Clones of 2-step nilpotent groups

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To the 70th birthday of G. Grätzer and E. T. Schmidt

Abstract. We prove that if G is a 2-step nilpotent group, then an operation $f\colon G^n \to G$ is a local term operation of G if and only if f preserves the subgroups of G^4.

1. Introduction

The **clone** of an algebra A in a language L is the collection $\mathrm{Clo}(A)$ of operations on A that are the interpretations of the L-terms. The members of $\mathrm{Clo}(A)$ are called **term operations** of A. A κ-ary **algebraic relation** on A is a subalgebra of A^κ. It is known that an operation on A belongs to $\mathrm{Clo}(A)$ if and only if it preserves the $|A|^m$-ary algebraic relations of A for all $m < \omega$. The following three properties of an operation $f\colon A^n \to A$ are related by the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ $(\mathrm{iii})_d$:

(i) f preserves the algebraic relations of A (i.e., $f \in \mathrm{Clo}(A)$);
(ii) f preserves the finitary algebraic relations of A;
$(\mathrm{iii})_d$ f preserves the d-ary algebraic relations of A (for some fixed $d < \omega$).

These implications are irreversible in general, but from the third sentence of this paragraph (i) and (ii) are equivalent for every f if A is finite.

If, for a particular algebra A, there is a finite d such that every operation f on A satisfies (i) $\Leftrightarrow$ $(\mathrm{iii})_d$, then we say that $\mathrm{Clo}(A)$ is **determined** by the d-ary algebraic relations of A. When A is finite, this means exactly that there is a $d < \omega$ such that $(\mathrm{iii})_d \Rightarrow$ (ii) holds for every operation f on A. We consider it desirable for $\mathrm{Clo}(A)$ to be determined by the d-ary algebraic relations for some $d < \omega$ when A is finite, since this yields a useful finite description of $\mathrm{Clo}(A)$.

It is proved in [1] that if G is a finite group with abelian Sylow subgroups, then $\mathrm{Clo}(G)$ is determined by the 3-ary algebraic relations of G. It is also proved that if G is a finite nilpotent group, then $\mathrm{Clo}(G)$ is determined by the d-ary algebraic relations of G for $d = |G|^{[G:Z(G)]-1}$. In this paper we refine a special case of the last result to show that if G is a finite 2-step nilpotent group, then $\mathrm{Clo}(G)$ is determined by the 4-ary algebraic relations of G. (It is shown in [1] that if Q is the quaternion group, then $\mathrm{Clo}(Q)$ is *not* determined by the 3-ary algebraic relations of Q.)

In fact, our arguments also apply to some infinite groups. For this, denote the collection of all operations on A that preserve the finitary algebraic relations of A by $\overline{\mathrm{Clo}}(A)$, and call this collection the **local clone** of A. The members of $\overline{\mathrm{Clo}}(A)$ are called **local term operations** of A. We prove that if G is a (possibly infinite) 2-step nilpotent group, then the local clone of G is determined by

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its 4-ary algebraic relations (i.e., $f\colon G^n \to G$ is a local term operation of G if and only if f preserves the 4-ary algebraic relations of G).

2. Statements of Results

If H is a group, then $H^{2\times 2}$ will denote the group of 2×2 matrices of elements of H under pointwise operations. This is a subgroup of the fourth power of H, hence is a 4-ary algebraic relation of H. If A and B are normal subgroups of H, then $\mathsf{M}_H(A,B)$ will denote the subgroup of $H^{2\times 2}$ generated by the set

$$\left\{\begin{bmatrix} a & a \\ b & b \end{bmatrix}, \begin{bmatrix} c & d \\ c & d \end{bmatrix} \in H^{2\times 2} \ : \ a \equiv b \pmod{A} \text{ and } c \equiv d \pmod{B}\right\}. \tag{2.1}$$

For a group G, we will use the phrase **the basic relations of** G to refer to the collection of all binary algebraic relations of G together with all 4-ary algebraic relations of the form $\mathsf{M}_H(A,B)$ where H is a subgroup in G and A and B are normal in H.

Every local term operation of a group G preserves all finitary algebraic relations of G, hence preserves the basic relations of G. The main result of this paper is that if G is 2-step nilpotent, then the converse statement also holds.

Theorem 2.1. *If G is a 2-step nilpotent group, then every operation on G that preserves the basic relations of G is a local term operation of G.*

We will prove Theorem 2.1 in a stronger form which explicitly describes the local term operations of G. To be able to state that version of the result we need some notation. Let $\widehat{\mathbb{Z}}$ denote the profinite completion of the ring $\mathbb{Z}$. To describe this object, let Λ be the directed set whose elements are the positive integers and whose order is defined by $m \preceq n$ if m divides n. Let $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, and for $k \in \mathbb{Z}$ let $\overline{k}$ denote its residue class in $\mathbb{Z}_m$. $\widehat{\mathbb{Z}}$ is the inverse limit of the inverse system $(\mathbb{Z}_m)_{m\in\Lambda}$ where the transition maps $\varphi_{n,m}\colon \mathbb{Z}_n \to \mathbb{Z}_m$ $(m \preceq n)$ are the natural surjective homomorphisms. This inverse limit may be realized concretely as the subring of the product $\prod_{m\in\Lambda} \mathbb{Z}_m$ consisting of all tuples $(\overline{k}_m)_{m\in\Lambda}$ satisfying

$$k_m \equiv k_n \pmod{m} \quad \text{whenever } m \preceq n.$$

The elements of $\widehat{\mathbb{Z}}$ will be called **profinite integers**.

Let G be a group, and let $a \in G$ be an element of finite order $|a|$. For any profinite integer $\kappa = (\bar{k}_m)_{m\in\Lambda} \in \widehat{\mathbb{Z}}$ the sequence $(a^{k_m})_{m\in\Lambda}$ is a tuple that is eventually constant with respect to the direction on Λ. Namely, for any $n \in \Lambda$ that is above $|a|$ in the $\preceq$-order we have $k_{|a|} \equiv k_n \pmod{|a|}$, hence $a^{k_{|a|}} = a^{k_n}$ whenever $|a| \preceq n$. We will denote the eventually constant value $a^{k_{|a|}}$ of the sequence $(a^{k_m})_{m\in\Lambda}$ by a^κ.

Claim 2.2. *If G is a torsion group and κ is a profinite integer, then the unary operation $u(x) = x^\kappa$ is a local term operation of G.*

Proof. Suppose that $R \leq G^n$ is a finitary algebraic relation of G. R itself is a torsion subgroup, since if $\mathbf{a} = (a_1, \dots, a_n) \in R$, then $|\mathbf{a}| = \operatorname{lcm}(|a_1|, \dots, |a_n|)$ $(=: r)$. We have

$$u(\mathbf{a}) = \mathbf{a}^\kappa = (a_1^\kappa, \dots, a_n^\kappa) = (a_1^{k_{|a_1|}}, \dots, a_n^{k_{|a_n|}}) = (a_1^{k_r}, \dots, a_n^{k_r}) = \mathbf{a}^{k_r},$$

and $\mathbf{a}^{k_r} \in R$ since R is a subgroup containing $\mathbf{a}$. Thus, u preserves R. Since R was an arbitrary finitary algebraic relation of G, u is a local term operation. □

Theorem 2.1 follows from the stronger statement below.

Theorem 2.3. *Let G be a 2-step nilpotent group, and let f be an n-ary operation on G. The following conditions on f are equivalent.*

(a) *f preserves the basic relations of G.*
(b) *f is a local term operation of G.*
(c)
$$f(x_1,\ldots,x_n) = x_1^{\kappa_1}\cdots x_n^{\kappa_n}\prod_{1\le i<j\le n}[x_i,x_j]^{\lambda_{ij}}$$

where

- *κ_i $(1\le i\le n)$ are profinite integers if G is a torsion group and ordinary integers otherwise; and*
- *λ_{ij} $(1\le i<j\le n)$ are profinite integers if G' is a torsion group and ordinary integers otherwise.*

It follows from Claim 2.2 that the operations in (c) are local term operations of G. Therefore (c) $\Rightarrow$ (b) in Theorem 2.3. The implication (b) $\Rightarrow$ (a) holds because local term operations preserve all finitary algebraic relations. The main assertion of Theorem 2.3 is therefore the implication (a) $\Rightarrow$ (c) whose proof will occupy the remaining sections of this paper.

3. Some reductions

In this section we will reduce the proof of implication (a) $\Rightarrow$ (c) in Theorem 2.3 to a study of some special operations f that we will call 'absorptive', and will establish some basic properties of these operations. We will also discuss statements that will allow us to carry over results proved for operations on certain subgroups or quotients of G to operations on G.

An operation $f(x_1,\ldots,x_n)$ on G is **absorptive** in variable i if

$$f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n)=1 \quad \text{for all } x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n\in G. \tag{3.1}$$

An operation is **absorptive** if it is absorptive in variable i for all i. According to this definition a nullary operation c is absorptive if $c=1$, and a unary operation $f(x)$ is absorptive if $f(1)=1$.

Lemma 3.1. *Let G be a group, and let $\mathcal{C}$ be a clone on the set G. If $\mathrm{Clo}(G)\subseteq\mathcal{C}$, then every operation $f\in\mathcal{C}$ such that $f(1,\ldots,1)=1$ can be written as a product of absorptive operations from $\mathcal{C}$.*

Proof. Suppose that the lemma is false, and that n is the least arity of an operation $f\in\mathcal{C}$ satisfying $f(1,\ldots,1)=1$ that cannot be written as a product of absorptive operations of $\mathcal{C}$. Necessarily $n>1$. Among such operations of arity n in $\mathcal{C}$, choose a particular operation f that is absorptive in a maximum number of variables. Rearranging variables if necessary, it may be assumed that (3.1) holds for $1\le i<k$, but that any operation in $\mathcal{C}$ of arity n that satisfies (3.1) for k or more variables is a product of absorptive operations of $\mathcal{C}$.

Let $g(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)=f(x_1,\ldots,x_{k-1},1,x_{k+1},\ldots,x_n)$. Since $g\in\mathcal{C}$ has arity less than n, our assumptions guarantee that g is a product of absorptive operations from $\mathcal{C}$. The operation

$$h(x_1,\ldots,x_n) := g(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)^{-1}\cdot f(x_1,\ldots,x_n)\in\mathcal{C}$$

is n-ary and satisfies (3.1) in its first k variables, hence is also a product of absorptive operations from $\mathcal{C}$. Thus $f=gh$ is a product of absorptive operations from $\mathcal{C}$, contrary to assumption. $\square$

Lemma 3.1 allows us to restrict our attention to absorptive operations. The next lemma shows that implication (a) $\Rightarrow$ (c) in Theorem 2.3 holds for unary operations.

Lemma 3.2. *Let G be a group, and let f be a unary operation on G such that f preserves the subgroups of G^2.*

(1) *If G is a torsion group, then there exists a profinite integer κ such that $f(x) = x^\kappa$ for all $x \in G$.*
(2) *If G has an element of infinite order, then there exists an integer k such that $f(x) = x^k$ for all $x \in G$.*

Proof. For this proof let $|a|$ denote the order of an element $a \in G$ of finite order, and let $|a| = 0$ if $a \in G$ is of infinite order. Furthermore, we extend the notation $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ and $\equiv \pmod{m}$ to the case $m = 0$ in the natural way: $\mathbb{Z}_0$ is the ring of integers, and $\equiv \pmod{0}$ is the equality relation on $\mathbb{Z}$.

To prove the lemma let f be a unary operation on G that preserves the subgroups of G^2. For every cyclic subgroup $C = \langle a \rangle$ of G, f preserves C^2 and therefore also C, so $f(C) \subseteq C$. Hence there exists an integer $r(a)$ such that $f(a) = a^{r(a)}$. Similarly, for arbitrary cyclic subgroup $H = \langle (a, b) \rangle$ of G^2 we have $f(H) \subseteq H$, therefore

$$f\big((a, b)\big) = (a, b)^k = (a^k, b^k) \qquad \text{for some integer } k.$$

However,

$$f\big((a, b)\big) = \big(f(a), f(b)\big) = (a^{r(a)}, b^{r(b)}),$$

so

$$r(a) \equiv k \pmod{|a|} \quad \text{and} \quad r(b) \equiv k \pmod{|b|} \qquad \text{for some integer } k. \tag{3.2}$$

In particular, it follows that

$$r(a) \equiv r(b) \pmod{|a|} \qquad \text{whenever } |a| \text{ divides } |b|. \tag{3.3}$$

If G has an element b of infinite order, then $|b| = 0$ and $|a|$ divides $|b|$ for all elements $a \in G$. Therefore we get from (3.3) that $f(a) = a^{r(a)} = a^{r(b)}$ for all $a \in G$. This proves that statement (2) holds for $k = r(b)$.

From now on we will assume that G is a torsion group, that is, $|a|$ is a positive integer for all $a \in G$. Let I denote the subset of Λ that consists of all positive integers $|a|$ $(a \in G)$. It follows from (3.3) that $r(a) \equiv r(b) \pmod{m}$ whenever $m = |a| = |b|$. Hence there exists a sequence $\delta = (\bar{d}_m)_{m \in I} \in \prod_{m \in I} \mathbb{Z}_m$ such that

$$r(a) \equiv d_{|a|} \pmod{|a|} \qquad \text{for all } a \in G. \tag{3.4}$$

Moreover, (3.2) implies that δ has the following property:

($*$) the system

$$z \equiv d_m \pmod{m}$$
$$z \equiv d_n \pmod{n}$$

of simultaneous congruences has a solution for all $m, n \in I$.

(3.4) implies that $f(a) = a^{r(a)} = a^{d_{|a|}}$ for all $a \in G$. Using property ($*$) of δ we will show in Claim 3.3 below that there exists a profinite integer $\kappa = (\bar{k}_m)_{m \in \Lambda} \in \widehat{\mathbb{Z}}$ such that $\bar{d}_m = \bar{k}_m$ for all $m \in I$. Thus

$$f(a) = a^{d_{|a|}} = a^{k_{|a|}} = a^\kappa \qquad \text{for all } a \in G,$$

proving statement (1).

So, it remains to verify the following claim.

Claim 3.3. *If I is a subset of Λ and $\delta = (\bar{d}_m)_{m \in I} \in \prod_{m \in I} \mathbb{Z}_m$ satisfies condition ($*$), then there exists a profinite integer $\kappa = (\bar{k}_m)_{m \in \Lambda} \in \widehat{\mathbb{Z}}$ such that $\bar{d}_m = \bar{k}_m$ for all $m \in I$.*

For each $i \in I$ let $N^{(i)}$ denote the set of all profinite integers $\kappa = (\bar{k}_m)_{m\in\Lambda} \in \widehat{\mathbb{Z}}$ such that $\bar{d}_i = \bar{k}_i$. Since δ satisfies condition $(*)$, the Chinese Remainder Theorem implies that for every finite subset F of I the system

$$z \equiv d_m \pmod{m}, \qquad m \in F,$$

of $|F|$ simultaneous congruences has a solution k^F. Hence $\kappa^F = \left(\overline{k^F}\right)_{m\in\Lambda} \in \widehat{\mathbb{Z}}$ is a profinite integer such that $\bar{d}_m = \overline{k^F}$ for all $m \in F$. Thus $\kappa^F \in \bigcap_{m\in F} N^{(m)}$. This shows that $\bigcap_{m\in F} N^{(m)} \neq \emptyset$ for all finite subsets F of I. We will use a compactness argument to conclude from this that $\bigcap_{m\in I} N^{(m)} \neq \emptyset$. This will finish the proof, because every profinite integer $\kappa = (\bar{k}_m)_{m\in\Lambda}$ in $\bigcap_{m\in I} N^{(m)}$ satisfies $\bar{d}_m = \bar{k}_m$ for all $m \in I$.

Each $\mathbb{Z}_m$ ($m \in \Lambda$) with the discrete topology is a compact Hausdorff space, therefore their product $\prod_{m\in\Lambda} \mathbb{Z}_m$ is a compact Hausdorff space. $\widehat{\mathbb{Z}}$ is a closed subset of $\prod_{m\in\Lambda} \mathbb{Z}_m$, so $\widehat{\mathbb{Z}}$ with the subspace topology is also a compact Hausdorff space. The sets $N^{(i)}$ are clopen in $\widehat{\mathbb{Z}}$. Therefore $\bigcap_{m\in I} N^{(m)}$ is an intersection of a family of closed sets in $\widehat{\mathbb{Z}}$ such that the intersection of any finite subfamily is nonempty. Since $\widehat{\mathbb{Z}}$ is compact, it follows that the intersection $\bigcap_{m\in I} N^{(m)}$ is not empty. This completes the proof of Claim 3.3 and also of Lemma 3.2. □

The next lemma will allow us to carry over results proved for operations on certain subgroups or quotients of G to operations on G.

Lemma 3.4. *Let G be a group, H a subgroup of G, and H/N a quotient of H. If f is an operation on G such that f preserves the basic relations of G, then*

(1) *restriction to H yields a natural action f^H of f on H, and f^H preserves the basic relations of H;*
(2) *f^H has a natural action on the set H/N, and the resulting operation $f^{H/N}$ preserves the basic relations of H/N.*

Moreover, if f is absorptive then so are f^H and $f^{H/N}$.

Proof. If f preserves all subgroups of G^2, then it preserves the subgroup H^2 as well, hence f preserves H. Thus f can be restricted to H to yield an operation f^H on H. Each basic relation of H is a basic relation of G, therefore f^H preserves the basic relations of H. This proves assertion (1).

To prove assertion (2) let $\nu\colon H \to H/N$ be the natural homomorphism. Its kernel $\ker(\nu)$ is the congruence of H that corresponds to the normal subgroup N, so $\ker(\nu)$ is a basic relation of H. Hence f^H preserves $\ker(\nu)$ by part (1) of this lemma. Thus $\ker(\nu)$ is a congruence of the algebra $(H; f^H)$. Hence there is a unique way to define an operation $f^{H/N}$ on the set H/N so that ν is a homomorphism from $(H; f^H)$ to $(H/N; f^{H/N})$; namely:

$$f^{H/N}(x_1N, \dots, x_nN) := f^H(x_1, \dots, x_n)N \quad \text{for all } x_1, \dots, x_n \in H.$$

To establish that $f^{H/N}$ preserves the basic relations of H/N we will use the following claim.

Claim 3.5. *For arbitrary subset S of $(H/N)^m$, $f^{H/N}$ preserves S if and only f^H preserves the preimage R of S under the coordinatewise action of ν.*

This claim is an easy consequence of the correspondence theorem for general algebras. To see this recall that $f^{H/N}$ preserves S means that S is a subalgebra of $(H/N; f^{H/N})^m$, while f^H preserves R means that R is a subalgebra of $(H; f^H)^m$. Since $f^{H/N}$ was defined so that $\nu\colon (H; f^H) \to (H/N; f^{H/N})$ is a homomorphism, it follows from the correspondence theorem that S is a subalgebra of $(H/N; f^{H/N})^m$ if and only if R is a subalgebra of $(H; f^H)^m$. Thus the claim follows.

Now we will show that $f^{H/N}$ preserves the basic relations of H/N. Let S be a basic relation of H/N, and let R denote the preimage of S under the coordinatewise action of ν. By Claim 3.5 it suffices to verify that f^H preserves R. If S is a subgroup of $(H/N)^2$, then R is a subgroup of H^2, so f^H preserves R by part (1) of this lemma.

Now let S be a quaternary basic relation of H/N of the form $S = \mathsf{M}_{H/N}(C, D)$ where C, D are normal subgroups of H/N. Hence $C = A/N$ and $D = B/N$ for some normal subgroups A, B of H that contain N. Therefore $\mathsf{M}_H(A, B)$ is a basic relation of H. For later reference we describe the relationship between $\mathsf{M}_H(A, B)$ and $\mathsf{M}_{H/N}(C, D) = \mathsf{M}_{H/N}(A/N, B/N)$ in a separate claim.

Claim 3.6. *If H is a subgroup of G and A, B, N are normal subgroups of H such that $N \leq A \cap B$, then*

1. *the natural homomorphism $\nu \colon H \to H/N$ maps $\mathsf{M}_H(A, B)$ onto $\mathsf{M}_{H/N}(A/N, B/N)$, and*
2. *the preimage of $\mathsf{M}_{H/N}(A/N, B/N)$ under ν consists of all 2×2 matrices that are related by $\ker(\nu)$ (entry-by-entry) to some matrix in $\mathsf{M}_H(A, B)$.*

Part (1) of the claim follows from the fact that ν maps the generating set (2.1) of $\mathsf{M}_H(A, B)$ onto the generating set

$$\left\{ \begin{bmatrix} aN & aN \\ bN & bN \end{bmatrix}, \begin{bmatrix} cN & dN \\ cN & dN \end{bmatrix} \in (H/N)^{2\times 2} \ : \ aN \equiv bN \pmod{A/N} \text{ and } cN \equiv dN \pmod{B/N} \right\}$$

of $\mathsf{M}_{H/N}(A/N, B/N)$. Part (2) is an easy consequence of (1).

By Claim 3.6 the preimage R of $S = \mathsf{M}_{H/N}(C, D) = \mathsf{M}_{H/N}(A/N, B/N)$ under ν is

$$R = \left\{ \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \in H^{2\times 2} \ : \ \text{there exists } \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \in \mathsf{M}_H(A, B) \text{ such that } (u_i, v_i) \in \ker(\nu) \text{ for } i = 1, 2, 3, 4 \right\}.$$

Here $\mathsf{M}_H(A, B)$ and $\ker(\nu)$ are basic relations of H, therefore f^H preserves these relations by part (1) of Lemma 3.4. Using this fact and the description of R above one can easily check that f^H also preserves R. This proves that $f^{H/N}$ preserves $S = \mathsf{M}_{H/N}(C, D)$.

Finally, let S be an arbitrary quaternary basic relation of H/N, that is, $S = \mathsf{M}_J(C, D)$ where J is a subgroup of H/N and C, D are normal subgroups of J. Hence $J = K/N$ for some subgroup K of H that contains N. Now we can apply the fact proved in the preceding paragraph to K in place of H to conclude that $f^{K/N}$ preserves $S = \mathsf{M}_{K/N}(C, D)$. Since $f^{K/N}$ is the restriction of $f^{H/N}$ to K/N we get that $f^{H/N}$ also preserves $S = \mathsf{M}_{K/N}(C, D)$. This completes the proof of part (2) of Lemma 3.4.

The last statement of Lemma 3.4 follows easily from the definitions of f^H and $f^{H/N}$. □

Later on in this section it will be useful to have an explicit description for the basic relations $\mathsf{M}_H(A, B)$ of G.

Lemma 3.7. *If H is a subgroup of a group G, and A, B are arbitrary normal subgroups of H, then*

$$\mathsf{M}_H(A, B) = \left\{ \begin{bmatrix} t & u \\ v & w \end{bmatrix} \in H^{2\times 2} \ : \ t \equiv v \pmod{A},\ t \equiv u \pmod{B}, \text{ and } tu^{-1} \equiv vw^{-1} \pmod{[A, B]} \right\}. \tag{3.5}$$

In particular, if $[A,B]=\{1\}$*, then for every matrix* $\begin{bmatrix} t & u \\ v & w \end{bmatrix} \in \mathsf{M}_H(A,B)$ *we have*

$$t = u \iff v = w.$$

Proof. Let R denote the subset of $H^{2\times 2}$ on the right hand side of (3.5). First we will verify the equality $\mathsf{M}_H(A,B) = R$ under the additional assumption that $[A,B]=\{1\}$. In this case

$$R = \left\{ \begin{bmatrix} t & u \\ v & w \end{bmatrix} \in H^{2\times 2} \ : \ t \equiv v \pmod{A},\ t \equiv u \pmod{B}, \ \text{ and } \ tu^{-1} = vw^{-1} \right\}.$$

Since R contains the generating set (2.1) of $\mathsf{M}_H(A,B)$, the inclusion $\mathsf{M}_H(A,B) \subseteq R$ will follow if we establish that R is a subgroup of $H^{2\times 2}$. It is easy to see that

$$\check{R} = \left\{ \begin{bmatrix} t & u \\ v & w \end{bmatrix} \in H^{2\times 2} \ : \ t \equiv v \pmod{A},\ t \equiv u \pmod{B} \right\}$$

is a subgroup of $H^{2\times 2}$ and $R \subseteq \check{R}$. Therefore if $\begin{bmatrix} t & u \\ v & w \end{bmatrix}, \begin{bmatrix} \tilde{t} & \tilde{u} \\ \tilde{v} & \tilde{w} \end{bmatrix} \in R$, then the matrices

$$\begin{bmatrix} t & u \\ v & w \end{bmatrix}^{-1} = \begin{bmatrix} t^{-1} & u^{-1} \\ v^{-1} & w^{-1} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} t & u \\ v & w \end{bmatrix} \cdot \begin{bmatrix} \tilde{t} & \tilde{u} \\ \tilde{v} & \tilde{w} \end{bmatrix} = \begin{bmatrix} t\tilde{t} & u\tilde{u} \\ v\tilde{v} & w\tilde{w} \end{bmatrix}$$

are in $\check{R}$. To see that they are in R as well, we need to show that $t^{-1}u = v^{-1}w$ and $t\tilde{t}(u\tilde{u})^{-1} = v\tilde{v}(w\tilde{w})^{-1}$. The assumptions that $\begin{bmatrix} t & u \\ v & w \end{bmatrix}, \begin{bmatrix} \tilde{t} & \tilde{u} \\ \tilde{v} & \tilde{w} \end{bmatrix} \in R$ and $[A,B]=\{1\}$ imply the following:

(1) $tu^{-1} = vw^{-1}$ and hence also $ut^{-1} = wv^{-1}$;
(2) $\tilde{t}\tilde{u}^{-1} = \tilde{v}\tilde{w}^{-1}$;

moreover, since $t^{-1}v \in A$ and $t^{-1}u, \tilde{t}\tilde{u}^{-1} \in B$, therefore

(3) $t^{-1}v$ commutes with $t^{-1}u$, and
(4) $t^{-1}v$ commutes with $\tilde{t}\tilde{u}^{-1}$.

Thus

$$v^{-1}w = v^{-1}wv^{-1}v \overset{(1)}{=} v^{-1}ut^{-1}v = v^{-1}t(t^{-1}u)(t^{-1}v) \overset{(3)}{=} v^{-1}t(t^{-1}v)(t^{-1}u) = t^{-1}u,$$

and

$$\begin{aligned} v\tilde{v}(w\tilde{w})^{-1} &= v\tilde{v}\tilde{w}^{-1}w^{-1} \overset{(2)}{=} v\tilde{t}\tilde{u}^{-1}w^{-1} = t(t^{-1}v)(\tilde{t}\tilde{u}^{-1})w^{-1} \\ &\overset{(4)}{=} t\tilde{t}\tilde{u}^{-1}t^{-1}vw^{-1} \overset{(1)}{=} t\tilde{t}\tilde{u}^{-1}t^{-1}tu^{-1} = t\tilde{t}\tilde{u}^{-1}u^{-1} = t\tilde{t}(u\tilde{u})^{-1}. \end{aligned}$$

This proves that R is a subgroup of $H^{2\times 2}$, and hence $\mathsf{M}_H(A,B) \subseteq R$. To prove the reverse inclusion we will argue that for every matrix $\begin{bmatrix} t & u \\ v & w \end{bmatrix} \in R$ the decomposition

$$\begin{bmatrix} t & u \\ v & w \end{bmatrix} = \begin{bmatrix} 1 & ut^{-1} \\ 1 & wv^{-1} \end{bmatrix} \cdot \begin{bmatrix} t & t \\ v & v \end{bmatrix}$$

on the right hand side has both factors in the generating set (2.1) of $\mathsf{M}_H(A,B)$. Since $\begin{bmatrix} t & u \\ v & w \end{bmatrix} \in R$, the entries satisfy $t \equiv v \pmod{A}$, $t \equiv u \pmod{B}$, and $tu^{-1} = vw^{-1}$. The first condition here shows that $\begin{bmatrix} t & t \\ v & v \end{bmatrix}$ belongs to the generating set (2.1). The second and third conditions imply that $ut^{-1} \equiv 1 \pmod{B}$ and $wv^{-1} = ut^{-1}$. Hence $\begin{bmatrix} 1 & ut^{-1} \\ 1 & wv^{-1} \end{bmatrix}$ also belongs to the generating set (2.1).

This completes the proof of equality (3.5) for the case when $[A,B]=\{1\}$. The last statement of Lemma 3.7 is an immediate consequence of this equality.

Now we will prove equality (3.5) for arbitrary normal subgroups A,B of H. As before, R will denote the right hand side of (3.5). Furthermore, let $N=[A,B]$ and let $\nu\colon H\to H/N$ denote the natural homomorphism. Since $[A/N,B/N]=\{1\}$ in H/N, the special case of Lemma 3.7 just established implies that $\mathsf{M}_{H/N}(A/N,B/N)=\nu(R)$. Clearly, R is the preimage of the subgroup $\mathsf{M}_{H/N}(A/N,B/N)=\nu(R)$ of $(H/N)^{2\times 2}$ under ν. By Claim 3.6 ν maps $\mathsf{M}_H(A,B)$ onto $\mathsf{M}_{H/N}(A/N,B/N)$. Thus it follows that $\mathsf{M}_H(A,B)$ is a subgroup of R. To prove the equality $\mathsf{M}_H(A,B)=R$ it remains to show that $\mathsf{M}_H(A,B)$ contains all matrices whose entries are in $N=[A,B]$.

For arbitrary elements $a\in A$ and $b\in B$

$$\begin{bmatrix}[a,b] & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}a^{-1} & a^{-1}\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}b^{-1} & 1\\ b^{-1} & 1\end{bmatrix}\cdot\begin{bmatrix}a & a\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}b & 1\\ b & 1\end{bmatrix},$$

and all four matrices on the right hand side belong to the generating set (2.1) of $\mathsf{M}_H(A,B)$. Hence $\begin{bmatrix}[a,b] & 1\\ 1 & 1\end{bmatrix}\in\mathsf{M}_H(A,B)$. Similarly, $\begin{bmatrix}1 & [a,b]\\ 1 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ [a,b] & 1\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 1 & [a,b]\end{bmatrix}\in\mathsf{M}_H(A,B)$. Since $\mathsf{M}_H(A,B)$ is closed under multiplication, it follows that $\mathsf{M}_H(A,B)$ contains all matrices whose entries are in $[A,B]$. This completes the proof of (3.5). $\square$

In the last two lemmas of this section (Lemmas 3.10 and 3.11) we will use Lemma 3.7 to establish some properties of absorptive operations that preserve the basic relations of a 2-step nilpotent group G. The following terminology will be useful.

Let G be a group and K be an abelian group. A function $h\colon G\times G\to K$ is **linear in its first variable** if $h(-,a)\colon G\to K,\ x\mapsto h(x,a)$ is a homomorphism for all $a\in G$, **linear in its second variable** if $h(a,-)\colon G\to K,\ x\mapsto h(a,x)$ is a homomorphism for all $a\in G$, and **bilinear** if it is linear in both variables. We call h **alternating** if $h(a,a)=1$ for all $a\in G$.

Lemma 3.8. *Let G be a group, K an abelian group, and let $h\colon G\times G\to K$ be a function.*

(1) *If h is alternating and bilinear, then for all $a,b,x\in G$*
  (i) $h(a,1)=1=h(1,a)$,
  (ii) $h(a,x^{-1})=h(a,x)^{-1}=h(a^{-1},x)$,
  (iii) $h(b,a)=h(a,b)^{-1}$, *and*
  (iv) *the homomorphisms $h(-,a),h(a,-)\colon G\to K$ have the same image.*

(2) *If h is bilinear, then $G\to K^G,\ a\mapsto h(a,-)$ is a group homomorphism.*

(3) *If h is bilinear, then for every subgroup $H\le K$,*

$$\{a\in G:\text{ the image of } h(a,-) \text{ is contained in } H\}$$

is a subgroup of G.

Proof. Items (i) and (ii) of (1) are parts of the definition of 'bilinear'. For item (iii), the bilinearity of h implies that $h(ab,ab)=h(a,a)h(a,b)h(b,a)h(b,b)$. This reduces to $1=h(a,b)h(b,a)$ if h is alternating, from which item (iii) follows. Item (iv) is a direct consequence of (iii).

For (2) we must show that $h(a,x)h(b,x)=h(ab,x)$ for all $a,b,x\in G$. This follows from the linearity of h in its first variable.

The set in (3) is the preimage of the subgroup $H^G\le K^G$ under the homomorphism from part (2), so it is a subgroup of G. $\square$

Of course, the claims in Lemma 3.8 (2) and (3) hold in the second variable of h as well as in the first.

Lemma 3.9. *If G is a 2-step nilpotent group, then*

(1) $G' \subseteq Z(G)$;
(2) *the commutator operation $[-,-]$ on G is alternating and bilinear;*
(3) *for an arbitrary element $a \in G$ the normal subgroup N_a generated by a is contained in $\langle a\rangle Z(G)$, and is therefore abelian;*
(4) *for an arbitrary element $a \in G$, $C_G(a) = C_G(N_a)$ is a normal subgroup of G.*

Proof. Items (1) and (2) are well known, so we prove (3) and (4) only. For (3), $N_a = \langle a^b \mid b \in G\rangle$ is the group generated by the conjugates $a^b := b^{-1}ab$ of a, and $a^b = b^{-1}ab = a(a^{-1}b^{-1}ab) = a[a,b] \in \langle a\rangle Z(G)$, so $N_a \subseteq \langle a\rangle Z(G)$.

For (4) we have $a \in N_a \subseteq \langle a\rangle Z(G)$, so

$$C_G(a) \supseteq C_G(N_a) \supseteq C_G(\langle a\rangle Z(G)) = C_G(a),$$

forcing $C_G(a) = C_G(N_a)$. This group is normal in G since the centralizer of any normal subgroup is normal. (The normality of $C_G(a)$ also follows from the fact that it is the kernel of the homomorphism $[a,-]$.) □

Lemma 3.10. *Let G be a 2-step nilpotent group, and let f be an absorptive operation of arity $n > 1$ that preserves the basic relations of G.*

(1) *The image of f is contained in G'.*
(2) *If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are n-tuples in G^n such that $a_1 = b_1$ and $a_i \equiv b_i \pmod{C_G(a_1)}$ for all i $(2 \le i \le n)$, then $f(a_1, a_2, \ldots, a_n) = f(b_1, b_2, \ldots, b_n)$.*

Proof. To prove part (1) let $a_1, a_2, \ldots, a_n$ be arbitrary elements of G. Since f preserves the basic relation $\mathsf{M}_G(G,G)$ and

$$\begin{bmatrix} a_1 & a_1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} a_i & 1 \\ a_i & 1 \end{bmatrix} \in \mathsf{M}_G(G,G) \qquad (i = 2, \ldots, n),$$

therefore the matrix obtained from these matrices by applying f also belongs to $\mathsf{M}_G(G,G)$. Using the assumptions that $n \ge 2$ and f is absorptive, we get that

$$\begin{bmatrix} f(a_1, a_2, \ldots, a_n) & 1 \\ 1 & 1 \end{bmatrix} = f\left(\begin{bmatrix} a_1 & a_1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} a_2 & 1 \\ a_2 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} a_n & 1 \\ a_n & 1 \end{bmatrix}\right) \in \mathsf{M}_G(G,G).$$

Thus the description of $\mathsf{M}_G(G,G)$ in Lemma 3.7 yields that $f(a_1, a_2, \ldots, a_n) \in [G,G]$.

For part (2) we will use the basic relation $\mathsf{M}_G\big(N_{a_1}, C_G(a_1)\big)$ where N_{a_1} is the normal subgroup of G generated by a_1. Recall from Lemma 3.9 (4) that $C_G(a_1)$ is a normal subgroup of G and $C_G(a_1) = C_G(N_{a_1})$. The latter implies that $[N_{a_1}, C_G(a_1)] = \{1\}$.

If $a_1 = b_1$ and $a_i \equiv b_i \pmod{C_G(a_1)}$ for all i $(2 \le i \le n)$, then

$$\begin{bmatrix} a_1 & b_1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} a_i & b_i \\ a_i & b_i \end{bmatrix} \in \mathsf{M}_G\big(N_{a_1}, C_G(a_1)\big) \qquad (i = 2, \ldots, n).$$

Since f preserves $\mathsf{M}_G\big(N_{a_1}, C_G(a_1)\big)$ and f is absorptive of arity $n \ge 2$, we get as before that

$$\begin{bmatrix} f(a_1, a_2, \ldots, a_n) & f(b_1, b_2, \ldots, b_n) \\ 1 & 1 \end{bmatrix} \\ = f\left(\begin{bmatrix} a_1 & b_1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ a_2 & b_2 \end{bmatrix}, \ldots, \begin{bmatrix} a_n & b_n \\ a_n & b_n \end{bmatrix}\right) \in \mathsf{M}_G\big(N_{a_1}, C_G(a_1)\big).$$

Here $[N_{a_1}, C_G(a_1)] = \{1\}$, therefore the second statement of Lemma 3.7 implies that $f(a_1, a_2, \ldots, a_n) = f(b_1, b_2, \ldots, b_n)$. □

Lemma 3.11. *Let G be a 2-step nilpotent group, and let f be a binary absorptive operation on G that preserves the basic relations of G. For each $a \in G$ and for every abelian normal subgroup A of G the mapping*
$$\varphi\colon AC_G(a) \to G', \quad y \mapsto f(a,y)$$
is a group homomorphism whose kernel contains $C_G(a)$.

Proof. First we will prove the following claim.

Claim 3.12. *Under the same assumptions on G, a, and A, the mapping*
$$\psi\colon A \to G', \quad y \mapsto f(a,y)$$
is a group homomorphism whose kernel contains $A \cap C_G(a)$.

To prove the claim notice first that ψ maps into G' by Lemma 3.10 (1). Next we will show that ψ is a group homomorphism. Since A is abelian, therefore $[A,A] = \{1\}$. It follows from the description of $\mathsf{M}_G(A,A)$ in Lemma 3.7 that for arbitrary elements $x, y \in A$,
$$\begin{bmatrix} a & a \\ a & a \end{bmatrix}, \begin{bmatrix} xy & y \\ x & 1 \end{bmatrix} \in \mathsf{M}_G(A,A).$$
Since $f(a,1) = 1$ and f preserves $\mathsf{M}_G(A,A)$, we get that
$$\begin{bmatrix} f(a,xy) & f(a,y) \\ f(a,x) & 1 \end{bmatrix} = f\left(\begin{bmatrix} a & a \\ a & a \end{bmatrix}, \begin{bmatrix} xy & y \\ x & 1 \end{bmatrix}\right) \in \mathsf{M}_G(A,A).$$
Hence the description of $\mathsf{M}_G(A,A)$ in Lemma 3.7 yields that $f(a,xy)\cdot f(a,y)^{-1} = f(a,x)\cdot 1^{-1}$, that is, $f(a,xy) = f(a,x)f(a,y)$ for all $x, y \in A$. Thus ψ is a group homomorphism. It follows from Lemma 3.10 (2) that $f(a,x) = f(a,y)$ whenever $x \equiv y \pmod{C_G(a)}$. Thus $A \cap C_G(a)$ is contained in the kernel of ψ, as claimed.

Now we will prove that φ is a group homomorphism. Let $\nu\colon AC_G(a) \to AC_G(a)/C_G(a)$ and $\mu\colon A \to A/\big(A \cap C_G(a)\big)$ be the natural homomorphisms, and $\iota\colon AC_G(a)/C_G(a) \to A/\big(A \cap C_G(a)\big)$ the natural isomorphism. By Claim 3.12 the kernel of ψ contains $A \cap C_G(a)$, so there is a unique group homomorphism $\bar\psi\colon A/\big(A \cap C_G(a)\big) \to G'$ such that $\psi = \bar\psi \circ \mu$. In fact,
$$\bar\psi\big(x(A \cap C_G(a))\big) = f(a,x) \qquad \text{for all } x \in A.$$
It follows that
$$(\bar\psi \circ \iota \circ \nu)(x) = f(a,x) \qquad \text{for all } x \in A.$$
Thus $\bar\psi \circ \iota \circ \nu\colon AC_G(a) \to G'$ is a group homomorphism that is constant on each coset of $C_G(a)$ in $AC_G(a)$ and restricts to A as ψ. By definition the mapping $\varphi\colon AC_G(a) \to G'$ also restricts to A as ψ, and by Lemma 3.10 (2) φ is constant on each coset of $C_G(a)$ in $AC_G(a)$. Thus $\varphi = \bar\psi \circ \iota \circ \nu$, so φ is a group homomorphism whose kernel contains $C_G(a)$. ☐

4. Cyclic G'

In this section we will consider 2-step nilpotent groups G whose commutator subgroups are cyclic. Our aim is to prove that if f is an absorptive operation of arity $n \geq 2$ on such a group G and f preserves the basic relations of G, then either

- $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in G^n$, or
- f is binary and there exists an integer k such that $f(x,y) = [x,y]^k$ for all $x, y \in G$.

Lemma 4.1. *Let G be a 2-step nilpotent group such that G' is cyclic. If $\delta = [a,b]$ generates the image of $[a,-]$ then the set $\{c \in G : [a,c] = \delta\} = bC_G(a)$ generates G. Consequently $G = \langle b\rangle C_G(a)$.*

Proof. It is a general fact that if $f\colon G \to H$ is a group homomorphism, and $X \subseteq G$ has the property that $f(X)$ generates $f(G)$, then $f^{-1}(f(X)) = X \cdot \ker(f)$ generates $f^{-1}(f(G)) = G$. The first claim of this lemma is exactly this statement in the situation where $f = [a, -]$, $H = G$, and $X = \{b\}$. The second assertion of the lemma follows from the first. □

Lemma 4.2. *If G is a 2-step nilpotent group such that G' is cyclic, then every element of G' is a commutator.*

Proof. G' is cyclic, hence satisfies ACC on subgroups. Therefore we may choose and fix $a \in G$ so that the image H of $[a, -]$ is maximal among the images of all homomorphisms $[a', -]$ $(a' \in G)$. H is a subgroup of G', so H is cyclic. Let χ be a generator of H, and let b be an element of G such that $[a, b] = \chi$. It follows from Lemma 4.1 that $\{c \in G : [a, c] = \chi\} = bC_G(a)$ generates G.

Claim 4.3. $G = \{c \in G :$ *the image of* $[-, c]$ *is contained in* $H\}$.

To prove this equality let G_1 denote the right hand side. By Lemma 3.8 (3), G_1 is a subgroup of G. Since $bC_G(a)$ generates G, the equality $G = G_1$ will follow if we show that $bC_G(a) \subseteq G_1$.

Let $c \in bC_G(a)$. Since $[a, c] = \chi$, the image of the homomorphism $[-, c]$ contains χ, and hence H. But $[-, c]$ has the same image as $[c, -]$ by Lemma 3.8 (1)(iv), therefore the maximality of H implies that the image of $[-, c]$ is H. Thus $c \in G_1$, which completes the proof of Claim 4.3.

By Claim 4.3 $[x, c] \in H$ for all $x, c \in G$. Thus $G' \subseteq H$. Since $H \subseteq G'$, we get that $H = G'$. This implies that G' is the image of $[a, -]$ for the element $a \in G$ that was chosen at the beginning of the proof. Hence for every $\gamma \in G'$ there exists $b \in G$ such that $\gamma = [a, b]$. □

Lemma 4.4. *Let G be a 2-step nilpotent group such that G' is cyclic. If f is an absorptive binary operation on G that preserves the basic relations of G, then $f\colon G \times G \to G'$ is bilinear. In fact, for every element $a \in G$ there exists an integer k such that $f(a, x) = [a, x]^k$ for all $x \in G$.*

Proof. It suffices to prove the last statement. Choose $a \in G$. The image K of $[a, -]$ is cyclic, since it is a subgroup of G'. By Lemma 4.1, if $[a, b]$ generates K then $G = \langle b\rangle C_G(a)$. Hence also $G = N_b C_G(a)$. Here N_b is an abelian normal subgroup of G by Lemma 3.9 (3). Therefore Lemma 3.11 for $A = N_b$ implies that $f(a, -)\colon G \to G'$ is a group homomorphism whose kernel contains $C_G(a)$.

We claim that the image of $f(a, -)$ is contained in K. To show this let c be an arbitrary element of G. Lemma 3.4 (1), applied to the subgroup $H = \langle a, c\rangle$ of G, yields that the restriction f^H of f to H is an absorptive operation on H that preserves the basic relations of H. Hence by Lemma 3.10 (1) the image of f^H is contained in $H' = \langle [a, c]\rangle$. This implies that $f(a, c) \in \langle [a, c]\rangle \subseteq K$.

Thus $[a, -]\colon G \to K$ is a surjective homomorphism whose kernel is $C_G(a)$, while $f(a, -)\colon G \to K$ is a homomorphism whose kernel contains $C_G(a)$. It follows that $f(a, -) = \varphi \circ [a, -]$ for some homomorphism $\varphi\colon K \to K$. Since K is cyclic, there exists an integer k such that $\varphi(z) = z^k$ for all $z \in K$. Hence $f(a, x) = [a, x]^k$ for all $x \in G$. □

Now we are ready to prove the main result of this section.

Theorem 4.5. *Let G be a 2-step nilpotent group such that G' is cyclic, and let f be an absorptive operation of arity n that preserves the basic relations of G.*

(1) *If $n \geq 3$, then $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in G^n$.*
(2) *If $n = 2$, then there exists an integer k such that $f(x, y) = [x, y]^k$ for all $x, y \in G$.*

Proof. We first consider the case when $n \geq 3$. Let $(a_1, \ldots, a_n) \in G^n$ be an arbitrary n-tuple. Our aim is to prove that $f(a_1, \ldots, a_n) = 1$. The image of $[a_1, -]$ is cyclic, since it is a subgroup of G'. By Lemma 4.1, if b is chosen so that $[a_1, b]$ generates the image of $[a_1, -]$ then $G = \langle b\rangle C_G(a_1)$. Hence there exist integers m_j ($j = 2, \ldots, n$) such that $a_j \equiv b^{m_j} \pmod{C_G(a_1)}$ for all j ($2 \le j \le n$). It follows from Lemma 3.10 (2) that

$$f(a_1, a_2, \ldots, a_n) = f(a_1, b^{m_2}, \ldots, b^{m_n}).$$

Now we apply Lemma 3.10 (2) again in the second variable of f, using that $a_1 \equiv a_1 \pmod{C_G(b^{m_2})}$ and $b^{m_j} \equiv 1 \pmod{C_G(b^{m_2})}$ for all j ($3 \le j \le n$). Thus we get that

$$f(a_1, b^{m_2}, b^{m_3}, \ldots, b^{m_n}) = f(a_1, b^{m_2}, 1, \ldots, 1).$$

Since $n \ge 3$ and f is absorptive, the displayed equalities imply that $f(a_1, \ldots, a_n) = 1$, as claimed. This proves assertion (1).

Now we consider the case when $n = 2$. By Lemma 4.2 every element of G' is a commutator, therefore we may choose and fix $a, b \in G$ so that $[a, b]$ generates G'. By Lemma 4.4 there exists an integer k such that $f(a, y) = [a, y]^k$ for all $y \in G$. Now we define a binary function g on G as follows:

$$g(x, y) = f(x, y)[x, y]^{-k} \qquad (x, y \in G).$$

g preserves the basic relations of G, since it is obtained from f and from the group operations by composition. Furthermore, g is absorptive and satisfies $g(a, y) = 1$ for all $y \in G$.

Claim 4.6. $G = \{c \in G : \textit{the image of } g(-, c) \textit{ is } \{1\}\}$.

To prove this equality let G_1 denote the right hand side. Since g preserves the basic relations of G and is absorptive, we know from Lemma 4.4 that $g \colon G \times G \to G'$ is bilinear. Hence we get from Lemma 3.8 (3) that G_1 is a subgroup of G. Since $[a, b]$ generates G', Lemma 4.1 implies that $bC_G(a)$ generates G. Therefore the equality $G = G_1$ will follow if we show that $bC_G(a) \subseteq G_1$.

Let c be an arbitrary element of $bC_G(a)$. By Lemma 4.4 there exists an integer l such that $g(x, c) = [x, c]^l$ for all $x \in G$. Thus $g(a, c) = [a, c]^l$. Here $g(a, c) = 1$ because $g(a, y) = 1$ for all $y \in G$. Furthermore, $[a, c] = [a, b]$, because $c \equiv b \pmod{C_G(a)}$. Therefore we get that $1 = [a, b]^l$. Since $[a, b]$ generates G', this equality implies that l is a multiple of $|G'|$. Hence $g(x, c) = [x, c]^l = 1$ for all $x \in G$. This shows that $c \in G_1$, which completes the proof of Claim 4.6.

The equality in Claim 4.6 proves that $g(x, y) = 1$ for all $x, y \in G$. Hence we get from the definition of g that $f(x, y) = [x, y]^k$ for all $x, y \in G$. This completes the proof of assertion (2) in Theorem 4.5. □

5. General G'

In this section G will be an arbitrary 2-step nilpotent group. Our goal is to prove that if f is an absorptive operation of arity $n \ge 2$ on G and f preserves the basic relations of G, then either

- $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in G^n$, or
- f is binary and $f(x, y) = [x, y]^\kappa$ for all $x, y \in G$ where κ is a profinite integer or an ordinary integer according to whether G' is a torsion group or not.

We will start by proving these statements under the additional assumption that G' is finitely generated.

Lemma 5.1. *Let G be a 2-step nilpotent group such that G' is finitely generated. If f is an absorptive operation of arity $n \ge 3$ that preserves the basic relations of G, then $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in G^n$.*

Proof. G' is a finitely generated abelian group, therefore it is a direct product of finitely many cyclic subgroups. We will prove the lemma by induction on the number r of cyclic direct factors of G'. Let f be an absorptive operation of arity $n \ge 3$ that preserves all basic relations of G. If $r = 1$, that is, if G' is cyclic, then $f(\mathbf{x}) = 1$ holds for all $\mathbf{x} \in G^n$ by Theorem 4.5 (1). Therefore from now on we will assume that $r \geq 2$ and that the conclusion of the lemma is true for all 2-step nilpotent groups whose commutator subgroup is a direct product of fewer than r cyclic subgroups.

To prove that the conclusion of the lemma is true for G whose commutator subgroup is a direct product of r cyclic subgroups, we choose and fix a direct decomposition of G' with cyclic direct factors $C_1, \ldots, C_r$. For arbitrary i ($1 \leq i \leq n$), C_i is a normal subgroup of G because $C_i \subseteq G' \subseteq Z(G)$. Thus, by Lemma 3.4 (2), the operation f^{G/C_i} induced by f on G/C_i is absorptive and preserves the basic relations of the quotient group G/C_i. Furthermore, the group G/C_i is 2-step nilpotent, and its commutator subgroup $(G/C_i)' = G'/C_i$ is a direct product of $r-1$ cyclic subgroups. Hence it follows from the induction hypothesis that $f^{G/C_i}(\mathbf{u}) = 1$ for all $\mathbf{u} \in (G/C_i)^n$. By the definition of f^{G/C_i} this is equivalent to the condition that $f(\mathbf{x}) \in C_i$ for all $\mathbf{x} \in G^n$. Since i ($1 \leq i \leq r$) was chosen arbitrarily and $r \geq 2$, we get that $f(\mathbf{x}) \in C_1 \cap C_2 = \{1\}$ for all $\mathbf{x} \in G^n$. $\square$

Lemma 5.2. *Let G be a 2-step nilpotent group such that G' is finitely generated. If f is a binary absorptive operation on G such that f preserves the basic relations of G, then there exists an integer k such that $f(x, y) = [x, y]^k$ for all $x, y \in G$.*

Proof. First we will prove the lemma under the stronger assumption that G' is finite. We will proceed by induction on the order of G'. By Theorem 4.5 (2) the conclusion of the lemma is true if G' is cyclic. Therefore we may assume from now on that G' is not cyclic, and that the conclusion of the lemma is true for all 2-step nilpotent groups with commutator subgroups of order less than $|G'|$.

To prove that the conclusion of the lemma is true for G' we choose and fix a prime p such that the Sylow p-subgroup P of G' is not cyclic. Such a p exists, since G' is not cyclic. The condition that P is not cyclic implies that P has an elementary abelian subgroup of order p^2, and hence P has $p+1$ subgroups $C_0, C_1, \ldots, C_p$ of order p such that $C_i \cap C_j = \{1\}$ for all $0 \leq i < j \leq p$. As in the preceding lemma we get that for arbitrary i ($0 \leq i \leq p$), C_i is a normal subgroup of G. By Lemma 3.4 (2), the operation f^{G/C_i} induced by f on G/C_i is absorptive and preserves the basic relations of the quotient group G/C_i. Furthermore, the group G/C_i is 2-step nilpotent, and its commutator subgroup $(G/C_i)' = G'/C_i$ has order less than $|G'|$. Hence we get from the induction hypothesis that there exists an integer k_i such that $f^{G/C_i}(u, v) = [u, v]^{k_i}$ for all $u, v \in G/C_i$. By the definition of f^{G/C_i} this is equivalent to the condition that

$$f(x, y) \in [x, y]^{k_i} C_i \qquad \text{for all } x, y \in G. \tag{5.1}$$

Since $C_i \subseteq Z(G)$ and $|C_i| = p$, we get that

$$f(x, y)^p = [x, y]^{k_i p} \qquad \text{for all } x, y \in G.$$

This holds for each i ($0 \leq i \leq p$), hence

$$[x, y]^{k_0 p} = [x, y]^{k_1 p} = \cdots = [x, y]^{k_p p} \qquad \text{for all } x, y \in G.$$

Let e denote the exponent of G'. Since the Sylow p-subgroup of G' is nontrivial, e is a multiple of p. By the last displayed equalities the numbers $k_0 p, k_1 p, \ldots, k_p p$ are congruent modulo the order of each pure commutator $[x, y] \in G'$. Since these elements generate G', the numbers $k_0 p, k_1 p, \ldots, k_p p$ are congruent modulo e. Equivalently,

$$k_0 \equiv k_1 \equiv \cdots \equiv k_p \pmod{e/p}.$$

Every number that is congruent to k_0 modulo e/p is congruent to one of the numbers $k_0 + (e/p)l$ ($l = 0, 1, \ldots, p-1$) modulo e. These are p different numbers, while $k_0, k_1, \ldots, k_p$ are $p+1$ numbers each of which is congruent to k_0 modulo e/p. Thus there exist $r \neq s$ $(0 \leq r, s \leq p)$ such that $k_r \equiv k_s \pmod{e}$. Consequently, $[x,y]^{k_r} = [x,y]^{k_s}$ for all $x, y \in G$. Now we get from (5.1) that

$$f(x,y)[x,y]^{-k_r} = f(x,y)[x,y]^{-k_s} \in C_r \cap C_s = \{1\} \qquad \text{for all } x, y \in G.$$

Hence $f(x,y) = [x,y]^{k_r}$ for all $x, y \in G$. This proves Lemma 5.2 for the case when G' is finite.

Now assume that G' is infinite. Since G' is a finitely generated abelian group, it has an element of infinite order. Therefore for some $a, b \in G$ the commutator $\gamma = [a,b]$ has infinite order. The subgroup $H = \langle a, b\rangle$ of G has cyclic commutator subgroup $H' = \langle [a,b]\rangle$, and by Lemma 3.4 (1) the restriction f^H of f to H is an absorptive operation that preserves the basic relations of H. Thus, by Theorem 4.5 (2), there exists an integer k such that $f(x,y) = f^H(x,y) = [x,y]^k$ for all x, y in H.

Our aim is to show that $f(x,y) = [x,y]^k$ for all x, y in G. Suppose that this is not the case, select $c, d \in G$ such that $f(c,d) \neq [c,d]^k$, and let $K = \langle c, d\rangle$. Repeating the argument in the preceding paragraph for c, d in place of a, b we get that there exists an integer l such that $f(x,y) = f^K(x,y) = [x,y]^l$ for all x, y in K. Since $[c,d]^k \neq f(c,d) = [c,d]^l$, therefore $l \neq k$; moreover, l and k are incongruent modulo the order of $[c,d]$ if $[c,d]$ has finite order.

To continue the proof we need to establish a fact that is true for any finitely generated abelian group F. First, for any abelian group A and positive integer n let $A^{[n]}$ denote the group of all n-th powers of elements of A. The fact we need is that there is a 'degree' function d assigning to each nontorsion element $u \in F$ a positive integer $d(u)$ such that for any positive integer n that is a multiple of $d(u)$ the order of $uF^{[n]}$ in $F/F^{[n]}$ is a multiple of $n/d(u)$. We prove this first when F is torsion free. We begin by defining the set of 'possible degrees' of $u \in F \setminus \{1\}$: let $D(u)$ be the set of positive integers r such that $x^r = u$ has a solution in F. $D(u)$ is closed under least common multiple since F is abelian, and $D(u)$ is finite since F is finitely generated. We take $d(u) = \mathrm{lcm}\big(D(u)\big)$. Let $u_0 \in F$ be such that $u_0^{d(u)} = u$. We have $d(u_0) = 1$, because if $v \in F$ is such that $v^{d(u_0)} = u_0$, then $v^{d(u_0)d(u)} = u$, and hence, by the definition of $d(u)$, $d(u_0)d(u)$ is a divisor of $d(u)$. Now let n be a multiple of $d(u)$, say $n = d(u)n'$, and assume that $uF^{[n]}$ has order r in $F/F^{[n]}$. This implies that $u^r = z^n$ for some $z \in F$. Hence $(u_0^r)^{d(u)} = u_0^{d(u)r} = u^r = z^n = z^{d(u)n'} = (z^{n'})^{d(u)}$. Since F is torsion free, we get that $u_0^r = z^{n'}$. Thus $r, n' \in D(u_0^r)$, which implies that $m := \mathrm{lcm}(r, n')$ is an element of $D(u_0^r)$. Therefore $u_0^r = y^m$ for some $y \in F$, so we get that $u_0^r = (y^{m/r})^r$. Since F is torsion free, we conclude that $u_0 = y^{m/r}$. But $d(u_0) = 1$, therefore $m/r = 1$. Hence $\mathrm{lcm}(r, n') = m = r$, and it follows that r is divisible by $n' = n/d(u)$, as claimed. This proves our claim for the case when F is torsion free. For the general case, let T be the torsion subgroup of F, let $\nu\colon F \to F/T$ be the natural map, and let $d_{F/T}$ denote the degree function on the torsion free group F/T described earlier in this paragraph. We define the degree function on F to be $d = d_{F/T} \circ \nu$. The function d is defined for all nontorsion elements $u \in F$, since $\nu(u) = uT$ is a nontorsion element of F/T and $d_{F/T}$ is defined for such elements. If u is a nontorsion element of F and n is a multiple of $d(u)$, then uT is a nontorsion element of F/T and n is a multiple of $d_{F/T}(uT)$. Hence by the claim proved for the torsion free case we know that the order of $(uT)(F/T)^{[n]}$ in $(F/T)/(F/T)^{[n]}$ is a multiple of $n/d_{F/T}(uT)$. Since $F/F^{[n]} \to (F/T)/(F/T)^{[n]}$, $xF^{[n]} \mapsto (xT)(F/T)^{[n]}$ is a homomorphism that maps $uF^{[n]}$ to $(uT)(F/T)^{[n]}$, it follows that the order of $uF^{[n]}$ in $F/F^{[n]}$ is also a multiple of $n/d_{F/T}(uT) = n/d(u)$.

We return to our main argument. Recall that our assumptions ensure that G' is a finitely generated abelian group, hence the torsion subgroup of G' is finite. Furthermore, we have selected elements $a, b, c, d \in G$ such that $[a,b]$ is a nontorsion element of G' and $[c,d]$ is another element of G' such that $f(a,b) = [a,b]^k$, $f(c,d) = [c,d]^l$, $k \neq l$, and l and k are incongruent modulo the order of $[c,d]$ if $[c,d]$ is a torsion element. Using the degree function on G' introduced in the preceding paragraph we now choose and fix a positive integer m that satisfies the following conditions: $m > |k-l|$, m is a multiple of the exponent of the torsion subgroup of G', m is a multiple of the degree of the nontorsion element $[a,b] \in G'$, and if $[c,d]$ is a nontorsion element, then m is also a multiple of the degree of $[c,d]$. Since G is 2-step nilpotent, $(G')^{[m^2]}$ is a normal subgroup of G. In $\bar{G} := G/(G')^{[m^2]}$ the commutator subgroup is finitely generated and has exponent dividing m^2, so it is finite. By Lemma 3.4 (2), the operation $f^{\bar{G}}$ is absorptive and preserves the basic relations of $\bar{G}$, so there is an integer r such that $f^{\bar{G}}(\bar{x},\bar{y}) = [\bar{x},\bar{y}]^r$ for all $\bar{x} = x(G')^{[m^2]}$ and $\bar{y} = y(G')^{[m^2]}$ in $\bar{G}$ (by the first part of this proof). This implies that

$$[\bar{a},\bar{b}]^r = f^{\bar{G}}(\bar{a},\bar{b}) = [\bar{a},\bar{b}]^k \qquad \text{and} \qquad [\bar{c},\bar{d}]^r = f^{\bar{G}}(\bar{c},\bar{d}) = [\bar{c},\bar{d}]^l. \tag{5.2}$$

Since the degree of $[a,b] \in G'$ divides m, the claim we proved in the preceding paragraph implies that the order of $[\bar{a},\bar{b}]$ in $\bar{G} = G/(G')^{[m^2]}$ is a multiple of m. Therefore the first equality in (5.2) implies that $r \equiv k \pmod{m}$. Similarly, if $[c,d]$ is a nontorsion element in G', then the order of $[\bar{c},\bar{d}]$ in $\bar{G} = G/(G')^{[m^2]}$ is a multiple of m, so the second equality in (5.2) yields that $r \equiv l \pmod{m}$. Hence in this case $k \equiv l \pmod{m}$, which is impossible, since $k \neq l$ and $m > |k-l|$. Therefore $[c,d]$ is a torsion element in G'. As m is a multiple of the exponent of the torsion subgroup of G', it follows that the order of $[c,d]$ is a divisor of m and is equal to the order of $[\bar{c},\bar{d}]$ in $\bar{G} = G/(G')^{[m^2]}$. So in this case the second equality in (5.2) yields that $r \equiv l \pmod{|[c,d]|}$. Since m is a multiple of $|[c,d]|$ and $r \equiv k \pmod{m}$, we get that $k \equiv l \pmod{|[c,d]|}$. But this contradicts the fact that k and l are incongruent modulo the order of $[c,d]$ if $[c,d]$ is a torsion element. This contradiction completes the proof of the lemma. □

Now we are ready to prove the result stated at the beginning of this section.

Theorem 5.3. *Let G be a 2-step nilpotent group, and let f be an absorptive operation of arity $n \geq 2$ that preserves the basic relations of G.*

(1) *If $n \geq 3$, then $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in G^n$.*
(2) *If $n = 2$ and G' is a torsion group, then there exists a profinite integer κ such that $f(x,y) = [x,y]^\kappa$ for all $x,y \in G$.*
(3) *If $n = 2$ and G' has an element of infinite order, then there exists an integer k such that $f(x,y) = [x,y]^k$ for all $x,y \in G$.*

Proof. By Lemma 3.4 (1) f restricts to every subgroup H of G as an absorptive operation f^H that preserves all basic relations of H^2. Since H is 2-step nilpotent, the function $[-,-]\colon H \times H \to H'$ is bilinear. Hence if H is finitely generated, say $H = \langle c_1,\dots,c_p\rangle$, then H' is generated by the commutators $[c_i,c_j]$ $(1 \leq i < j \leq p)$, so H' is also finitely generated. Therefore Lemmas 5.1 and 5.2 imply the following.

Claim 5.4. *Let H be a finitely generated subgroup of G.*

(1) *If f has arity $n \geq 3$, then $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in H^n$.*
(2) *If f is binary, then there exists an integer k such that $f(x,y) = [x,y]^k$ for all $x,y \in H$.*

To prove part (1) of Theorem 5.3 assume that $n \geq 3$, and let $a_1,\dots,a_n$ be arbitrary elements of G. We have to verify that $f(a_1,\dots,a_n) = 1$. Let $H = \langle a_1,\dots,a_n\rangle$. By Claim 5.4 $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in H^n$. In particular, $f(a_1,\dots,a_n) = 1$, which completes the proof of (1).

To prove parts (2) and (3) of Theorem 5.3 let f be binary. We will use the conventions introduced in the first paragraph of the proof of Lemma 3.2. Applying Claim 5.4 to an arbitrary 2-generated subgroup $H = \langle a,b\rangle$ of G we see that there exists an integer $r(a,b)$ such that $f(a,b) = [a,b]^{r(a,b)}$. Similarly, applying Claim 5.4 to an arbitrary 4-generated subgroup $H = \langle a,b,c,d\rangle$ of G we get that

$$f(a,b) = [a,b]^k \quad \text{and} \quad f(c,d) = [c,d]^k \qquad \text{for some integer } k.$$

But
$$f(a,b) = [a,b]^{r(a,b)} \quad \text{and} \quad f(c,d) = [c,d]^{r(c,d)},$$
so
$$r(a,b) \equiv k \pmod{|[a,b]|} \quad \text{and} \quad r(c,d) \equiv k \pmod{|[c,d]|} \qquad \text{for some integer } k. \tag{5.3}$$
In particular, it follows that
$$r(a,b) \equiv r(c,d) \pmod{|[a,b]|} \qquad \text{whenever } |[a,b]| \text{ divides } |[c,d]|. \tag{5.4}$$

G' is abelian, and each of its elements is a product of finitely many commutators of elements of G. These properties imply that if G' has an element of infinite order, then some commutator $[c,d]$ $(c,d \in G)$ is of infinite order. Hence $|[c,d]| = 0$ and $|[a,b]|$ divides $|[c,d]|$ for all elements $a,b \in G$. Therefore we get from (5.4) that $f(a,b) = [a,b]^{r(a,b)} = [a,b]^{r(c,d)}$ for all $a,b \in G$. This proves that statement (3) holds for $k = r(c,d)$.

From now on we will assume that G' is a torsion group, that is, $|[a,b]|$ is a positive integer for all $a,b \in G$. Let I denote the subset of Λ that consists of all positive integers $|[a,b]|$ $(a,b \in G)$. It follows from (5.4) that $r(a,b) \equiv r(c,d) \pmod{m}$ whenever $m = |[a,b]| = |[c,d]|$. Hence there exists a sequence $\delta = (\bar{d}_m)_{m \in I} \in \prod_{m \in I} \mathbb{Z}_m$ such that
$$r(a,b) \equiv d_{|[a,b]|} \pmod{|[a,b]|} \qquad \text{for all } a,b \in G. \tag{5.5}$$
Moreover, (5.3) implies that δ has the following property:

($*$) the system
$$z \equiv d_m \pmod{m}$$
$$z \equiv d_n \pmod{n}$$
of simultaneous congruences has a solution for all $m,n \in I$.

It follows from Claim 3.3 that there exists a profinite integer $\kappa = (\bar{k}_m)_{m \in \Lambda} \in \widehat{\mathbb{Z}}$ such that $\bar{d}_m = \bar{k}_m$ for all $m \in I$. This fact together with (5.5) implies that
$$f(a,b) = [a,b]^{r(a,b)} = [a,b]^{d_{|[a,b]|}} = [a,b]^{k_{|[a,b]|}} = [a,b]^{\kappa} \qquad \text{for all } a,b \in G.$$
This completes the proof of statement (2) and hence of Theorem 5.3. □

The main result of this paper, Theorem 2.3, follows from Lemmas 3.1, 3.2, and Theorem 5.3, as we will show below.

Proof of Theorem 2.3. At the end of Section 2 we argued that (c) ⇒ (b) and (b) ⇒ (a). Now we will prove that (a) ⇒ (c). Let $f(\mathbf{x}) = f(x_1,\dots,x_n)$ be an n-ary operation on G that preserves the basic relations of G. Since f preserves the one-element subgroup $\{(1,1)\}$ of G^2, it follows that $f(1,\dots,1) = 1$. Applying Lemma 3.1 to the clone $\mathcal{C}$ of all operations that preserve the basic relations of G we get that $f(\mathbf{x})$ can be written as a product
$$f(\mathbf{x}) = \prod_q f_q(\mathbf{x}_q) \tag{5.6}$$
of absorptive operations $f_q(\mathbf{x}_q)$ that preserve the basic relations of G. Assume that some variable $y \neq x_1,\dots,x_n$ occurs on the right hand side of (5.6). If we substitute 1 for all such variables y, the factors on the right hand side that involve such variables become 1 (and can be omitted), while the other factors and the left hand side remain unchanged. Therefore we may assume without loss of generality that each f_q depends only on some of the variables $x_1,\dots,x_n$.

By Theorem 5.3 (1) each f_q with more than two variables satisfies $f_q(\mathbf{x}_q) = 1$ for all arguments $\mathbf{x}_q$ from G, therefore each such factor can be omitted from the product in (5.6). So we may assume without loss of generality that each f_q is either unary or binary. By Lemma 3.2 each unary f_q has the form $f_q(x_i) = x_i^\kappa$ for some i $(1 \le i \le n)$ and κ where κ is a profinite integer if G is a torsion group, and an ordinary integer otherwise. Similarly, by Theorem 5.3 (2) and (3), each binary f_q has the form $f_q(x_i, x_j) = [x_i, x_j]^\lambda$ for some $i < j$ $(1 \le i, j \le n)$ and λ where λ is a profinite integer if G' is a torsion group, and an ordinary integer otherwise.

The well-known identities

$$\begin{gathered} y^\beta x^\alpha = x^\alpha y^\beta [y^\beta, x^\alpha] = x^\alpha y^\beta [y,x]^{\alpha\beta} = x^\alpha y^\beta [x,y]^{-\alpha\beta}, \\ [x,y]^\gamma z = z[x,y]^\gamma, \\ x^\alpha x^\beta = x^{\alpha+\beta}, \qquad x^0 = 1, \\ [x,y]^\alpha [x,y]^\beta = [x,y]^{\alpha+\beta}, \qquad [x,y]^0 = 1 \end{gathered}$$

for 2-step nilpotent groups hold even if some of the exponents α, β, γ are profinite integers, because if G or G', respectively, are torsion groups, then on their finitely generated subgroups the profinite integer exponents can be replaced by ordinary integer exponents. Using these identities we get that the product representation of f in (5.6) can be rewritten in the form described in (c). This completes the proof of Theorem 2.3. □

References

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