Phil. 2440
Course Requirements, What is logic?

To discuss today:

About the class:

Some general course information
Who should take this class?
Course mechanics
What you need to do

About logic:

Why is it important?

About the course:

Some general course information

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Office hours: MWF, 1-2, in Prufrock's.
Web page: http://home.sprynet.com/~owl1/244.htm

Subject matter of the course:

Propositional logic
Predicate logic
Set theory
Metalogic + Gödel's Theorem

Course requirements:

Tests.
Homework problems. Guidelines (see syllabus):
- May discuss, do not copy
- Lateness: 2/3 credit
- Sending by email
- Grading

Miscellaneous guidelines for the course:

Come on time.
Come to office hours.
Question.
Grading: the curve:

(Adjusted grade) = (Raw score)(n) + 100(1- n)

What do you need to do now?

- Get the course reader.
- Read the syllabus.
- Read chapter 1.
- For Friday: do questions on chapter 1

About logic:

Why is logic important for philosophers?

- The importance of arguments in philosophy
- Logic teaches us about the structure of propositions.
  - Many philosophical theses/issues could not be formulated without modern, formal logic.
  - You should be able to understand modern philosophers.
- Can logic help us make progress in philosophy?
  - To think about: how did modern science make progress?
  - The role of mathematics in modern science.

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Chapter 1: General Introduction

To discuss today:

- What logic is
- Arguments
- Basic concepts used in logic
- Some silly-sounding principles of logic

The subject matter of logic:

- What is logic?
- What is reasoning?
- The importance of reasoning
- 'Correct' vs. 'incorrect' reasoning. Two kinds of mistakes:
  - False premises
  - Invalid reasoning
- Logical vs. psychological questions.
  - Logical: is this a good argument for that?
  - Psychological: why do people believe this?

About Arguments:

- What are they?
- Premises & conclusions
- Validity and soundness
  - 'Valid' arguments: It is impossible that the premises all be true and the conclusion be false.
  - 'Sound' arguments: Valid + all true premises
- Deductive, inductive, and other arguments
  - Deductive: purports to be valid
  - Non-deductive: purports to support conclusion but not to be valid. Renders conclusion more probable.
  - Inductive: example: "All ravens so far observed have been black. So (probably) all ravens are black.

Important logical concepts & distinctions

- Statements vs. sentences
- Statements, beliefs, and propositions
- What is truth?
  - Aristotle: "To say of what is, that it is, is true."

Logical possibility--the received view. Which of the following are possible?
- "The solar system has nine planets."
- "The solar system has 62 planets."
- "My cat wins the world chess championship next year."
- "The law of conservation of mass/energy is false."
- "My car is completely red and completely green at the same time."
- "Sam is a married bachelor."
- "2 + 2 = 7."
- "It is raining and it is not raining."
- "Some things are neither red nor not red."

What is wrong with the received view

Other senses of 'possible'
- Epistemic possibility
- Physical possibility
- Metaphysical possibility

Logical truth and falsity

Contingent propositions

Contradictions

Entailment

Logical equivalence

Silly doctrines of modern logic

Is this valid:
- It is raining.
- It is not raining.
- Therefore, Skeezix is furry.

Is this valid:
- All men are mortal.
- Socrates is a man.
- Therefore, it is either raining or not raining.

Three definitions of "valid"
- If the premises are true, the conclusion must be true.
- The conclusion follows from the premises.
- It is not possible that the premises be true and the conclusion be false.

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Chapter 2: Propositional Calculus Symbolizations

To discuss today:

Formal systems in general.
How to symbolize things in propositional logic.
Miscellaneous logical terminology/concepts.

About formal systems

What's a formal system?
What are formal systems good for?
The propositional calculus
Compound vs. atomic sentences

Propositional calculus symbols

symbol	what it means	example	other comments
A	Stands for any atomic sentence.	"Alice got a haircut" can be symbolized as A. "Bert owns a cat" can be symbolized as B.	You can use any capital letters, not just "A".
∨	or	"Bill has an elephant in his apartment, or he's very fat" = (E ∨ F)	This symbol is called a "vel".
&	and	"I went to the store today and I bought a cow" = (S & C)	Sometimes people use "•" or "∧" (without the quote marks) for this.
~	not	"I did not go to the store today" = ~S	This one is called a "tilde". Sometimes they use "¬".
→	If ... then ...	"If Bill Clinton was a great President, then I'm a monkey's uncle" = (G → M)	People also use "⊃".
↔	... if and only if ...	"I will go to the party if and only if you go" = (I ↔ Y)	People also use "≡".

symbol	what it means	example	other comments
( )	Parentheses are used to avoid ambiguity (see below).	"If Liz and Sue go, I will go" = ((L & S) → I) "Liz will go, and if Sue goes I will go" = (L & (S → I))	Used when you join together two other sentences with "∨", "&", "→", or "↔"

Things to notice:

Use parentheses to avoid ambiguity.

Inclusive 'or'.

If and only if

"And" in English

Other terminology

"propositional constant"

"connective"

"conjunction", "conjunct"

"disjunction", "disjunct"

"negation", "negatum"

"conditional", "antecedent", "consequent"

"biconditional"

Other English connectives

"but", "so", "although"

"A only if B"

"A if B"

"provided that", "assuming"

"unless"

"neither ... nor"

"not both"

Miscellaneous stuff

Well-formed formulas

Compound vs. atomic sentences: compound sentence contains connective(s)

The main connective

Propositional variables

Forms: What are they?

Substitution instances

What are sentence forms good for?

A sentence can have many different forms

- Example: $(B \leftrightarrow (B \& C))$
- Forms:
  - $[p \leftrightarrow (p \& q)]$
  - $[p \leftrightarrow (q \& r)]$
  - $(p \leftrightarrow q)$
  - p

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Chapter 3: Truth Tables

To Discuss Today:
- Truth tables:
  - Defining connectives with them
  - Using them to evaluate arguments
- Limitations of propositional logic

Truth Tables for Defining Connectives

Background concepts:
- Truth values
- Functions
- Truth-functions & "truth-functional" connectives

What is a truth-table?

	p	q	(p & q)
1.	T	T	T
2.	T	F	F
3.	F	T	F
4.	F	F	F

p	~p
T	F
F	T

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

The material conditional & material equivalence:

"If A then B" = "Not: (A and not-B)"

Truth-table:

p	q	p → q
T	T	T
T	F	F
F	T	T

p	q	p ↔ q
T	T	T
T	F	F
F	T	F

F	F	T

F	F	T

Defining connectives in terms of other connectives

(A ↔ B) = (A → B) & (B → A)

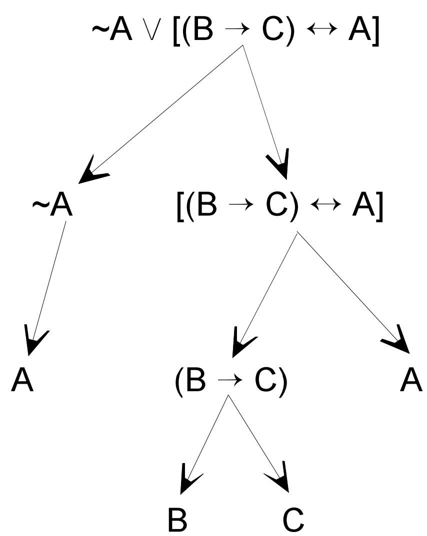
(A → B) = ~(A & ~B)

(A & B) = ~(~A ∨ ~B)

Q: Can all the connectives be defined in terms of a single connective?

Truth-Tables for Evaluating Arguments

Breaking a complex sentence into parts



Result: A, B, C, ~A, (B → C), [(B → C) ↔ A], {~A ∨ [(B → C) ↔ A]}

Truth-tables for complex sentences

We need 2^n lines in the table, n=# of atomic sentences

Columns for each part of the sentence

Fill in T's and F's for atomic sentences

	A	B	C	~A	B → C	(B → C) ↔ A	~A ∨ [(B → C) ↔ A]
1.							
2.							
3.							
4.							
5.							
6.							
7.							
8.							

	A	B	C	~A	B → C	(B → C) ↔ A	~A ∨ [(B → C) ↔ A]
1.	T	T	T				
2.	T	T	F				
3.	T	F	T				
4.	T	F	F				
5.	F	T	T				
6.	F	T	F				
7.	F	F	T				
8.	F	F	F				

	A	B	C	~A	B → C	(B → C) ↔ A	~A ∨ [(B → C) ↔ A]
1.	T	T	T	F	T		
2.	T	T	F	F	F		
3.	T	F	T	F	T		
4.	T	F	F	F	T		
5.	F	T	T	T	T		
6.	F	T	F	T	F		
7.	F	F	T	T	T		
8.	F	F	F	T	T		

	A	B	C	~A	B → C	(B → C) ↔ A	~A ∨ [(B → C) ↔ A]
1.	T	T	T	F	T	T	T
2.	T	T	F	F	F	F	F
3.	T	F	T	F	T	T	T
4.	T	F	F	F	T	T	T
5.	F	T	T	T	T	F	T
6.	F	T	F	T	F	T	T
7.	F	F	T	T	T	F	T
8.	F	F	F	T	T	F	T

More compact way of doing truth tables:

Stage 1:

	A	B	C	~A ∨ [(B → C) ↔ A]
1.	T	T	T	F T
2.	T	T	F	F F
3.	T	F	T	F T
4.	T	F	F	F T
5.	F	T	T	T T
6.	F	T	F	T F
7.	F	F	T	T T
8.	F	F	F	T T

Stage 2:

	A	B	C	~A ∨ [(B → C) ↔ A]
1.	T	T	T	F T T
2.	T	T	F	F F F
3.	T	F	T	F T T
4.	T	F	F	F T T
5.	F	T	T	T T F
6.	F	T	F	T F T
7.	F	F	T	T T F
8.	F	F	F	T T F

Stage 3:

	A	B	C	~A ∨ [(B → C) ↔ A]
1.	T	T	T	F T T T
2.	T	T	F	F F F F

3.	T	F	T	F T T T
4.	T	F	F	F T T T
5.	F	T	T	T T T F
6.	F	T	F	T T F T
7.	F	F	T	T T T F
8.	F	F	F	T T T F

Testing validity

Is there a line in which all premises are true & conclusion is false?

Example: Is this valid?:

~(A → B)

∴ (A ∨ B)

	A	B	A → B	~(A → B)	A ∨ B
1.	T	T	T	F	T
2.	T	F	F	T	T
3.	F	T	T	F	T
4.	F	F	T	F	F

Other uses:

Identifying contradictions

Identifying tautologies

Contingent propositions

Limitations of the Propositional Calculus

The material conditional: Does it correspond to the ordinary meaning of "if...then"? Are these valid:

Example 1:

If I put sugar in my coffee, it will taste fine.

∴ If I put sugar and motor oil in my coffee, it will taste fine.

Example 2:

I have no orange juice in the refrigerator.

∴ If I have orange juice in the refrigerator, then the world will come to an end.

Example 3:

It's not the case that if God exists, the universe is the product of blind chance.

∴ God exists.

The test of validity

Example: Is this valid?:

Socrates is a man.

All men are mortal.

∴ Socrates is mortal.

Symbolization:

S

A

∴ M

Truth table:

		S	A	M
	1.	T	T	T
☞	2.	T	T	F
	3.	T	F	T
	4.	T	F	F
	5.	F	T	T
	6.	F	T	F
	7.	F	F	T
	8.	F	F	F

Wait for the predicate calculus.

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Chapter 4: Propositional Logic Proofs

To Discuss Today:

How to do proofs
A bunch of inference rules
Reductio ad absurdum & conditional proof

What Are Inference Rules?

What is a rule of inference?
Implications versus equivalences

Seven Simple Rules

Addition (add):

$$\frac{p}{p \vee q} \qquad \frac{q}{p \vee q}$$

Conjunction (conj):

$$\frac{\begin{matrix} p \\ q \end{matrix}}{p \mathbin{\&} q}$$

Commutative Law (comm):

$p \mathbin{\&} q \equiv q \mathbin{\&} p$
$p \vee q \equiv q \vee p$

Double Negation (DN):

$p \equiv {\sim}{\sim}p$

Material Implication (impl):

$p \rightarrow q \equiv {\sim}p \vee q$

Material Equivalence (equiv):

$p \leftrightarrow q \equiv (p \rightarrow q) \mathbin{\&} (q \rightarrow p)$
$p \leftrightarrow q \equiv (p \mathbin{\&} q) \vee ({\sim}p \mathbin{\&} {\sim}q)$

DeMorgan's Law (DeM)

${\sim}(p \mathbin{\&} q) \equiv ({\sim}p \vee {\sim}q)$
${\sim}(p \vee q) \equiv ({\sim}p \mathbin{\&} {\sim}q)$

Using the rules in a proof

Example:

Given: ~A, B.
To prove: ~(B → A).

1.	~A	**premise**

2.	B	**premise**
3.	~A & B	1,2 **conj**
4.	~~(~A & B)	3 **DN**
5.	~(~~A ∨ ~B)	4 **DeM**
6.	~(A ∨ ~B)	5 **DN**
7.	~(~B ∨ A)	6 **comm**
8.	~(B → A)	7 **impl**

Reductio ad Absurdum & Conditional Proof

The idea of reductio ad absurdum

p	~p
⋮	⋮
q & ~q	q & ~q
~p	p

The idea of conditional proof

p
⋮
q
―――
p → q

Examples: proofs for 3 famous laws of logic

Example 1: Law of Excluded Middle: A ∨ ~A.

→1.	~(A ∨ ~A)	**Assumption**
2.	~A & ~~A	1 **DeM**
3.	(A ∨ ~A)	1-2 **RAA**

Example 2: Law of Non-Contradiction: ~(A & ~A).

→1.	(A & ~A)	a.
2.	~(A & ~A)	1-1 RAA

Example 3: Not really the Law of Identity: (A ↔ A).

→1. A	a.
2. A → A	1-1 CP
3. (A → A) & (A → A)	2,2 conj.
4. (A ↔ A)	3 equiv.

Using assumptions properly

Rules for use of assumptions:

- All assumptions must be discharged
- After assum. is discharged: Do not use steps from inside its scope
- Conclusion must be outside the scope of any assumptions
- Using multiple assumptions: The brackets should not cross

Example: What is wrong with this?

1. A	premise
→2. ~A	a.
3. A & ~A	1,2 conj
4. A	2-3 RAA
→5. ~B	a.
6. A & ~A	1,2 conj
7. B	5-6 RAA

More Rules of Inference

Disjunctive Syllogism (DS):

$$\frac{p \lor q \quad {\sim}p}{q} \qquad \frac{p \lor q \quad {\sim}q}{p}$$

Modus Ponens (MP):

$$\frac{p \to q \quad p}{q}$$

Simplification (simp):

$$\frac{p \,\&\, q}{p} \qquad \frac{p \,\&\, q}{q}$$

Exportation (exp):

$(p \& q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

Modus Tollens (MT):

$$\frac{\begin{array}{l} p \rightarrow q \\ \sim q \end{array}}{\sim p}$$

Hypothetical Syllogism (HS):

$$\frac{\begin{array}{l} p \rightarrow q \\ q \rightarrow r \end{array}}{p \rightarrow r}$$

Constructive Dilemma (CD):

$$\frac{\begin{array}{l} p \rightarrow q \\ r \rightarrow s \\ p \lor r \end{array}}{q \lor s}$$

Contraposition (contra):

$p \rightarrow q \equiv \sim q \rightarrow \sim p$

Tautology (taut):

$p \equiv p \lor p$

$p \equiv p \& p$

Associative Law (assoc):

$p \& (q \& r) \equiv (p \& q) \& r$

$p \lor (q \lor r) \equiv (p \lor q) \lor r$

Distributive Law (dist):

$p \& (q \lor r) \equiv (p \& q) \lor (p \& r)$

$p \lor (q \& r) \equiv (p \lor q) \& (p \lor r)$

Miscellaneous Stuff

Theorems of propositional logic

What are they?

Proof strategy

Memorize all the rules

Start by writing premises

RAA and CP are very useful. CP for proving conditionals.

Look for premises that haven't been used

Try to get A, ~A, B, ~B, etc.

~(A v B) or ~(A & B): use DeM

(A & B): use Simp.

(A → B): use MP or Impl.

~(A → B): convert to A and ~B
Work backwards
Combine the last step with something earlier
Look back at previous steps for pairs that can be combined
Always remember the conclusion
Check your work over

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Chapter 5: Predicate Logic Symbolizations

To discuss today:

Atomic sentences in predicate logic
Quantifiers
Some important kinds of sentences
Well-formed formulas

About the predicate calculus

What is the predicate calculus?
Why is the predicate calculus better than the propositional calculus?

Atomic propositions and their structure

- Predicates and subjects
  - "All cats are furry"
  - Logical vs. grammatical predicates & subjects
  - "It is raining"
- Relations
- Symbolizing things
  - Individuals: a, b, ..., t
  - Predicates: A, B, ...
  - Atomic sentences: Fa, Cb, Rca, ...
- Variables & open sentences
  - Individual variables: x, y, z, u, v, w
  - Open sentences: Fx, Rxy, ...

Quantified sentences

- Quantifiers in English. Examples:
  - All cats are furry.
  - Some cats are furry.
  - Most cats are furry.
  - No cats are furry.
- Quantifiers in predicate logic:
  - Universal quantifier: (∀x), (x), (∀y), (y), ...
  - Existential quantifier: (∃x), (∃y), ...
- Examples:
  - (∃x) Fx
  - (x) Cx
- Domain of a quantifier:
  - "Drinks for everyone!" Interpretations:
    - (x) Dix
    - (x) ((Px & Rx) → Dix)
- Multiple quantifiers:
  - "Someone loves everyone"

= (∃x)(x loves everyone)
= (∃x)(y)(x loves y)
= (∃x)(y) Lxy

Quantifier scope:
A quantifier goes with the first complete sentence following it.
"Bound" vs. "free" variables
Examples:
(∃x) (Cx & Fx)
(∃x) Cx & Fx
(∃x) Cx & (x) Fx
(x) Lxy

Important kinds of sentences and how to symbolize them

"All A's are B" = (x)(Ax → Bx)
"Some A's are B" = (∃x)(Ax & Bx)
"Some A's are non-B" = (∃x)(Ax & ~Bx)
"No A's are B" = (x)(Ax → ~Bx) = ~(∃x)(Ax & Bx)
Existential import
"Only" and "unless":
"Only A's are B" = (x)(Bx → Ax) = (x)(~Ax → ~Bx)
"A thing is A unless it is B" = (x)(~Bx → Ax)
Times and places:
"Someday I'll be famous" = (∃x)(Dx & Fix)
"God is everywhere" = (x)(Px → Lgx)

Well-formed formulas

Include open sentences & complete sentences.
Rules for wff's:

1. Atomic formulas are wff's.
2. If "ϕ" is a wff, then "~ϕ" is a wff.
3. If "ϕ" and "ψ" are wff's then "(ϕ ∨ ψ)", "(ϕ & ψ)", "(ϕ → ψ)", and "(ϕ ↔ ψ)" are wff's.
4. If "ϕ" is a wff, then "(x) ϕ", "(∃x) ϕ", "(y) ϕ", "(∃y) ϕ", etc., are wff's.

Examples:

(Hx → P)	~Acac
(x) y → Fx	(Ha ∨ Fy)
Ax & Fy ∨ Ba	(Ha ∨ (x)Fy)
Ca (∃x)	(∃y)(z) Ax

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Chapter 6: Predicate Logic Proofs

To discuss today:
5 new inference rules
Strategy for predicate logic proofs

Applying the old rules to predicate logic sentences

All the old rules still apply.
Implicational rules: only apply to whole lines.
Examples: which of the following are good?

Example 1:
1. (∃x) Fx → Ga
2. (∃x) Fx
3. Ga   1,2 MP

Example 2:
1. (∃x) (Fx → Ga)
2. (∃x) Fx
3. Ga   1,2 MP

Example 3:
1. (∃x) (Fx ∨ Gx)
2. (∃x) ~Fx
3. (∃x) Gx   1,2 DS

Example 4:
1. (x) (~Fx & ~Gx)
2. (x) ~(Fx ∨ Gx)   1 DeM

Quantifier Negation (QN)

Rule: One kind of quantifier can be switched to the other kind, while adding/subtracting a "~" on both sides of it. Thus:

(x) ϕ ≡ ~(∃x) ~ϕ
(∃x) ϕ ≡ ~(x) ~ϕ
(∃x) ~ϕ ≡ ~(x) ϕ
(x) ~ϕ ≡ ~(∃x) ϕ

Example:

~(∃x) Fx → ~(∃x) ~Gx
(x) ~Fx
∴ (x) Gx

1. ~(∃x) Fx → ~(∃x) ~Gx   p
2. (x) ~Fx   p
3. (x) ~Fx → ~(∃x) ~Gx   1 QN
4. ~(∃x) ~Gx   2,3 MP
5. (x) Gx   4 QN

Existential Instantiation (EI)

Rule: Remove existential quantifier and substitute for every variable under its scope an unknown symbol:

(∃v) ϕ(v)

ϕ($\underline{u}$)

Using 'unknown' symbols.

Examples:

Example 1:

1. (∃x) (Fx & Gx)
2. F$\underline{a}$ & G$\underline{a}$ 1 EI

Example 2:

1. (∃x) (Fx & Gx)
2. F$\underline{b}$ & G$\underline{b}$ 1 EI

Example 3:

1. (∃x) [Fx & (y) (Fy → Gx)]
2. F$\underline{a}$ & (y) (Fy → G$\underline{a}$) 1 EI

Restriction on EI: $\underline{u}$ cannot appear previously in the proof

Example 4:

1. (∃x) Hx p
2. (∃x) Px p
3. H$\underline{a}$ 1 EI
4. P$\underline{a}$ 2 EI
5. H$\underline{a}$ & P$\underline{a}$ 3,4 conj.

Example 5:

1. (∃x) Hx premise
2. (∃x) Px premise
3. H$\underline{a}$ 1 EI
4. P$\underline{b}$ 2 EI

Note: cannot apply EI to part of a line.

Example 6:

1. (∃x) (Fx & Gx) ∨ (∃y) Ay
2. (F$\underline{a}$ & G$\underline{a}$) ∨ (∃y) Ay 1 EI

Example 7:

1. (x)(∃y) Myx
2. (x) M$\underline{b}$x 1 EI

Existential Generalization (EG)

Rule: Replace one or more occurrences of a constant/unknown with a variable, and add an existential quantifier to the sentence.

ϕ($\underline{u}$)	ϕ(c)
(∃v) ϕ(v)	(∃v) ϕ(v)

Example 1:

1. Fc
2. (∃x) Fx 1 EG

Example 2:

1. $Fa \,\&\, Ga$
2. $(\exists x)\,(Fx \,\&\, Gx)$ 1 EG

Example 3:

1. $Fa \,\&\, Ga$
2. $(\exists x)\,(Fx \,\&\, Ga)$ 1 EG

Universal Instantiation (UI)

Rule: Remove a universal quantifier and replace all variables under its scope with a constant/unknown symbol.

$$\frac{(v)\ \phi(v)}{\phi(\underline{u})} \qquad\qquad \frac{(v)\ \phi(v)}{\phi(c)}$$

Example:

1. $(x)\ Fx$
2. Fa 1 UI

Universal Generalization (UG)

Rule: Replace every occurrence of an unknown symbol with a variable, and add a universal quantifier to the sentence.

$$\frac{\phi(\underline{u})}{(v)\ \phi(v)}$$

Restrictions:

Does not work on constants.

$\underline{u}$ does not occur previously in a line obtained by EI

$\underline{u}$ does not occur in an undischarged assumption

Examples: which of these are correct uses of UG?

Example 1:

1.	$(x)\ (Sx \rightarrow Dx)$	p
2.	$(x)\ Sx$	p
3.	$\sim D\underline{a}$	a
4.	$S\underline{a} \rightarrow D\underline{a}$	1 UI
5.	$S\underline{a}$	2 UI
6.	$D\underline{a}$	4,5 MP
7.	$D\underline{a} \,\&\, \sim D\underline{a}$	3,6 conj.
8.	$D\underline{a}$	3-7 RAA
9.	$(x)\ Dx$	8 UG

Example 2:

1.	$(x)\ (Sx \rightarrow Dx)$	p
2.	$(x)\ Sx$	p
3.	$S\underline{a} \rightarrow D\underline{a}$	1 UI
4.	$S\underline{a}$	2 UI
5.	$D\underline{a}$	3,4 MP
6.	$(x)\ Dx$	5 UG

Example 3:

1. (∃x) Fx	p
2. F<u>b</u>	1 EI
3. (x) Fx	2 UG

Example 4:

1. ~(x) Fx	p
→2. F<u>b</u>	a
3. (x) Fx	2 UG
4. (x) Fx & ~(x) Fx	1,3 conj.
5. ~F<u>b</u>	2-4 RAA
6. (x) ~Fx	5 UG

Miscellaneous Stuff

Remembering the names of the rules.
General predicate-logic proof strategy.
Use EI first (before UI).
Example 1:

(x) (Ax ∨ Bx)
(x) (Bx → Ax)
∴ (x) Ax

1. (x) (Ax ∨ Bx)	p
2. (x) (Bx → Ax)	p
3. A<u>a</u> ∨ B<u>a</u>	1 UI
4. B<u>a</u> → A<u>a</u>	2 UI
5. ~A<u>a</u>	a
6. ~B<u>a</u>	4,5 MT
7. A<u>a</u>	3,6 DS
8. A<u>a</u> & ~A<u>a</u>	7,5 conj
9. A<u>a</u>	5-8 RAA
10. (x) Ax	9 UG

Example 2:

(∃x)Ax → (x)(Bx → Cx)
Am & Bm
∴ Cm

1. (∃x)Ax → (x)(Bx → Cx)	p
2. Am & Bm	p
3. Am	2 simp
4. Bm	2 simp
5. (∃x) Ax	3 EG
6. (x) (Bx → Cx)	1,5 MP
7. Bm → Cm	6 UI
8. Cm	4,7 MP

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Chapter 7: Relations and Identity

To Discuss Today:
Logical properties of relations
Symbolizations involving identity
Logical laws of identity

Properties of relations

Symmetry:
Symmetric: Rxy ⊢ Ryx.
"x is next to y"
Asymmetric: Rxy ⊢ ~Ryx
"x is bigger than y"
Non-symmetric: (neither symmetric nor asymmetric)
"x hits y"
Transitivity:
Transitive: Rxy & Ryz ⊢ Rxz
"x is bigger than y"
Intransitive: Rxy & Ryz ⊢ ~Rxz
"x is the daughter of y"
Non-transitive: (neither transitive nor intransitive)
"x is a friend of y"
Reflexivity:
Reflexive: Rxy ⊢ Rxx
"x lives in the same house as y"
Irreflexive: ~Rxx
"x is older than y"
Non-reflexive: (neither reflexive nor irreflexive)
"x loves y"
Equivalence relations:
Have the properties of 'equivalence':
Reflexive, symmetric, transitive

Identity

General points about identity:
Numerical vs. type identity
Properties of identity:
Equivalence relation
Symbol for identity:
"x = y": x is numerically identical with y
Note: "=" is a 2-place predicate.
Numerical statements:
There is exactly 1 cat:
(∃x)[Cx & (y)(Cy → y=x)]

There are exactly 2 cats:

$(\exists x)(\exists y)\ [(Cx \ \&\ Cy) \ \&\ x \neq y \ \&\ (z)\ (Cz \rightarrow [z=x \vee z=y])]$

There are exactly 3 cats:

$(\exists x)(\exists y)(\exists z)\ [(Cx \ \&\ Cy \ \&\ Cz) \ \&\ (x \neq y \ \&\ y \neq z \ \&\ x \neq z) \ \&\ (w)(Cw \rightarrow (w=x \vee w=y \vee w=z))]$

There is at most 1 cat:

$\sim(\exists x)(\exists y)\ [(Cx \ \&\ Cy) \ \&\ x \neq y]$

Definite descriptions:

The King of France is bald:

$(\exists x)[Kxf \ \&\ (y)(Kyf \rightarrow y=x)] \ \&\ (x)(Kxf \rightarrow Bx)$

$(\exists x)[(Kxf \ \&\ Bx) \ \&\ (y)(Kyf \rightarrow y=x)]$

Aside: Strawson's criticism of Russell's analysis

Logical laws of identity

The Law of Identity (Id):

Intuitive statement: everything is identical to itself.

Rule: Write down "$\alpha=\alpha$" at any stage, where α is any constant or unknown symbol.

Leibniz' Law (LL):

If x=y, then any property of x is a property of y and vice versa.

Rule: From "$\phi(\alpha)$" and "$\alpha=\beta$", deduce "$\phi(\beta)$".

Example:

To prove: $(x)(y)(x=y \rightarrow y=x)$. (Identity is symmetric.)

1. $\underline{a}=\underline{b}$ a. (for CP)
2. $\underline{a}=\underline{a}$ Id
3. $\underline{b}=\underline{a}$ 1,2 LL
4. $\underline{a}=\underline{b} \rightarrow \underline{b}=\underline{a}$ 1-3 CP
5. $(y)\ (\underline{a}=y \rightarrow y=\underline{a})$ 4 UG
6. $(x)(y)\ (x=y \rightarrow y=x)$ 5 UG

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Chapter 9: Naive Set Theory

To Discuss Today:
What are sets
Axioms of naive set theory
Set theoretic terminology
Theorems
What sets aren't.

About Set Theory

A little history
Why it's interesting
Basis of mathematics?
Used in defining:
Numbers
Geometrical objects
Functions
Probabilities
Philosophical objects: properties, propositions
Used in understanding infinity
Fun & famous paradoxes

What is a set?

A set is a collection/group?
Problem: Empty set? Singleton sets?
Sets are 'primitive'?
Problem: How are we supposed to have this concept?
Sets are implicitly defined by the axioms of set theory?
Existence condition
Identity condition

The Axioms of Naive Set Theory

The Naive Comprehension Axiom
$(\exists s)(x)\ (x \in s \leftrightarrow \phi(x))$
Examples:
There is a set of all cats: $(\exists s)(x)\ (x \in s \leftrightarrow Cx)$
There is a set of all fat cats: $(\exists s)(x)\ (x \in s \leftrightarrow (Cx \ \&\ Fx))$
There is a set containing me and the Empire State Building: $(\exists s)(x)\ (x \in s \leftrightarrow (x=m \vee x=e))$
The Axiom of Extensionality
$(s)(r)\ [s=r \leftrightarrow (x)\ (x \in s \leftrightarrow x \in r)]$
Examples:
{2,3}
{3,2}
the set of all prime numbers less than 5
the set of all integers between 1 and 4

Set Theory Terminology

Representing sets:

{a, b, c}

{2, 4, 6, ...}

{x: Fx} or {x| Fx}

The empty set:

{}, ∅

The universal set:

U, V

Singleton set:

A set with exactly one member.

Example: {2}, {Mike}

Union of two sets:

$s \cup r = \{x: x \in s \vee x \in r\}$

Example: {a, b, c} ∪ {c, d} = {a, b, c, d}

Intersection of two sets:

$s \cap r = \{x: x \in s \,\&\, x \in r\}$

Example: {a, b, c} ∩ {c, d} = {c}

Complement of a set:

$s' = \{x: x \notin s\}$

s minus r:

$s - r = \{x: x \in s \,\&\, x \notin r\}$

Example: {a, b, c} - {c, d} = {a, b}

Subset:

$s \subseteq r \leftrightarrow (x)(x \in s \rightarrow x \in r)$

Example:

{a, b} ⊆ {a, b, c}

{a, b} ⊆ {a, b}

{} ⊆ {a, b}

Proper subset:

$s \subset r \leftrightarrow [(x)(x \in s \rightarrow x \in r) \,\&\, s \neq r]$

(Same as subset, except a set is not a proper subset of itself.)

Powerset:

$\wp s = \{x: x \subseteq s\}$

Example: s = {a, b}

℘s = {{a}, {b}, {a,b}, {}}

Union of a set of sets:

$\bigcup s = \{x: (\exists y)\, (y \in s \,\&\, x \in y)\}$

Example:

s = {{a}, {a,b}, {c}}

⋃s = {a, b, c}

Intersection of a set of sets:

$\bigcap s = \{x: (y)\, (y \in s \rightarrow x \in y)\}$

Examples:

s = {{a}, {a,b}, {c}}

⋂s = {}

r = {{b}, {a,b}, {c,b}}
⋂r = {b}

Disjoint sets:

r and s are disjoint when r ∩ s = ∅.

Example: {a, b} and {c} are disjoint.

Open, closed, and half-open intervals:

$(a, b) = \{x: a < x < b\}$

$[a, b] = \{x: a \leq x \leq b\}$

$[a, b) = \{x: a \leq x < b\}$

$(a, b] = \{x: a < x \leq b\}$

Example: [0,1) is the set containing all real numbers from 0 up to 1 (including 0 but not including 1).

Terms vs. formulas:

Terms: s ∪ r, s ∩ r, s', s - r, ℘s, ⋃s, ⋂s

Formulas: s ⊆ r, s ⊂ r

Theorems

Theorem 1:*

Given any open sentence, ϕ (with one free variable), there is exactly one set whose members are all and only the objects satisfying ϕ.

*This is not really true.

Theorem 2:

There is an empty set, i.e., a set with no members.

Theorem 3:*

There is a universal set, i.e., a set of which everything is a member.

Theorem 4:

Every set is a subset of itself: (s) s ⊆ s.

Theorem 5:

For any sets, s and r, s=r iff (s ⊆ r and r⊆ s).

Theorem 6:

Every pair of sets has a unique union, i.e., for all s, r, s ∪ r exists and is unique.

Theorem 7:

Every pair of sets has a unique intersection.

Theorem 8:*

Every set has a unique complement.

Theorem 9:

Every set has a unique powerset.

Theorem 10:

Every object has a singleton set, i.e., for all x, there exists the set {x}.

Theorem 11:

For every x, y, there exists the set {x, y}.

What Sets Are Not

Not aggregates/mereological sums

Not properties

Sets are defined extensionally

Properties are not

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Chapter 10: Applications of Set Theory

To Discuss Today:

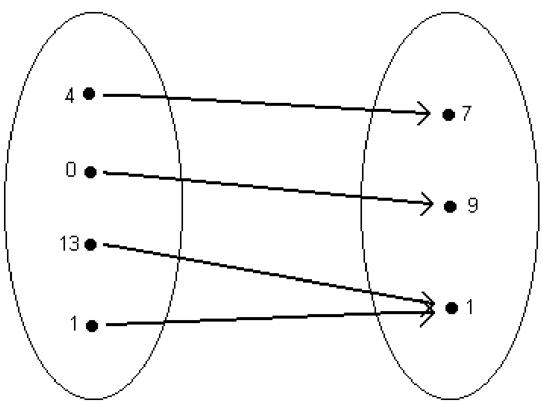
Ordered pairs
Functions
Natural numbers
Infinity

Ordered Pairs

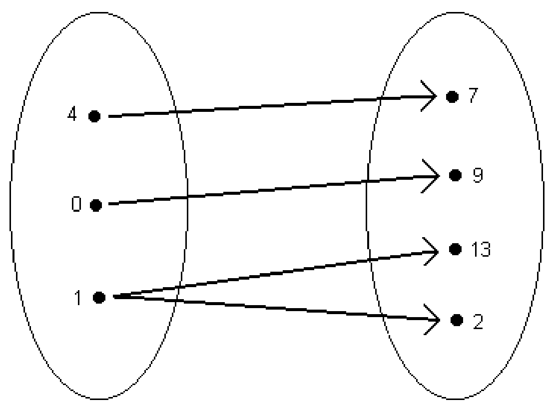
Like sets, but order matters
$\langle a, b \rangle = \{\{a\}, \{a, b\}\}$
$\langle a, b, c \rangle = \langle a, \langle b, c \rangle \rangle$
$\langle x_1, x_2, \ldots x_n \rangle = \langle x_1, \langle x_2, \ldots x_n \rangle \rangle$

Functions

Exactly one output for each input
Example: $y = x^2 + 4$:
y is a function of x
x not a function of y



A function. Notice how each item on the left (in the domain) has one arrow pointing away from it.



Not a function. Notice how one of the items on the left (in the domain) has two arrows pointing away from it.

Terminology:
argument(s)
values
domain
range
"from", "onto", "into"

Functions can also have multiple inputs.

Example:

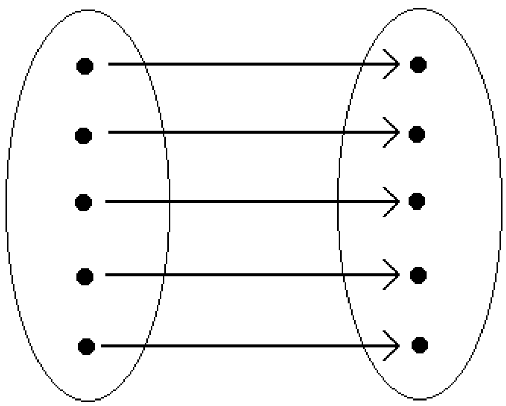
List the functions from {a, b} onto {c, d}
a → c
b → d

a → d
b → c

Not these (not *onto* {c,d}):
a → c
b → c

a → d
b → d

One-one function:
Every input is paired with a unique output, and vice versa



A one-one function. Each item on the left is correlated with one item on the right, and vice versa.

Natural Numbers

Numbers can be 'reduced' to set theory

Russell's approach:

Intuitive idea:

0 = the set of all 0-membered (empty) sets
1 = the set of all single-membered sets
2 = the set of all 2-membered sets
etc.

How to say that without using any number words?

x has the same cardinality as y:

There is a one-one function from x onto y. Also called: x and y are equinumerous

x has a lower cardinality than y:

There is no one-one function from x onto y, but there is a one-one function from x onto a subset of y.

x has a higher cardinality than y:

There is no one-one function from x onto y, but there is a one-one function from a subset of x onto y.

The numbers, again:

0 = the set of all sets that are equinumerous with {}
1 = the set of all sets that are equinumerous with {0}

2 = the set of all sets that are equinumerous with {0,1}
3 = the set of all sets that are equinumerous with {0,1,2}
etc.

Other ways of doing it?
Frege: uses 'concepts' instead of sets
Alternate way:
0 = {}
1 = {{}}
2 = {{{}}}
etc.

Philosophical question: Are these plausible accounts of numbers?

Countable Infinities

ω, $\aleph_0$:
The set of all sets that are equinumerous with {0, 1, 2, 3, ...}.
= The cardinality of the set of natural numbers
This is the first infinite "number".
Also: it is a 'countable infinity'.

More countably infinite sets:
{1, 2, 3, ...}
(consider f(x) = x + 1.)
{0, 2, 4, ...}
(consider f(x) = 2x.)
{... -2, -1, 0, 1, 2, ...}

0	1	-1	2	-2	3	-3	...
↕	↕	↕	↕	↕	↕	↕	
0	1	2	3	4	5	6	...

{x: x is prime}, {1, 2, 3, 5, 7, 11, ...}

1	2	3	5	7	11	13	...
↕	↕	↕	↕	↕	↕	↕	
0	1	2	3	4	5	6	...

Interesting characteristic of infinite sets:
An infinite set can be mapped one-one onto a proper subset of itself.

The Continuum

c:
The cardinality of the set of real numbers

$c > \omega$.
The natural #s can be mapped one-one onto a subset of the real #s. (obvious)
They cannot be mapped one-one onto all of the real #s. Cantor's "Diagonalization Argument":
Assume f is a one-one function from the natural #s onto the real #s between 0 and 1.

x	*f(x)*
0	. <u>5</u> 4 5 0 9 2 ...
1	. 4 <u>3</u> 6 2 1 4 ...
2	. 1 9 <u>7</u> 9 6 7 ...

3	. 8 4 9 4 6 5 ...
4	. 4 6 5 5 9 6 ...
5	. 6 5 4 6 5 0 ...
⋮	⋮

We can construct a real #, R, that is not one of the values of *f*.

x	*f(x)*	Digits of R
0	. 5 4 5 0 9 2 ...	. 6
1	. 4 3 6 2 1 4 ...	. 6 4
2	. 1 9 7 9 6 7 ...	. 6 4 8
3	. 8 4 9 4 6 5 ...	. 6 4 8 5
4	. 4 6 5 5 9 6 ...	. 6 4 8 5 0
5	. 6 5 4 6 5 0 ...	. 6 4 8 5 0 1
⋮	⋮	⋮

Therefore, *f* is not a one-one function from the natural #s onto the real #s (by RAA).
So there is no one-one function from the natural #s onto the real #s (by UG).

Other interesting result: there are many more infinite cardinals.
The 'powerset theorem': the powerset of A always has a higher cardinality than A.
Hence, there is an infinite hierarchy of infinite cardinals.

Philosophical Questions

Aristotle's doctrine
The impossibility of an 'actual' infinity.

Galileo's argument
Which is greater: the number of natural numbers, or the number of perfect squares?
First answer: There are more natural numbers than perfect squares. (Argument: natural numbers include the perfect squares, plus a lot more.)
Second answer: There are just as many perfect squares as natural numbers. (Argument: for every natural # n, there is a square, n^2.)
Conclusion: Infinite sets are neither greater, nor less, nor equal to, other infinite sets.
Further conclusion: Infinity is not a genuine number?

Calculus
Does not vindicate treatment of infinity as a number
Standard approach uses only real #s.
No infinities
No infinitesimals

Cantor's doctrine
Embraces the "one-one function" test
Dismisses Galileo's 'first answer' (the natural numbers include the perfect squares, plus a lot more)
There are infinite numbers, in the same sense that the natural #s are numbers
Is Cantor right?

Cantor’s conception is a *generalization* and *extension* of the intuitive notion of “greater than”.
Plausibility depends on the reduction of numbers to sets.

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Chapter 11: Less Naive Set Theory

To Discuss:
- Russell's paradox
- Responses:
  - The theory of types
  - New Foundations
  - Von Neumann
  - Zermelo-Fraenkel
- The Axiom of Choice

Russell's Paradox

Let r = {x: x ∉ x}

Question: r ∈ r?

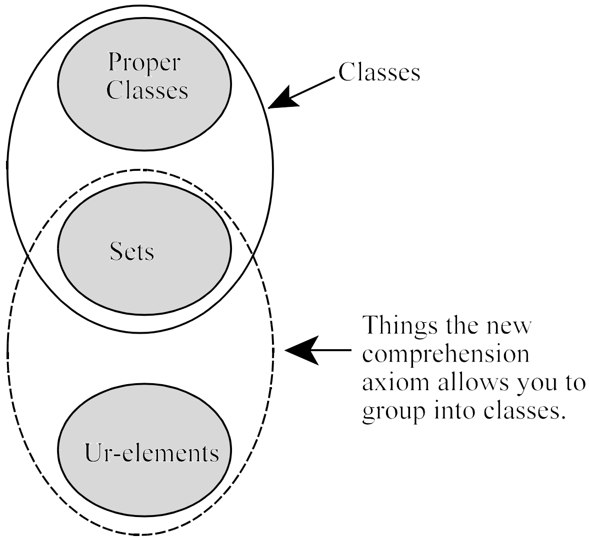
Formally:

1.	(∃s)(x) (x ∈ s ↔ x ∉ x)	Comprehension Axiom
2.	(x) (x ∈ r̲ ↔ x ∉ x)	1 EI
3.	r̲ ∈ r̲ ↔ r̲ ∉ r̲	2 UI

The Theory of Logical Types (Russell)
- Objects organized into a hierarchy
  - Type 0: ur-elements
  - Type 1: sets containing type 0 objects
  - Type 2: sets containing type 1 objects
  - etc.
- Predicates have type restrictions
  - "∈": object on right must have higher type than object on left
- Implications:
  - No Russell set
  - No universal set
  - No absolute complement of a set
- A better variant of type theory: cumulative types

New Foundations (Quine)
- Axiom of Comprehension:
  - (∃s)(x) (x ∈ s ↔ ϕ(x))
- holds when ϕ is a stratified predicate. Examples: which of these are stratified?
  - x ∉ x
  - x ∈ x
  - x = x
  - (y) x ∈ y
- Implications:
  - No Russell set
  - Allows universal set
  - Allows absolute complement

Von Neumann Set Theory



Divides objects into:

Classes: sets, proper classes

Ur-elements

Axiom of comprehension is restricted:

Only ur-elements and sets can be grouped into classes

$(\exists s)(x)\ [x \in s \leftrightarrow (\sim Px \ \&\ \phi(x))]$

Implications:

No Russell set. Why:

1.	$(\exists s)(x)\ (x \in s \leftrightarrow [\sim Px \ \&\ x \notin x])$	Von Neumann Comprehension Axiom
2.	$(x)\ (x \in \underline{r} \leftrightarrow [\sim Px \ \&\ x \notin x])$	1 EI
3.	$\underline{r} \in \underline{r} \leftrightarrow (\sim P\underline{r} \ \&\ \underline{r} \notin \underline{r})$	2 UI

So $\underline{r}$ is not a member of itself and is a proper class.

Zermelo-Fraenkel Set Theory (ZF or ZFC)

Is a 'pure' set theory. You get:

{}

{{}}

{{}, {{}}}

{{}, {{}}, {{}, {{}}}}

etc.

Axioms:

Axiom of Extensionality:

$(x)(y)\ [(z)(z \in x \leftrightarrow z \in y) \rightarrow x=y]$

Axiom of Separation:

$(x)(\exists y)(z)\ [z \in y \leftrightarrow (z \in x \ \&\ \phi(z))]$

Unordered Pair Axiom:

$(x)(y)(\exists z)(w)\ [w \in z \leftrightarrow (w=x \vee w=y)]$

Union Axiom:

$(x)(\exists y)(z)\,[z \in y \leftrightarrow (\exists w)\,(w \in x \,\&\, z \in w)]$

Powerset Axiom:

$(x)(\exists y)(z)\,[z \in y \leftrightarrow (w)\,(w \in z \rightarrow w \in x)]$

Axiom of Infinity:

$(\exists x)\,[(\exists y)\,(y \in x \,\&\, (z)\, z \notin y) \,\&\, (y)\,(y \in x \rightarrow (\exists z)[z \in x \,\&\, y \in z \,\&\, (w)(w \in z \rightarrow w=y)])]$

Axiom of Replacement: For any function, there exists a set containing all its values.

$(x)\,[(y)\,(y \in x \rightarrow (\exists! z)\,\phi(y,z)) \rightarrow (\exists w)(z)\,(z \in w \leftrightarrow (\exists y)\,[y \in x \,\&\, \phi(y,z)])]$

Axiom of Foundation: No set has a nonempty intersection with each of its own elements.

$(x)\,[(\exists y)\, y \in x \rightarrow (\exists y)\,(y \in x \,\&\, \sim(\exists z)\,[z \in y \,\&\, z \in x])]$

Rules out the likes of:

{{{...}}}

A = {B} and B = {A}

The Axiom of Choice

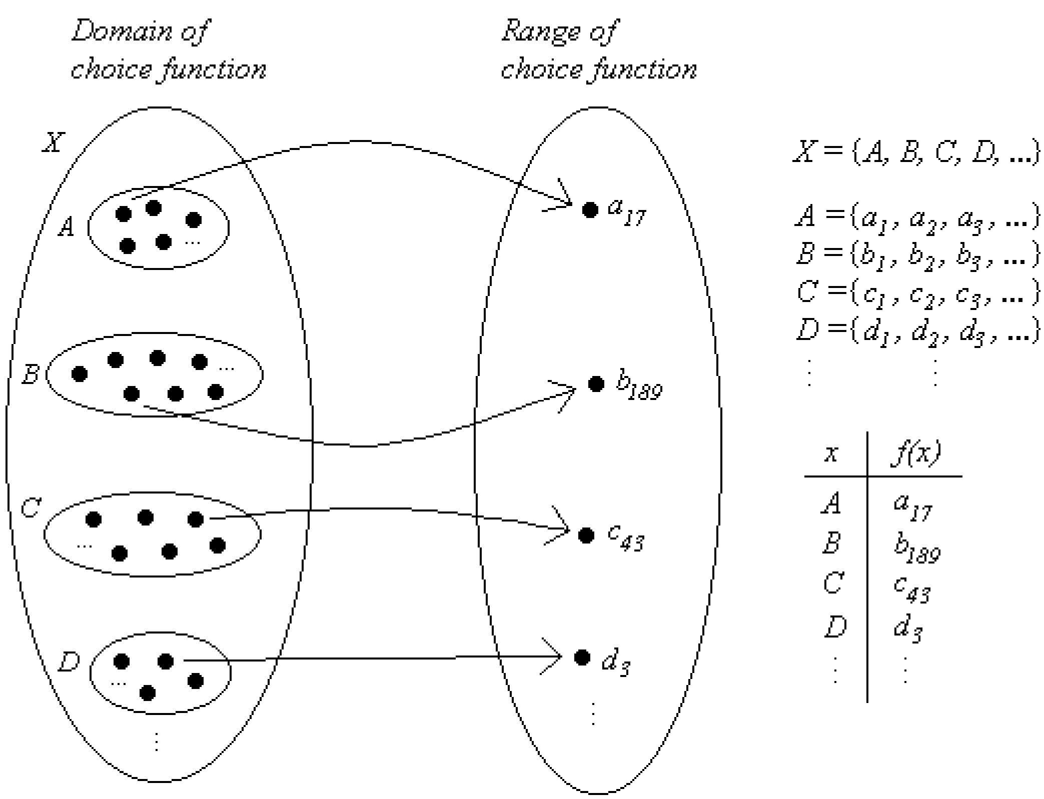
Two formulations:

If *X* is a (non-empty) set of (non-empty) sets, then there exists a function that maps each member of *X* onto a member of itself.

If X is a (non-empty) set of (non-empty, disjoint) sets, there exists a set which contains exactly one member from each member of X.

Intuitive idea: Enables us to ‘choose’ an element from each of the sets in X.

Illustration:



AC is controversial:

Most think it is intuitively obvious.
Some smart people think it is false. (e.g., Borel, Lebesgue, Brouwer)
Philosophical issue: Does a function require a specifiable rule? Does a set require a defining property?

Consequences of AC:
- Well-ordering principle
- Banach-Tarski paradox

The Independence of AC:
- Cannot be proven/disproven in ZF.

The Continuum Hypothesis

The next cardinality above ω is *c*.
This is independent of ZFC.

Philosophical Questions about Sets

Do non-constructible mathematical objects exist?
Do sets exist? Does the empty set exist?
Which version of set theory, if any, is correct?
How to decide whether to accept AC, or the continuum hypothesis?
Is the Frege/Russell reduction of numbers to sets good? Are numbers sets?
What is the best solution to Russell's Paradox?

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Chapter 12: Metalogic

To Discuss:
- Basic concepts of metalogic
- Metalogic for the propositional calculus
- Metalogic for the predicate calculus

Basic concepts of metalogic

Logic vs. meta-logic

Some of the defects of ordinary language:
- Systematic ambiguities.
  - Ex.: “All politicians are not honest.”
- Metaphysics
- Sentences with misleading grammatical structures.
  - Ex.: “The average man has 2.3 children.”

The idea of a ‘logically perfect language’:
- No systematic ambiguities.
- No meaningless sentences
- Grammatical structure reflects logical structure
- Logical properties can be read off the syntactic structure

Formal systems
- Formation rules
- Axioms
- Transformation rules

Other concepts:
- ‘Arguments’
- ‘Proofs’
- ‘Theorems’
- ‘Interpretation’ of a system
- ‘Model’ of a set of sentences

Desirable properties of formal systems

Completeness:
- Every sentence that is true in all intended interpretations is a theorem.

Consistency:
- No sentence of the form p & $\sim p$ is a theorem.

Soundness:
- No false sentences (in the intended interpretation) are theorems.

The consistency of the propositional calculus

Axioms:
- Law of excluded middle. $p \vee \sim p$
- Law of non-contradiction. $\sim(p$ & $\sim p)$

Interpretations in propositional logic:

Assign truth-values to atomic sentences

Consistency proof:

Lemma: In the propositional calculus, every theorem is a tautology, i.e., a proposition that is true in every intended interpretation.

A) All the axioms of the propositional calculus are tautologies.
B) Each of the transformation rules of the propositional calculus preserves tautologousness. That is, if you start from tautologies, they will enable you to derive only other tautologies.
C) Therefore, all the theorems of the propositional calculus are tautologies, since they are derived from the axioms using the transformation rules.

1. In the propositional calculus, every theorem is a tautology.
2. Some propositions are not tautologies.
3. Therefore, some propositions are not theorems of the propositional calculus. (from 1,2)
4. If the propositional calculus is inconsistent, then every proposition is a theorem of it.
5. Therefore, the propositional calculus is consistent. (from 3,4)

Completeness of the propositional calculus

Conjunctive normal form:

Basic idea: One or more conjuncts. Each conjunct is a disjunction of one or more sentences. Each disjunct is an atomic sentence of the negation of an atomic sentence.

More precisely:

a. There are no →'s or ↔'s.
b. All ~'s apply to atomic sentences.
c. All ∨'s apply to atomic sentences or negated atomic sentences.

Examples: which of these are in conjunctive normal form?

(A ∨ ~A)
A & (B ∨ C)
(B ∨ C) & (~C ∨ ~A)
(A ∨ B) & (B ∨ C ∨ A) & (C ∨ A)
A → (B ∨ C)
~(A & ~A)
(A & B) ∨ ~C

How to transform a sentence into conjunctive normal form:

Apply Impl. & Equiv.
Apply DeM
Apply Dist.

Example 1: Transform "A → ~(B ∨ C)" into conjunctive normal form.

1. A → ~(B ∨ C)
2. ~A ∨ ~(B ∨ C) 1 impl
3. ~A ∨ (~B ∨ ~C) 2 DeM

Example 2: Transform "~(A ↔ B) ∨ C" into conjunctive normal form.

1. ~(A ↔ B) ∨ C
2. ~[(A & B) ∨ (~A & ~B)] ∨ C 1 equiv
3. [~(A & B) & ~(~A & ~B)] ∨ C 2 DeM

4. [(~A ∨ ~B) & ~(~A & ~B)] ∨ C 3 DeM
5. [(~A ∨ ~B) & (~~A ∨ ~~B)] ∨ C 4 DeM
6. [(~A ∨ ~B) & (A ∨ B)] ∨ C 5 DN (twice)
7. [(~A ∨ ~B) ∨ C] & [(A ∨ B) ∨ C] 6 Dist
8. (~A ∨ ~B ∨ C) & (A ∨ B ∨ C) rewriting 7

Proving a sentence in conjunctive normal form:

Each conjunct must be tautologous

So each disjunction must be tautologous

So each disjunction must contain an atomic sentence & its negation

Example 3: To prove: A → (B → A)

1. A ∨ ~A axiom
2. ~A ∨ A 1 comm
3. (~A ∨ A) ∨ ~B 2 add
4. ~A ∨ (A ∨ ~B) 3 assoc
5. ~A ∨ (~B ∨ A) 4 comm
6. A → (~B ∨ A) 5 impl
7. A → (B → A) 6 impl

Metalogic for predicate calculus

Interpretations:

Domain of discourse

An object assigned to each constant

A set of objects assigned to each predicate (its extension)

For relational predicates: assign a set of ordered pairs (triples, etc.)

Models:

A model for a set of sentences = an interpretation that makes all the sentences true.

Example:

(∃x)(∃y) Rxy

(∃x)(y) ~Rxy

(x)(∃y) Ryx

A model:

Domain of discourse = all natural numbers. R = the "successor" relation.

Desirable properties of predicate logic:

Consistency

Completeness

Soundness

Phil. 2440
Chapter 13: Gödel's Theorem

To Discuss:
Gödel's Theorem
Gödel's Second Theorem

What Is Gödel's Theorem?
Originally a response to *Principia Mathematica*
Applies to any other formal system of arithmetic
Gödel's Theorem:
Any formal system capable of representing arithmetic on the natural numbers is either inconsistent or incomplete.
What G's Theorem does not say:
Every formal system is inconsistent or incomplete.
Anything about "knowledge".
There are truths of arithmetic that cannot be *proven* in the standard English sense.
There are truths of arithmetic that cannot be proven in any formal system.
Anything about limits to human reason, the human mind, etc.

Outline of the Proof Procedure
Background: The liar paradox
(S) Statement S is false.
A Gödel sentence:
(G) Statement G cannot be proven in *Principia Mathematica*.
More precisely: The Gödel sentence for PM is a sentence of arithmetic that must be true if and only if it is not possible to derive that sentence using the rules of PM.
A little more detail:
Step 1: Number the sentences (and arguments) of PM.
Step 2: Show that the Gödel # of any sentence will have a specific arithmetical property, if and only if the sentence can be proven in PM.
Step 3: Formulate a sentence of PM that says that its own Gödel # does not have that property.

Step 1: Gödel Numbering
Goal of this section: To assign numbers to sentences & arguments in a formal system. I.e., to map sentences/arguments one-one onto a subset of the natural #s.

Numbering the basic symbols:

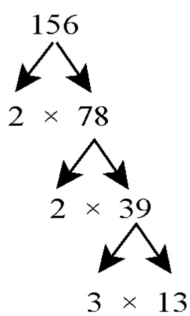
Symbol	Gödel number
(	1
)	2
∃	3
∨	4
~	5

0	6
s	7
=	8
+	9
×	10

Symbol	Gödel number	Symbol	Gödel number
x	11	*A*	12
y	13	*B*	14
z	15	*C*	16
⋮	⋮	⋮	⋮

Prime factorization:

All numbers have a unique prime factorization



Numbering the sentences:

Take the *n*th symbol in the sentence.
Find its Gödel #. Suppose it is *a*.
Take the *n*th prime number raised to the *a* power.

Example 1:

Find the Gödel number for the sentence, "0 = 0"

Answer:

String of symbols:	0	=	0
Gödel numbers for the symbols:	↓ 6	↓ 8	↓ 6
Series of prime numbers:	2	3	5
Gödel # for the string:	2^6	3^8	5^6

Answer: $2^6 \times 3^8 \times 5^6 = 6{,}561{,}000{,}000$

Example 2:

Find the sentence, if any, corresponding to the number 11,049,048,188,640.

Answer:

The prime factorization is $2^5 \times 3^2 \times 5^1 \times 7^8 \times 11^3$.

Prime factorization:	$2^5 \times 3^2 \times 5^1 \times 7^8 \times 11^3$
	↓ ↓ ↓ ↓ ↓
Gödel #s of symbols in the string:	5 2 1 8 3
The string:	~) (= ∃

Numbering Arguments:

Take the *n*th sentence in the argument.
Find its Gödel #. Suppose it is *a*.
Take the *n*th prime number raised to the *a* power.

Example 3:

Find the Gödel # for the argument:

(x) x+0 = x
0+0 = 0
(∃x) x+x = x

Answer:

Sentences in the proof	Gödel numbers	For short
(x) x+0 = x	$2^1 \times 3^{12} \times 5^2 \times 7^{12} \times 11^9 \times 13^6 \times 17^8 \times 19^{12}$	*a*
0+0 = 0	$2^6 \times 3^9 \times 5^6 \times 7^8 \times 11^6$	*b*
(∃x) x+x = x	$2^1 \times 3^3 \times 5^{12} \times 7^2 \times 11^{12} \times 13^9 \times 17^{12} \times 19^8 \times 23^{12}$	*c*

Answer:

$2^a \times 3^b \times 5^c =$

$$2^{(2^1\cdot 3^{12}\cdot 5^2\cdot 7^{12}\cdot 11^9\cdot 13^6\cdot 17^8\cdot 19^{12})}\cdot 3^{(2^6\cdot 3^9\cdot 5^6\cdot 7^8\cdot 11^6)}\cdot 5^{(2^1\cdot 3^3\cdot 5^{12}\cdot 7^2\cdot 11^{12}\cdot 13^9\cdot 17^{12}\cdot 19^8\cdot 23^{12})}$$

Step 2: Correlating syntactic properties of sentences with arithmetical properties of Gödel numbers

Goal of this section: To show that there is an arithmetical property possessed by the Gödel numbers of valid arguments in PM.

Each syntactic property (of a sentence) corresponds to an arithmetical property (of a Gödel #).
Examples:

Syntactic remark about sentence	Arithmetical statement about Gödel #
S begins with "(".	The Gödel number of S is divisible by 2 but not by 4.
S contains "~~" somewhere.	There are consecutive prime numbers *n* and *m*, such that the Gödel number of S is divisible by $(n^5 \times m^5)$ but not by n^6 or m^6.
⋮	⋮

Syntactic properties of *arguments* also correspond to arithmetical properties of Gödel #s.
Examples:

Argument A has 3 steps: The Gödel # of A is divisible by 2, 3, and 5, but not by any prime # greater than 5.

The operation of removing a double negation from the front of a sentence:

1. ~~0 = 0 $\quad 2^5 \times 3^5 \times 5^6 \times 7^8 \times 11^6$

2. 0 = 0 $\quad 2^6 \times 3^8 \times 5^6$

Getting an arithmetical property of the Gödel #s of theorems:

For any syntactic operation, there is a corresponding mathematical (arithmetic) operation.
So there is a mathematical relationship corresponding to each rule of the formal system.
So there is a mathematical relationship corresponding to *following the rules of the system*.
So there is a mathematical property of a sequence that follows the rules of the system.
So there is a mathematical property that the Gödel # of an argument has, if that argument is a proof in the system.
So there is a mathematical property that the Gödel # of a *sentence* has, if *there exists* a proof of that sentence in the system. Suppose this property is represented by ϕ(y).
So the formula

ϕ(y)

is true (in the intended interpretation of the formal system) if and only if y is a theorem of the system.

Step 3: Formulating a Gödel sentence

Goal of this section: To show how a Gödel sentence for a formal system can be constructed, given the result of the previous section.

The direct approach: What about something like

~ϕ(3097540239750934309)

where 3097540239750934309 is the Gödel # of "~ϕ(3097540239750934309)"?

The substitution operation:

Removing all occurrences of a given free variable in a formula, and replacing them with the symbol for a specific number.

Examples:

Formula	Variable letter to be replaced	Number symbol to replace it with	Result
x = ssy	x	s0	s0 = ssy
(∃x) x = ssy	y	sss0	(∃x) x = sssss0
y + sy = ss0	y	s0	s0 + ss0 = ss0
⋮	⋮	⋮	⋮

The Sub function:

The mathematical function that takes the Gödel # of a formula, the Gödel # of a variable letter, and a third number as inputs, and gives as output: the Gödel # of the formula that results from substituting the symbol for the third number for all occurrences of the variable in the formula.

Formula	Variable	Number	Result
x = ssy	x	s0	s0 = ssy
(∃x) x = ssy	y	sss0	(∃x) x = sssss0
y + sy = ss0	y	s0	s0 + ss0 = ss0
⋮	⋮	⋮	⋮

Important:

Sub(65,4,8) is a number.

"**Sub**(65,4,8)" is an expression in the formal system (where "**Sub**" is the formal system's representation of the Sub function).

"**Sub**(65,4,8)" refers to the number, Sub(65,4,8).

So, we have:

Inputs of Sub function			Outputs of Sub function
$2^{11} \cdot 3^8 \cdot 5^7 \cdot 7^7 \cdot 11^{13}$	11	1	$2^7 \cdot 3^6 \cdot 5^8 \cdot 7^7 \cdot 11^7 \cdot 13^{13}$
$2^1 \cdot 3^3 \cdot 5^{11} \cdot 7^2 \cdot 11^{11} \cdot 13^8 \cdot 17^7 \cdot 19^7 \cdot 23^{13}$	13	3	$2^1 \cdot 3^3 \cdot 5^{11} \cdot 7^2 \cdot 11^{11} \cdot 13^8 \cdot 17^7 \cdot 19^7 \cdot 23^7 \cdot 29^7 \cdot 31^7 \cdot 37^6$
$2^{13} \cdot 3^9 \cdot 5^7 \cdot 7^{13} \cdot 11^8 \cdot 13^7 \cdot 17^7 \cdot 19^{13}$	13	1	$2^7 \cdot 3^6 \cdot 5^9 \cdot 7^7 \cdot 11^7 \cdot 13^6 \cdot 17^8 \cdot 19^7 \cdot 23^7 \cdot 29^6$
⋮	⋮	⋮	⋮

How to find the value of Sub(x,y,z):

a) Find the wff with Gödel number x.
b) Find the variable with Gödel number y.
c) Find the symbol that represents the number z in the formal system.
d) In the wff mentioned in (a): take all occurrences of the variable mentioned in (b), and substitute the symbol mentioned in (c).
e) Then find the Gödel # of the resulting sentence.

Some interesting formulas:

~ϕ[**Sub**(y,13,y)] **(1)**

Suppose the Gödel number of formula (1) is *n*. Now consider:

~ϕ[**Sub**(*'n'*,13,*'n'*)] **(3)**

What is the value of **Sub**(*'n'*,13,*'n'*)?

a) Find the wff with Gödel number *n*. That is formula (1).
b) Find the variable with Gödel number 13. That is "*y*".
c) Find the symbol that represents the number *n* in the formal system. That is '*n*'.
d) In formula (1): Take all occurrences of "y", and substitute 'n'. The result is formula (3) itself.
e) So the value of **Sub**(*'n'*,13,*'n'*) is the Gödel # of formula (3).

This is interesting:

Formula (3) then says that *its own* Gödel # does not have property ϕ.

ϕ is the property that the Gödel #'s of all the theorems of the formal system have.

So formula (3) says that formula (3) itself is not a theorem of the system.

Conclusion of the proof

Suppose formula (3) is true:

Then it is true but not a theorem. → The system is incomplete.

Suppose formula (3) is false:

Then it is false and is a theorem. → The system is unsound.

Gödel's Second Theorem

No consistent formal system, capable of representing arithmetic, can be used to prove its own consistency.

What this does not say:

Anything about 'knowledge'

Anything about proof in the standard English sense.

Anything about limitations of the human mind

Anything about the imperfections of mathematics

That a given system's consistency cannot be proven in *any* formal system. (It can be proven in a stronger system.)