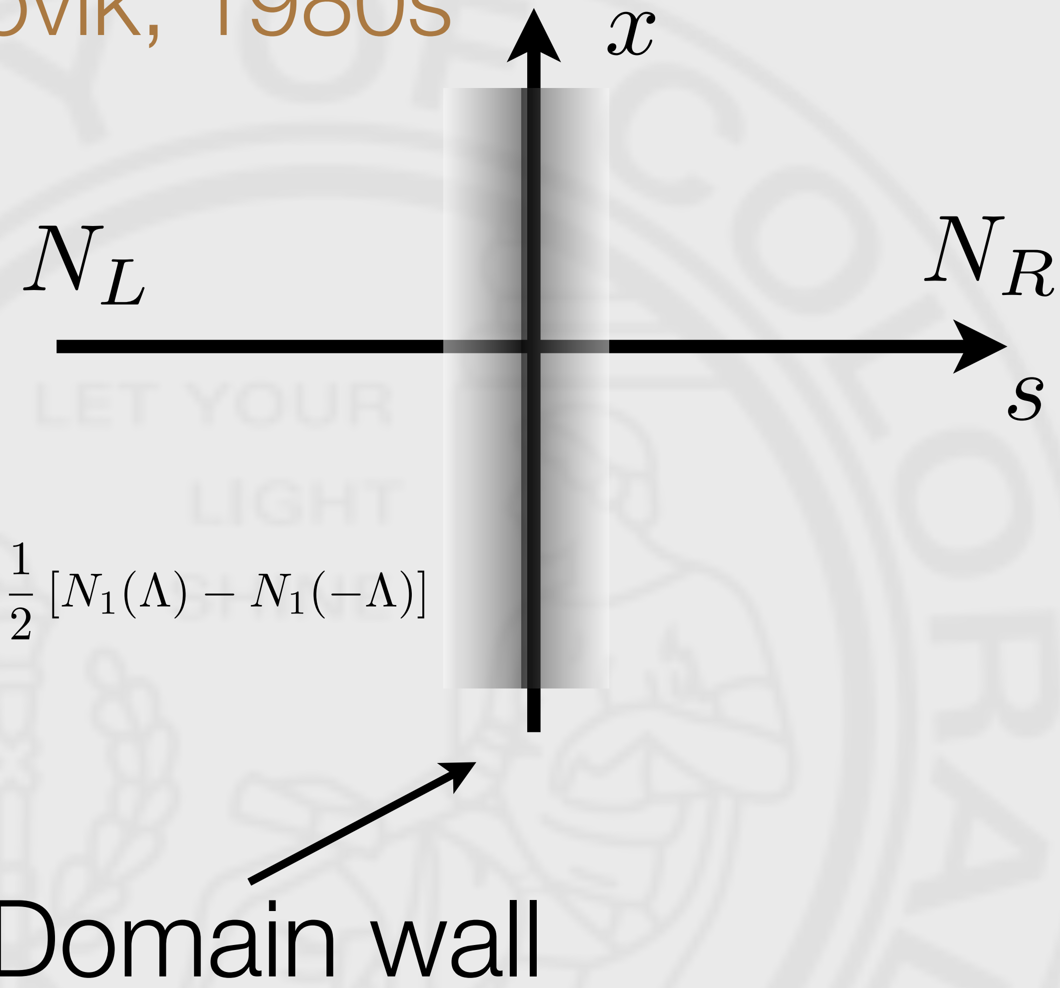


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## Gradient (Moyal product) expansion

$$K(\omega;p_x;p_s,s) = G^{-1} + \frac{1}{2i}G^{-1}\left(\partial_s G\, G^{-1}\, \partial_{p_s} G - \partial_{p_s} G\, G^{-1}\, \partial_s G\right) G^{-1} + \ldots$$

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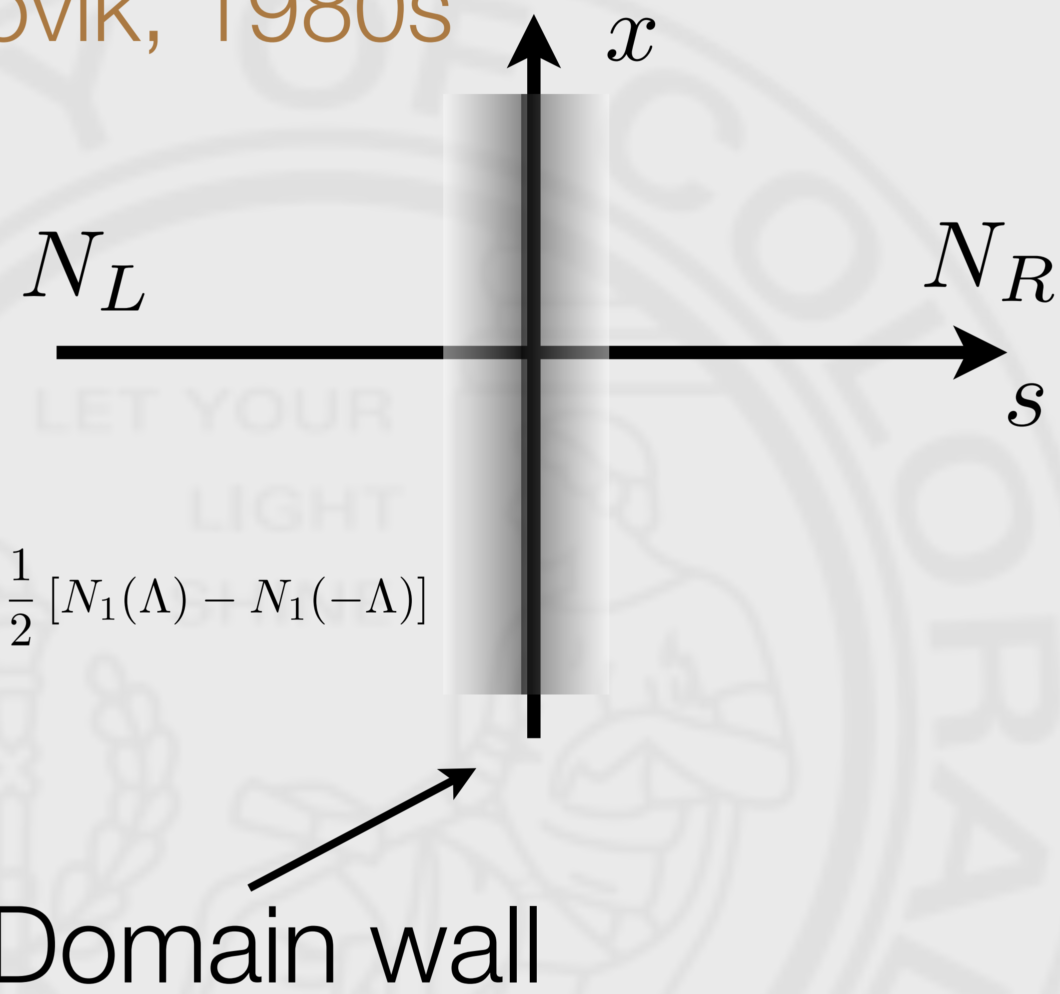


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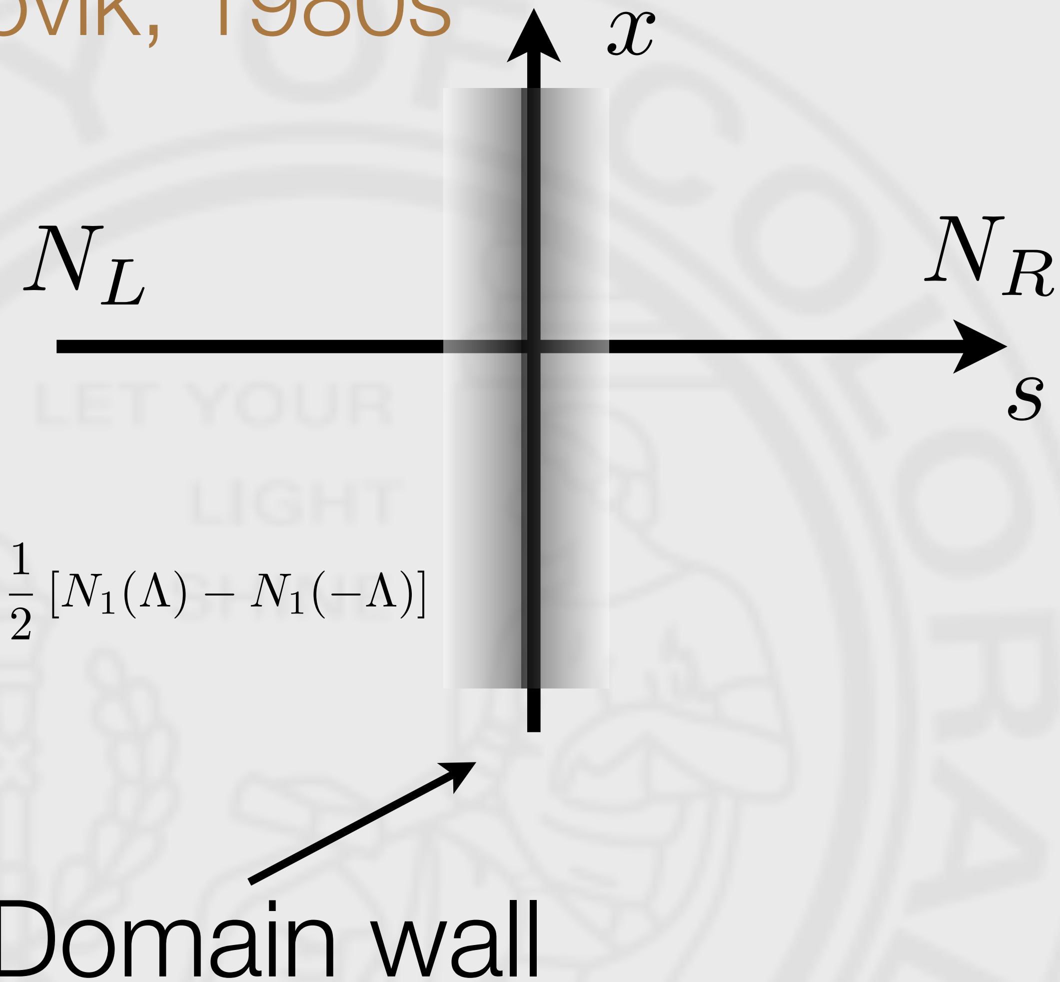


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$$n_\alpha = \sum_{\beta,\gamma,\delta} \epsilon_{\alpha\beta\gamma\delta}\, G^{-1}\partial_\beta G\, G^{-1}\partial_\gamma G\, G^{-1}\partial_\delta G$$

4-dim vector, space **ω**, $p_x$, $p_s$, $s$

$$\#(\text{zero modes}) = \int dS_\alpha\, n_\alpha$$

# d=2; Classes A, D, C

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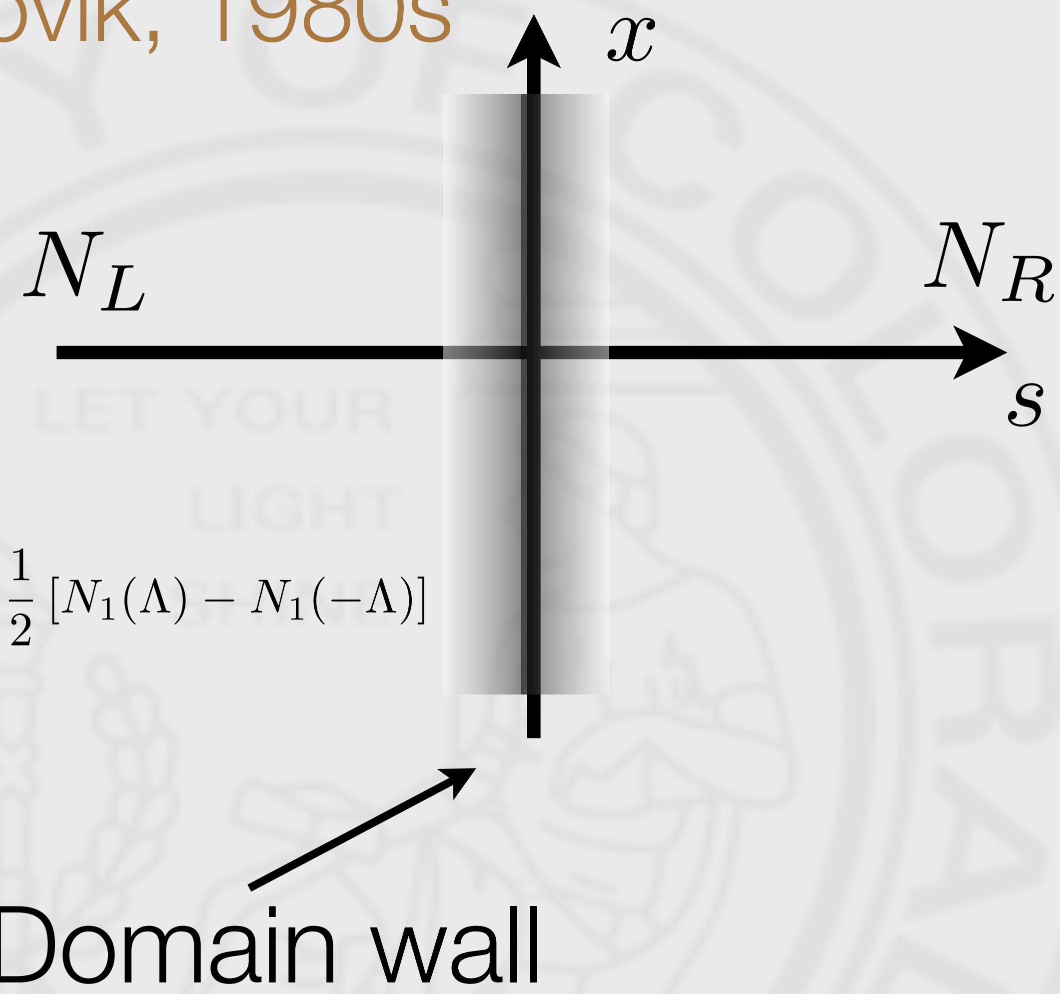


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$$\#(\text{zero modes}) = \int dS_\alpha\, n_\alpha$$

$$\#(\text{zero modes}) = N_R - N_L$$

# d=2; Classes A, D, C

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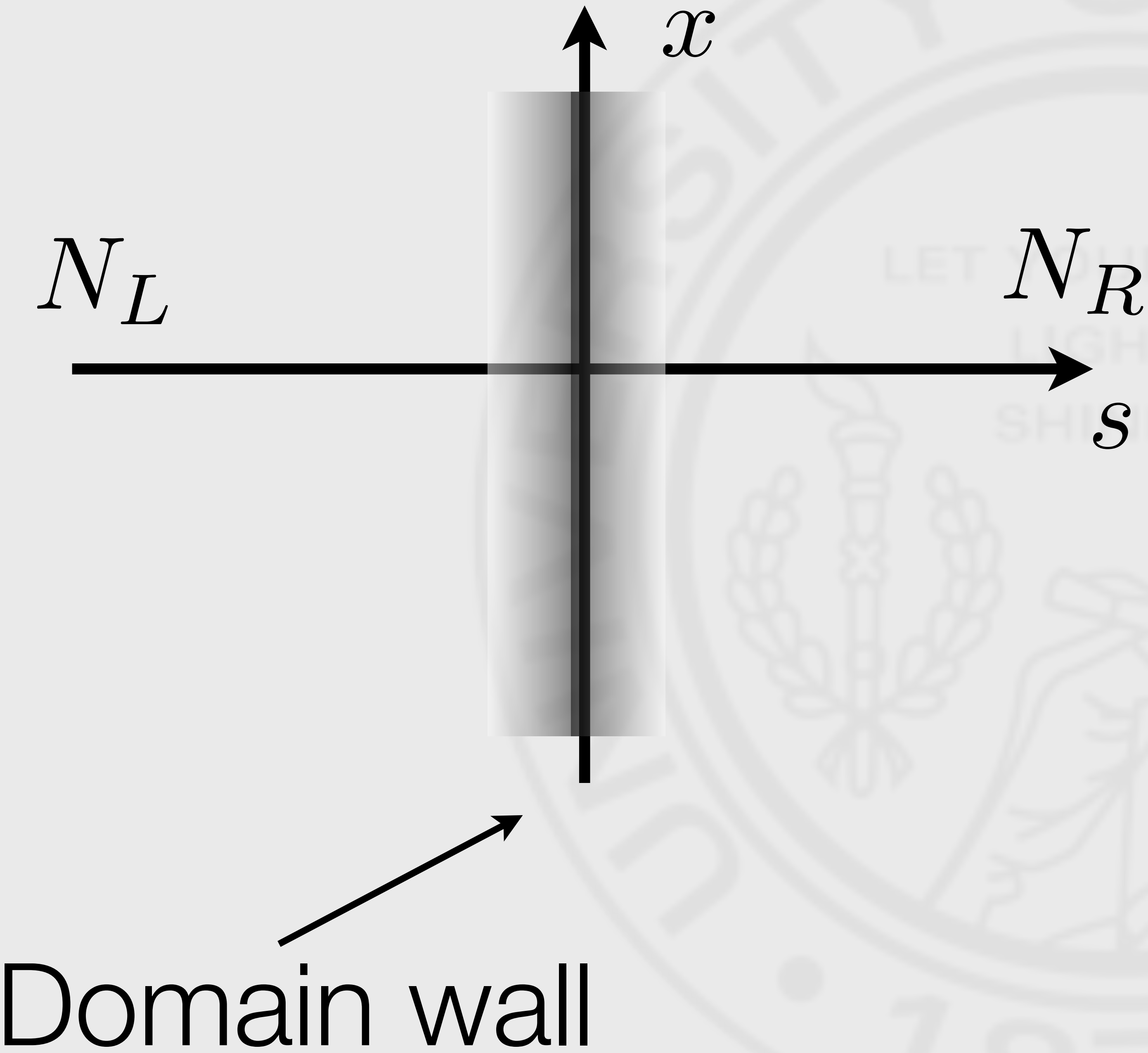


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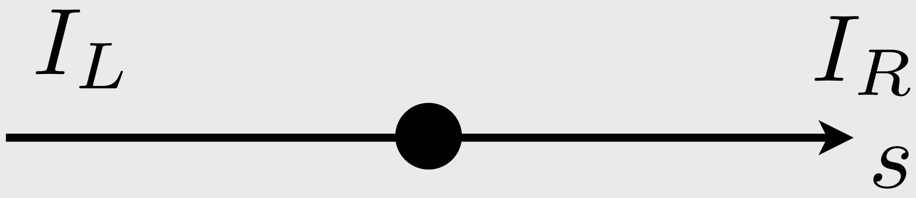
# d=1, classes AIII, BDI, CII




$$Q(\omega, \mathbf{p}_s, s) = G^{-1}(\omega, \mathbf{p}_s, s)\,\Sigma\, G(\omega, \mathbf{p}_s, s)$$

$$I(s) = \frac{1}{16\pi i}\mathrm{tr}\int_0^\infty d\omega \int_{-\infty}^\infty dp_s Q\left(\partial_\omega Q \partial_{p_s} Q - \partial_{p_s} Q \partial_\omega Q\right)$$

$$I(L) - I(-L) = \frac{1}{16\pi i}\lim_{\omega\to 0}\mathrm{tr}\int dx dk\, Q\left(\partial_x Q \partial_{p_s} Q - \partial_{p_s} Q \partial_x Q\right)$$

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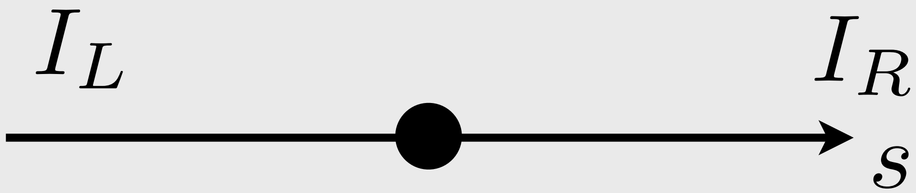


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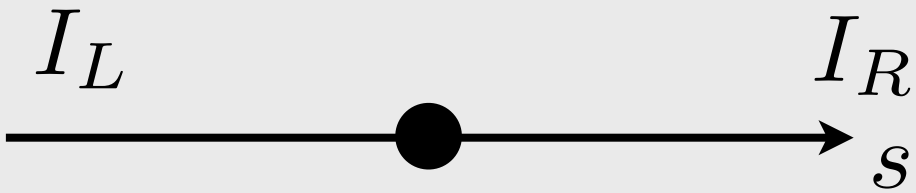


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gradient expansion

$$\#\,(\text{zero modes}) = I(L) - I(-L)$$

# Other classes of topological insulators



Relationship between the edge states and the Green's function topological invariant

1. All nonchiral classes in even d higher than 2:
A.W.W. Ludwig, A. Essin, VG, 2010 (in preparation)

2. Chiral classes in odd d higher than 1.
A. Essin, VG, 2010 (in preparation)

3. $Z_2$ topological invariants,
A. Essin, VG, 2010 (in preparation)

# Topological invariants in the presence of interactions

# The invariant at *d*=0, *D*=1 with interactions

VG, 2010

$$N_1 = \mathrm{tr} \int_{-\infty}^{\infty} \frac{d\omega}{\pi i} G^{-1} \partial_\omega G \; = \int_{-\infty}^{\infty} \frac{d\omega}{\pi i} \, \partial_\omega \det G$$

$$\boxed{N_1 = \sum_n \mathrm{sign}\, \epsilon_n}$$

No interactions

$$\det G = \prod_n \frac{1}{i\omega - \epsilon_n}$$

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In the presence of interactions

$$G_{ij}(\omega) = \sum_{n, \epsilon_n > 0} \frac{\langle 0 | \hat{a}_i | n \rangle \langle n | \hat{a}_j^\dagger | 0 \rangle}{i\omega - \epsilon_n} + \sum_{n, \epsilon_n < 0} \frac{\langle 0 | \hat{a}_j^\dagger | n \rangle \langle n | \hat{a}_i | 0 \rangle}{i\omega - \epsilon_n}$$

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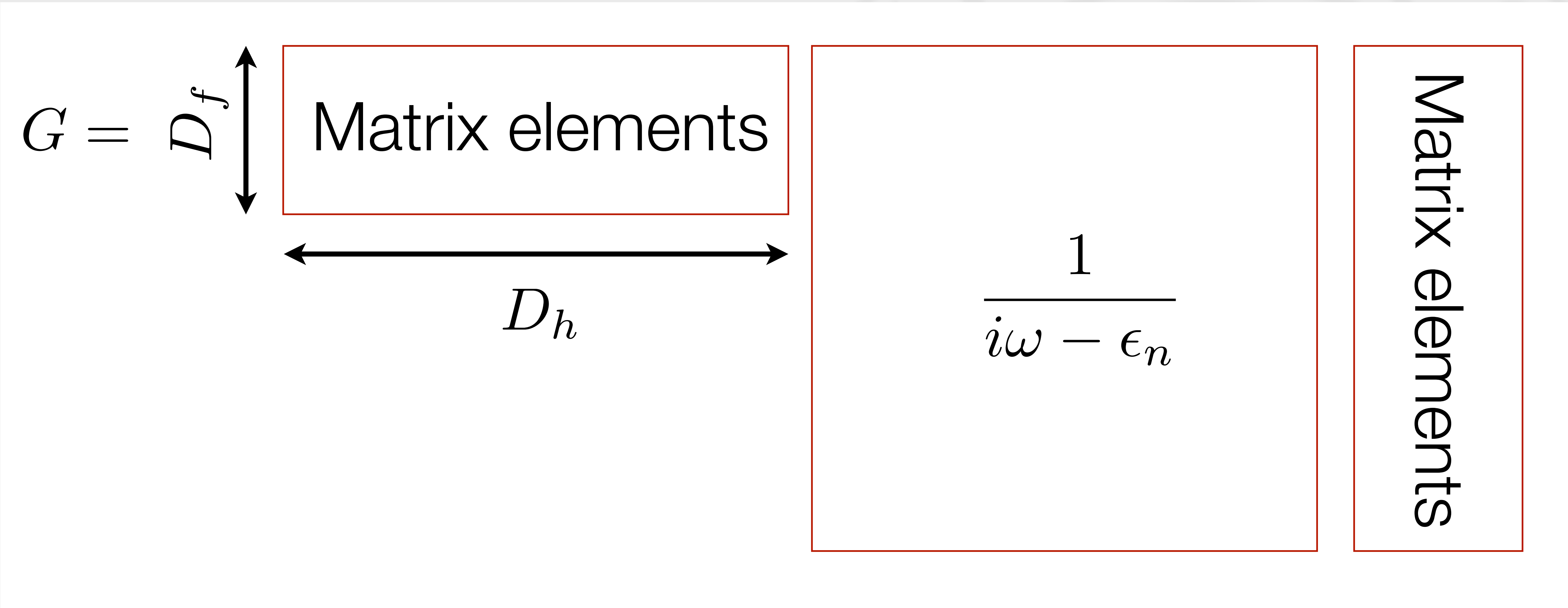


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$$\det G = \frac{\prod_{n=1}^{D_h - D_f} (i\omega - r_n)}{\prod_{n=1}^{D_h} (i\omega - \epsilon_n)}$$

← zeroes of the Green's function (numerator)

← poles of the Green's function (denominator)

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Switching on interactions

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Poles and zeroes can emerge and disappear in pairs

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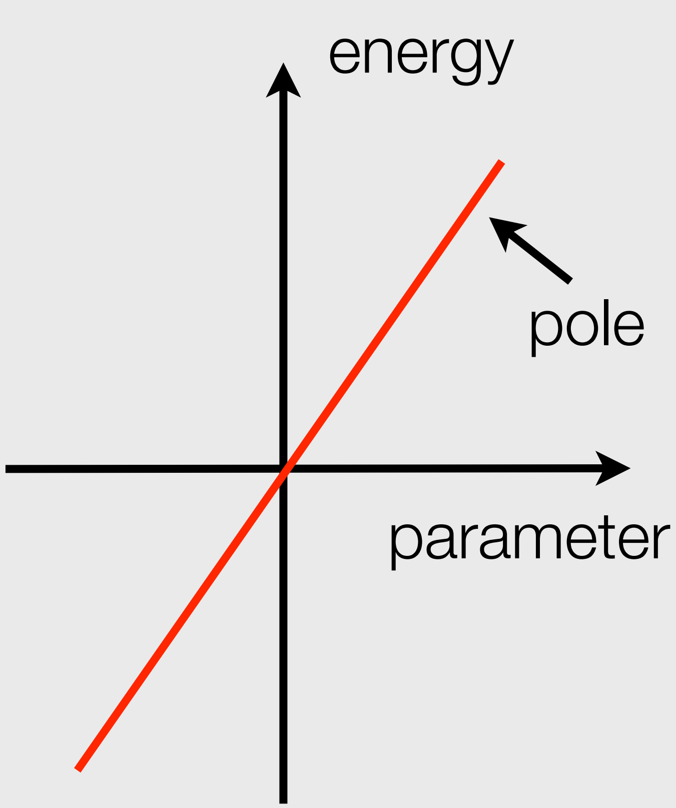

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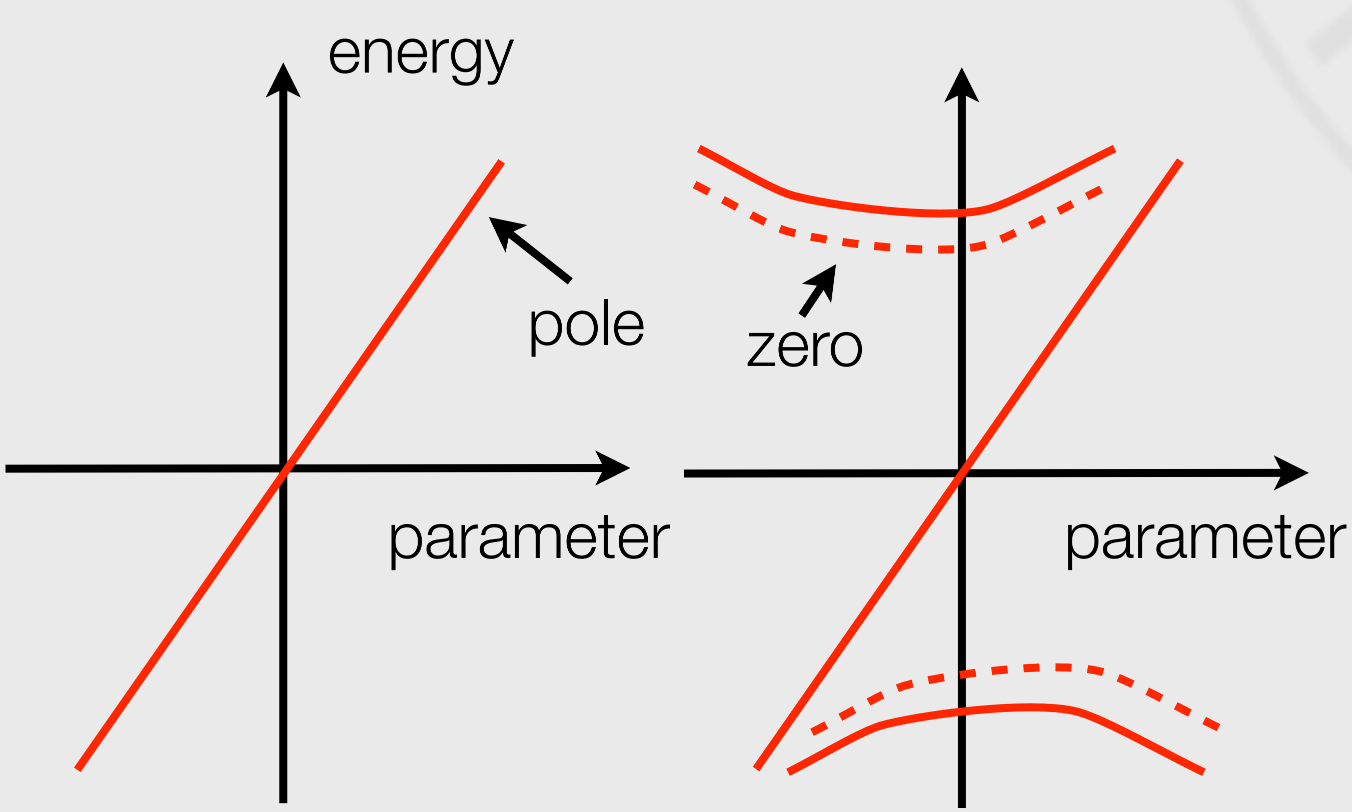

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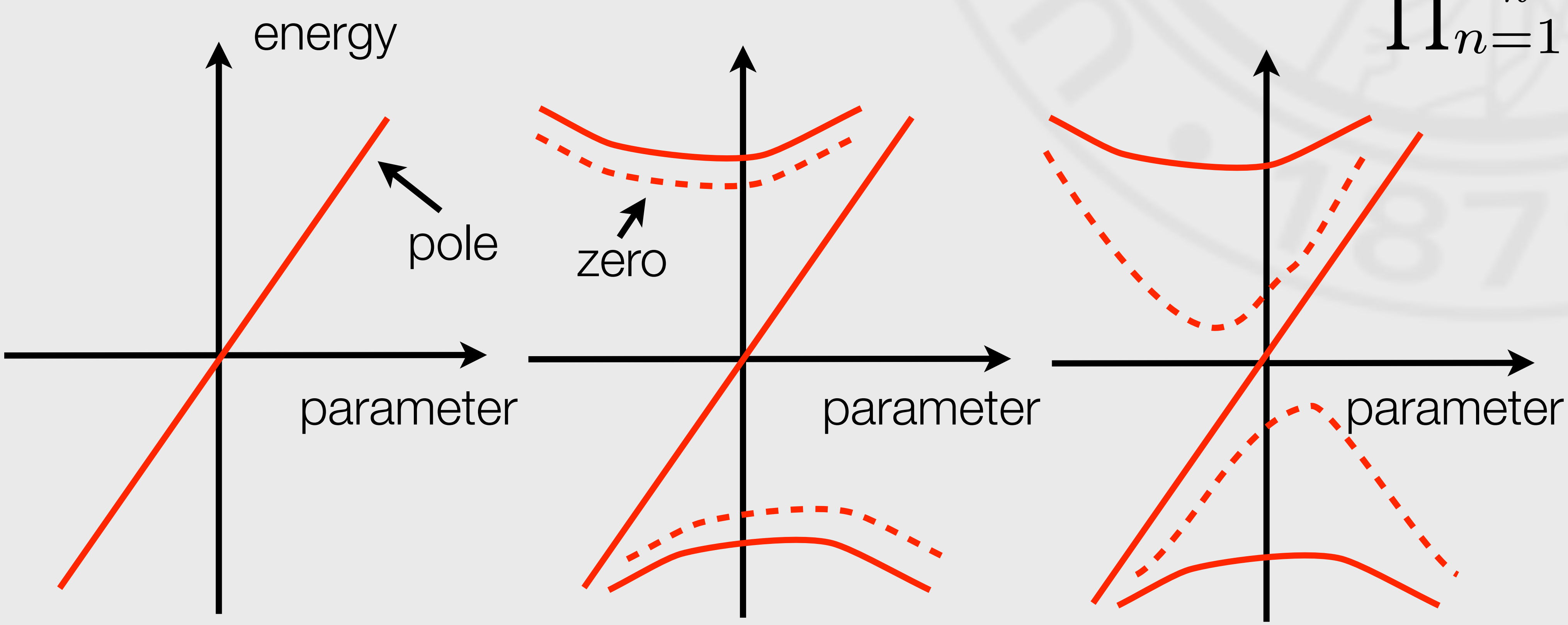

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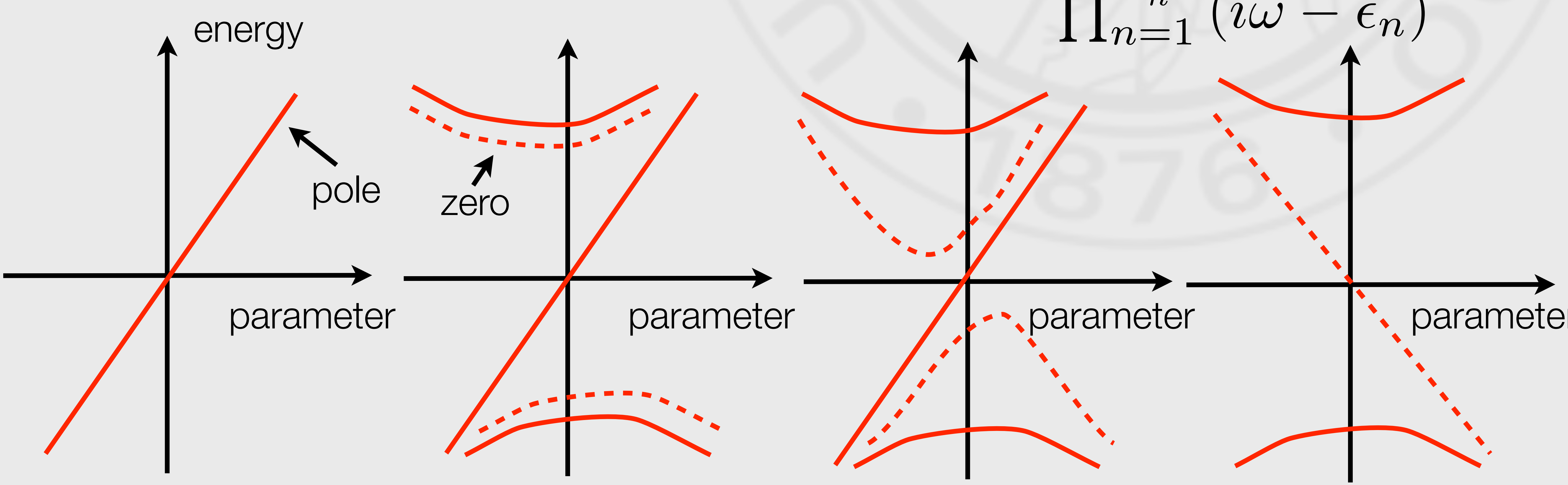
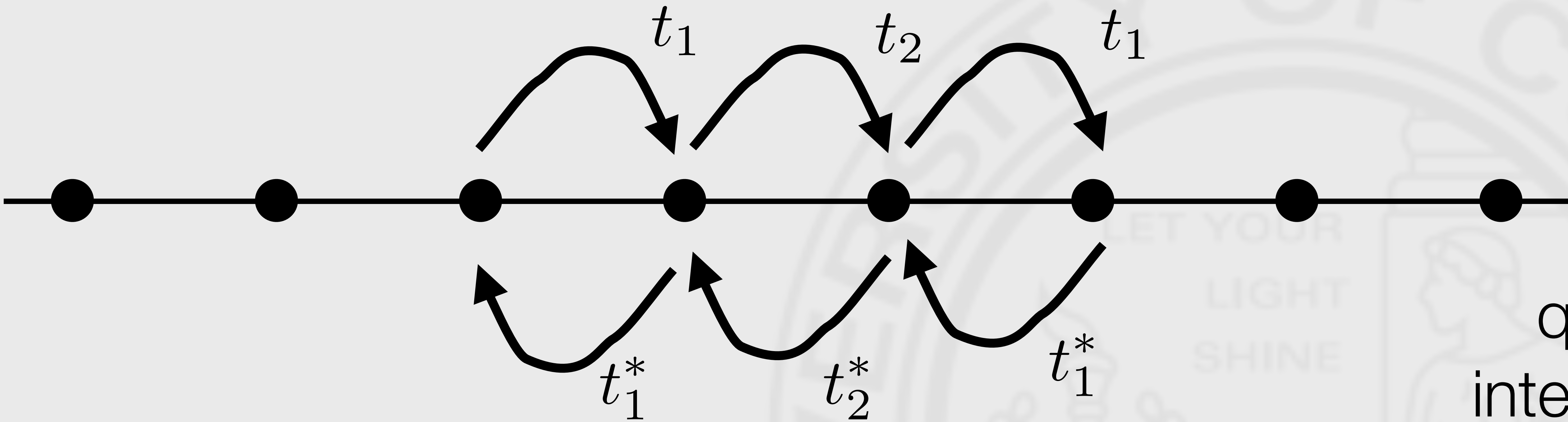
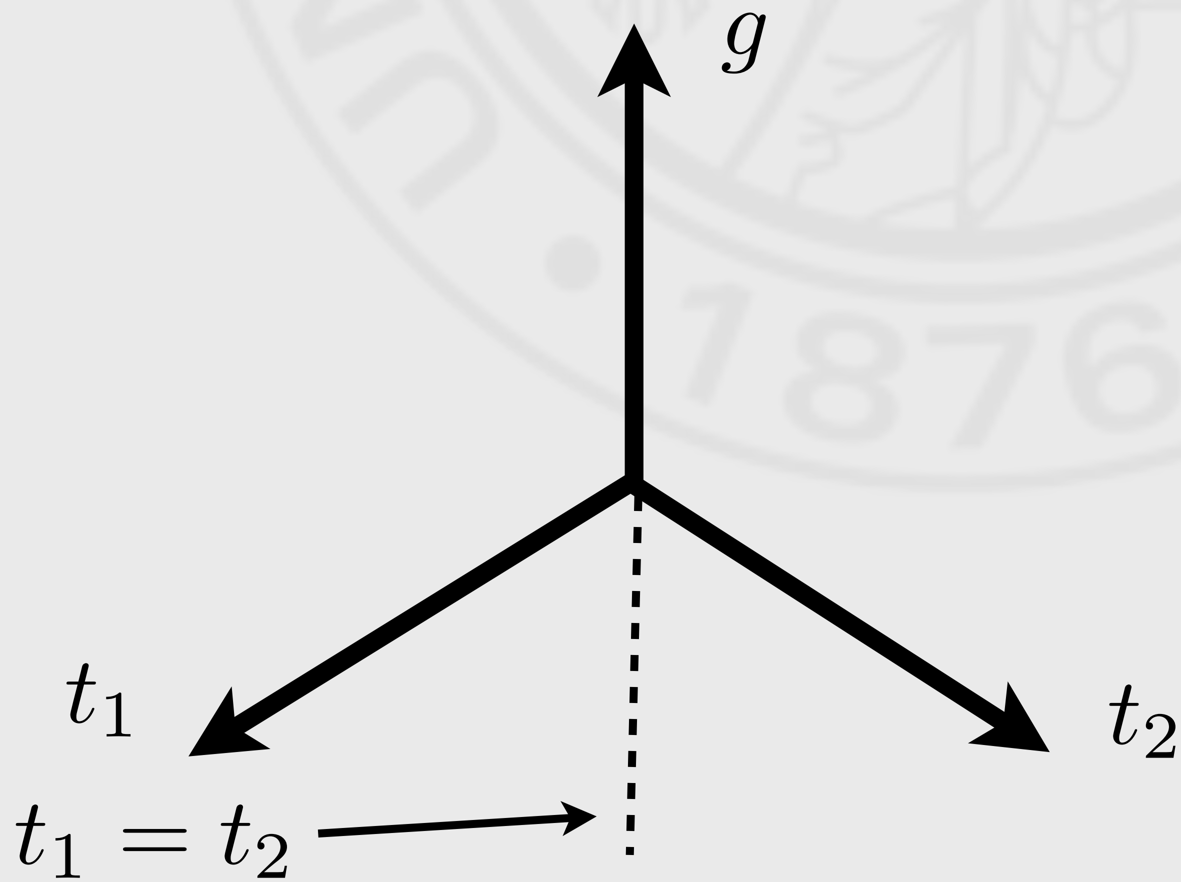


# Fidkowski-Kitaev model (2010)




quartic interactions

$$\hat{H} = \sum_x \sum_{\mu=1}^{4} \left[ t_1\, \hat{a}^\dagger_{x+1,\mu} \hat{a}_{x,\mu} + t_2\, \hat{a}^\dagger_{x+2,\mu} \hat{a}_{x+1,\mu} \right] + h.c. + gW$$



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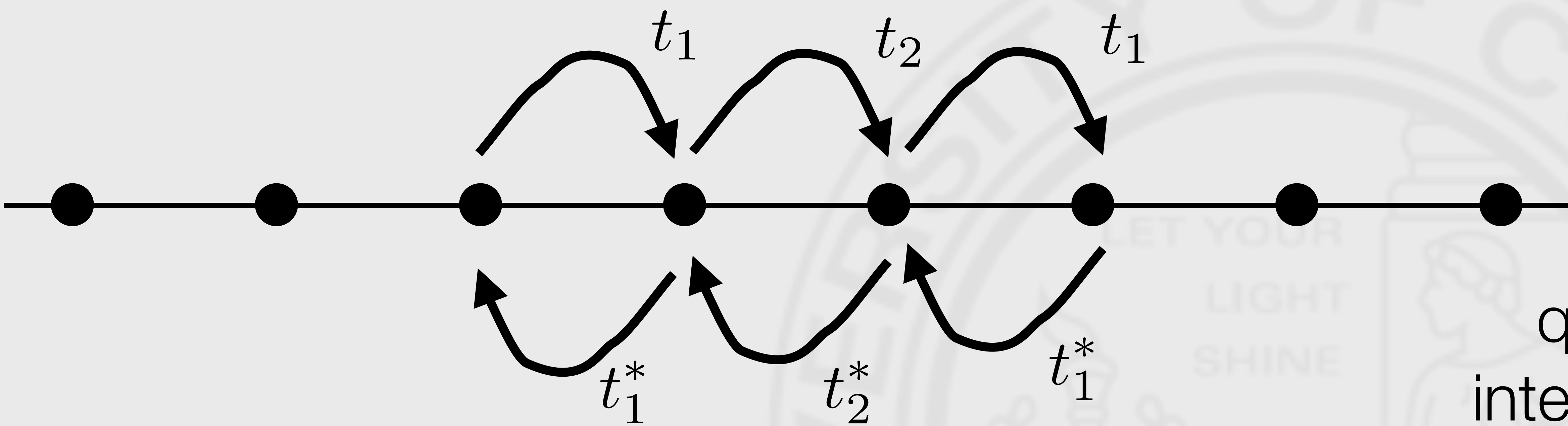


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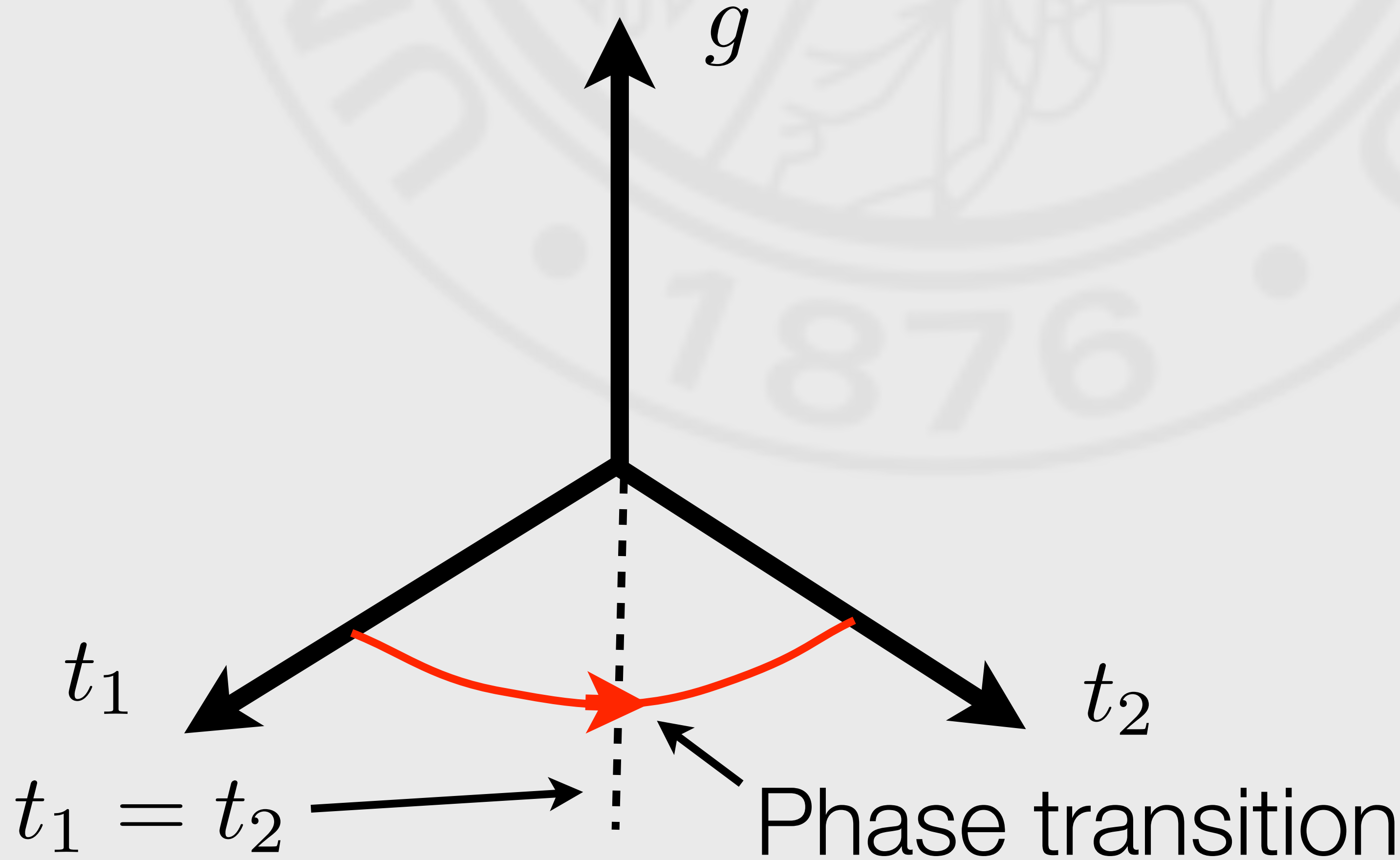

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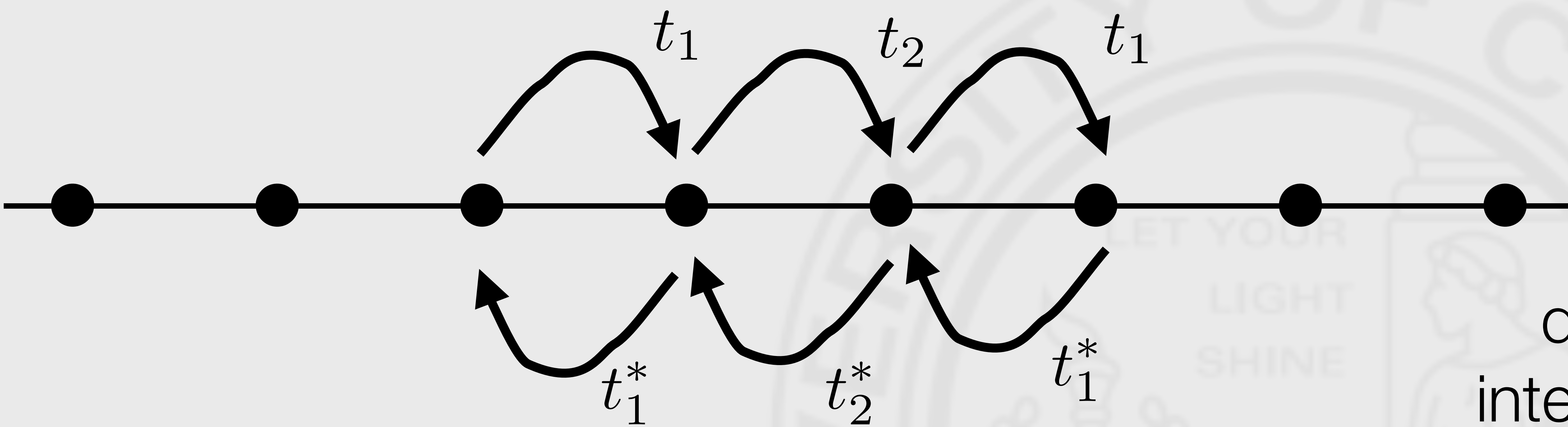


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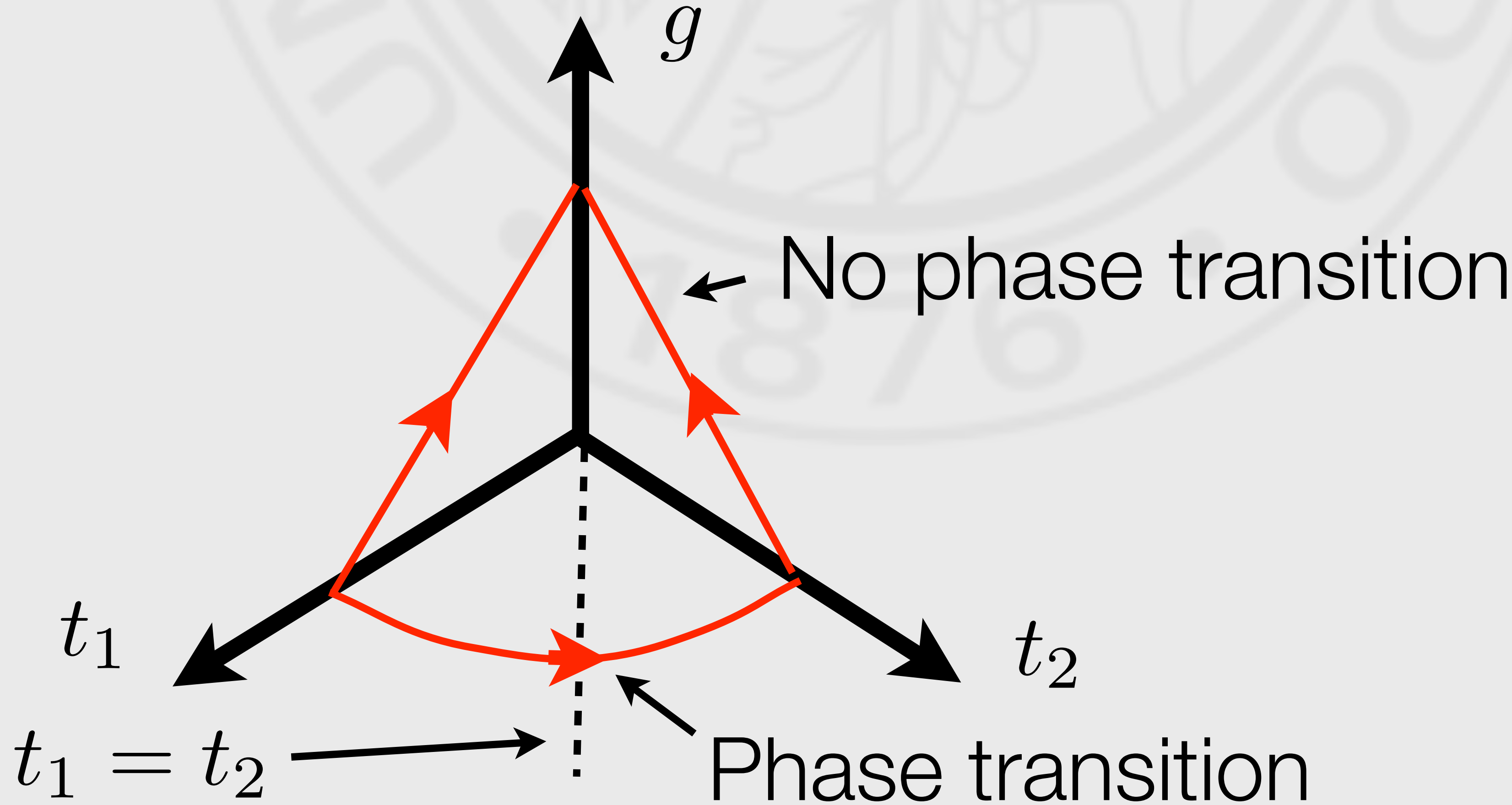

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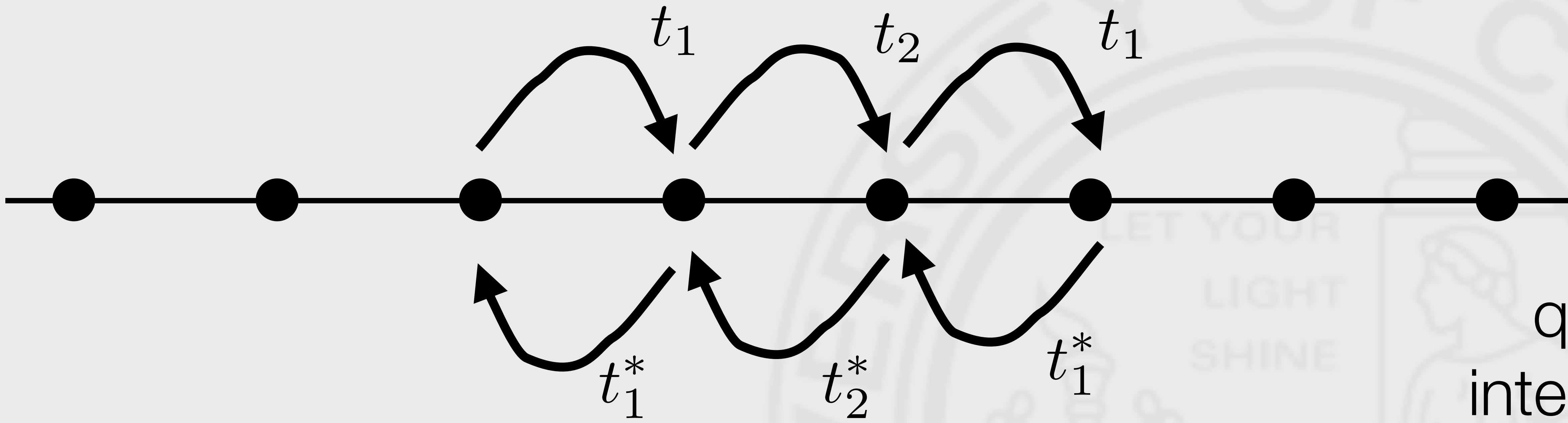


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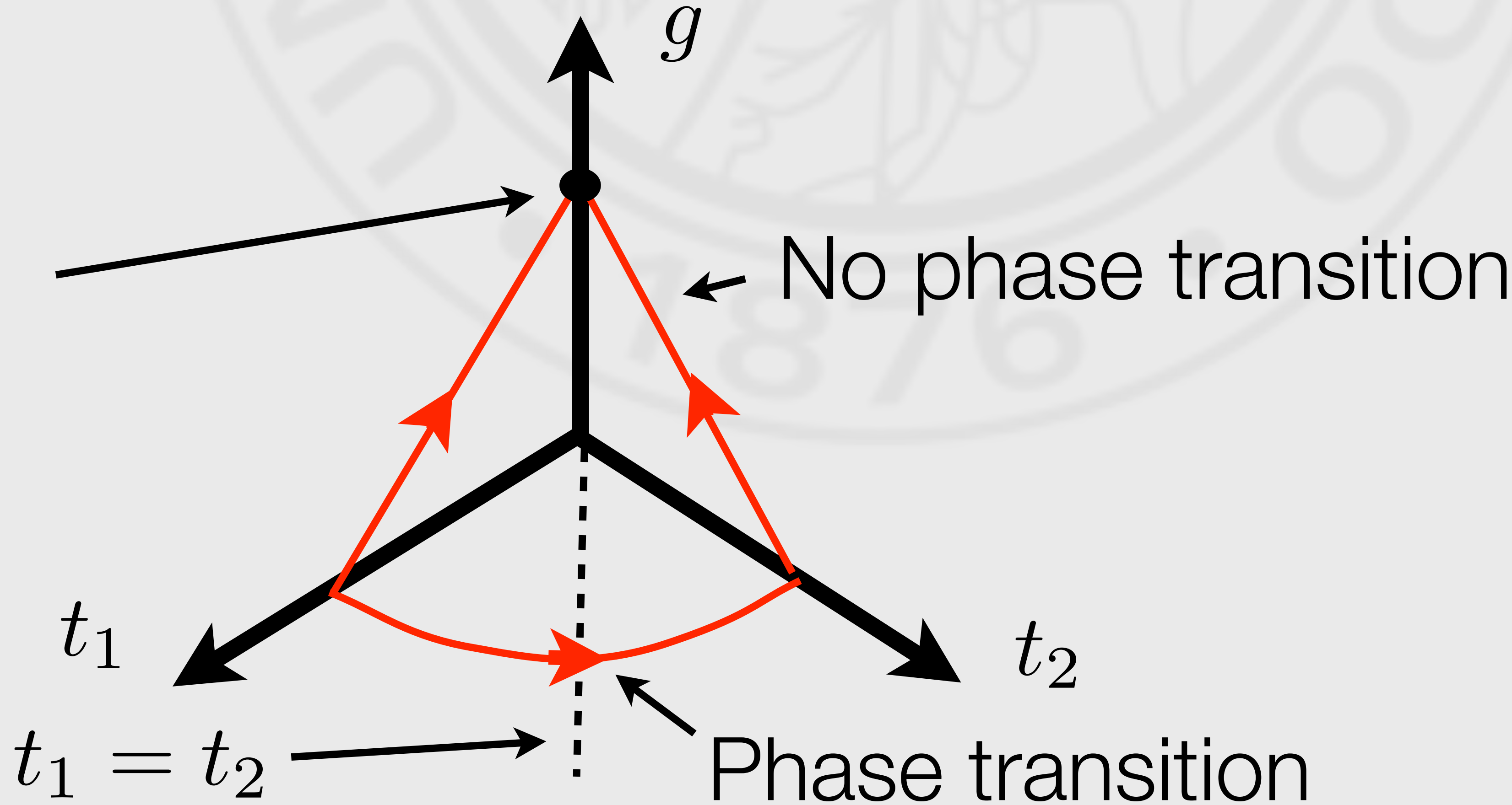
A. Essin, VG:

Green's function has a zero at zero energy



# Conclusions and open questions



1. Single particle Green's functions - a powerful tool to understand topological insulators without or even with interactions.

2. Zeroes of the Green's functions. What are they, when do they appear, how can they be detected, why are they important for interacting topological insulators?

# The end