EXTENDING UFDS TO PIDS WITHOUT ADDING UNITS

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ABSTRACT. If U is a UFD, then there is a PID P containing U that has the same unit group as U. Moreover, P can be taken so that its field of fractions is a pure transcendental extension of the field of fractions of U with transcendence degree at most $|U|$.

1. Introduction. At a recent conference, Anna Romanowska raised the question of whether there is a PID P such that (i) P is a proper subring of the real numbers, (ii) P properly contains the ring of integers, and (iii) P has unit group $P^\times = \{\pm 1\}$. She was presenting her joint paper with Gábor Czédli, [**2**], which studies convex sets in generalized affine spaces. A classical real affine space may be described algebraically as an $\mathbb{R}$-module equipped with the affine operations $ax + (1-a)y$ for $a \in \mathbb{R}$. The convex subsets are those closed under those operations $ax + (1-a)y$ where $a \in [0,1]$. The Czédli-Romanowska generalization replaces $\mathbb{R}$ with a subring $P \leq \mathbb{R}$ that is a PID. Affine spaces over P are P-modules equipped with the operations $ax + (1-a)y$, $a \in P$, and convex subsets of such faithful spaces are the subsets closed under those operations $ax + (1-a)y$ where $a \in P \cap [0,1]$. It turns out that any invertible element in $P \cap [0,1]$ gives rise to a congruence (called an "aiming congruence") on $C \times C$ for each convex subset C of an affine space over P (Section 5 of [**2**]). Such congruences play an essential role in the algebraic description of the topological closure of C. This explains the source of Romanowska's question: Can the PID $P \subseteq \mathbb{R}$ be chosen so that its notion of convexity is nontrivial (i.e., $P \neq \mathbb{Z}$), and such that all aiming congruences are trivial (i.e., $P^\times = \{\pm 1\}$)?

In this note Romanowska's question is considered as a question of pure commutative ring theory, and the question is answered affirmatively. In fact, it will be shown that if U is any UFD, then there is a PID P containing U that has the same unit group as U. Moreover, P can be taken so that its field of fractions is a pure transcendental extension of the field of fractions of U with transcendence degree at most $|U|$. This answers Romanowska's question as follows: $\mathbb{Z}[\pi]$ is a UFD that is a subring of $\mathbb{R}$ that properly contains $\mathbb{Z}$ and has only ± 1 as units. Extend $\mathbb{Z}[\pi]$ to a PID P without adding units using the theorem of this paper. The field of fractions of P will be a pure transcendental extension of $\mathbb{Q}(\pi)$ of countable degree, hence will be embeddable in $\mathbb{R}$ since the field extension $\mathbb{R}/\mathbb{Q}(\pi)$ has uncountable transcendence degree. Hence there is a PID P contained properly between $\mathbb{Z}$ and $\mathbb{R}$ whose only units are ± 1.

The main result is proved in Section 2. This note concludes with Section 3 where the following observations are explained: (i) if P is any PID answering Romanowska's question, then every number in the difference $P - \mathbb{Z}$ must be transcendental, (ii) there are integral domains that cannot be extended to PID's without adding units, and (iii) there are UFD's that can be extended to PID's without adding units but which cannot be extended further to Euclidean domains without adding units.

2. The proof. If A is an integral domain, then $\widehat{A}$ denotes its field of fractions and $A^\times$ denotes its group of units. If $S \subseteq A$ is a multiplicatively closed subset, then the localization of A at S is

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denoted $S^{-1}A$, although if $S = \{b^n\}_{n\geq 0}$ is generated by a single element b then we typically write A_b for $S^{-1}A$.

Lemma 2.1. *If U is a UFD, $a, b \in U$ are coprime, and X and Y are indeterminates, then*

$$U[X,Y]/(aX+bY-1)$$

is a UFD that extends U. The field of fractions of $U[X,Y]/(aX+bY-1)$ is a pure transcendental extension of $\widehat{U}$ of transcendence degree 1.

Proof. First let's see that $U[X,Y]/(aX+bY-1)$ is a domain that extends U. $U[X]$ is a UFD whose prime elements are the irreducibles of U together with those polynomials in $U[X]$ having content 1 that are irreducible over $\widehat{U}$. One such is the linear polynomial $(aX-1)$, since the content is $\gcd(a,-1) = 1$. Now $U[X,Y] = (U[X])[Y]$ is a UFD whose prime elements are the irreducibles of $U[X]$ together with those polynomials in $U[X,Y]$ having content 1 that are irreducible over $\widehat{U}(X)$. One such is the linear polynomial $bY + (aX-1)$, since the content is $\gcd(b, aX-1) = 1$. Since $aX+bY-1$ is prime in $U[X,Y]$, $U[X,Y]/(aX+bY-1)$ is a domain. To see that it extends U, it suffices to note that the ideal $(aX+bY-1)$ restricts trivially to the subring $U \leq U[X,Y]$ consisting of constant polynomials. This follows from the fact that every nonzero element of $(aX+bY-1)$ has degree at least 1 with respect to X or Y.

Next let's see that $U[X,Y]/(aX+bY-1)$ satisfies the ascending chain condition on principal ideals (ACCP). Suppose that $(d_1) \subseteq (d_2) \subseteq \cdots$ is an ascending chain of principal ideals in $U[X,Y]/(aX+bY-1)$. Choose elements $e_{k+1} \in U[X,Y]/(aX+bY-1)$ such that $d_k = d_{k+1}e_{k+1}$. Writing $U[X,Y]/(aX+bY-1)$ in the form $U[X, \frac{1-aX}{b}]$, consider it to be a subring of the localization

$$(U[X, \tfrac{1-aX}{b}])_b = U_b[X]. \tag{2.1}$$

In the larger ring, $U_b[X]$, which is a UFD, the chain must stabilize. Assume that $(d_k) = (d_{k+1}) = \cdots$, so for sufficiently large k there exist elements $f_k/b^{n_k} \in U_b[X]$ with $f_k \in U[X,Y]/(aX+bY-1)$ such that $d_k \cdot (f_k/b^{n_k}) = d_{k+1}$. Since $U_b[X]$ is a domain in which $d_k = d_{k+1}e_{k+1}$ and $d_k \cdot (f_k/b^{n_k}) = d_{k+1}$, it must be that $e_{k+1} \cdot (f_k/b^{n_k}) = 1$ in $U_b[X]$, or $e_{k+1}f_k = b^{n_k}$ in $U[X,Y]/(aX+bY-1)$. This shows that for sufficiently large k the element e_{k+1} divides a power of b in $U[X,Y]/(aX+bY-1)$. A similar argument shows that for sufficiently large k the element e_{k+1} divides a power of a in $U[X,Y]/(aX+bY-1)$. Since $aX+bY = 1$ in $U[X,Y]/(aX+bY-1)$, e_{k+1} is a unit for sufficiently large k. Since $d_k = d_{k+1}e_{k+1}$ it follows that $(d_k) = (d_{k+1}) = \cdots$ in $U[X,Y]/(aX+bY-1)$ for sufficiently large k.

Next we claim that if q is a prime divisor of b in U, then q remains prime in $U[X,Y]/(aX+bY-1)$, i.e., (q) is a prime ideal in $U[X,Y]/(aX+bY-1)$. For this it suffices to establish the primeness of the ideal $(q, aX+bY-1) = (q, aX-1)$ in $U[X,Y]$. Now $U[X,Y]/(q,aX-1) \cong U/(q)[X,Y]/(aX-1) \cong U/(q)[Y]_a$, where the last ring may be constructed in steps: form the quotient $U/(q)$; form the polynomial ring $U/(q)[Y]$; then localize at the powers of a, $U/(q)[Y]_a$. U itself was a domain, factoring by the prime ideal (q) preserves/creates the domain property, forming the polynomial ring $U/(q)[Y]$ preserves the domain property, then localizing at the nonzero element a, $U/(q)[Y]_a$, also preserves the domain property. (That a is nonzero in $U/(q)[Y]$ follows from the fact that q does not divide a in U, since $q \mid b$ and $\gcd(a,b) = 1$.) This shows that $U[X,Y]/(q,aX-1)$ is a domain, so $(q, aX-1)$ is prime in $U[X,Y]$ and so q is prime in $U[X,Y]/(aX+bY-1)$.

Nagata's Criterion states that if A is an integral domain with ACCP, S is a multiplicatively closed subset of A that is generated by prime elements, and the localization $S^{-1}A$ is a UFD, then A itself

is a UFD. Apply this to the ring $A = U[X,Y]/(aX+bY-1)$ with S equal to the multiplicatively closed subset of $U[X,Y]/(aX+bY-1)$ that is generated by the set of all prime divisors of b in U. Here it helps to write $A = U[X,Y]/(aX+bY-1)$ in the form $U[X, \frac{1-aX}{b}]$. It has been shown that A has ACCP. In the localization $S^{-1}A$ the element b is a unit, hence

$$S^{-1}A = S^{-1}(U[X, \tfrac{1-aX}{b}]) = S^{-1}U[X, 1-aX] = S^{-1}U[X],$$

which is a UFD since it is a localization of a polynomial ring over a UFD. By Nagata's Criterion, $A = U[X,Y]/(aX+bY-1)$ is itself a UFD.

For the final statement of the theorem, the element b becomes a unit in the field of fractions of $U[X,Y]/(aX+bY-1)$. Hence the field of fractions of $U[X,Y]/(aX+bY-1)$ is the same as the field of fractions of the ring $(U[X,Y]/(aX+bY-1))_b = U_b[X]$. This field of fractions is easily seen to be $\widehat{U}(X)$, which has transcendence degree 1 over $\widehat{U}$. □

Lemma 2.2. *Assume that U is a UFD and $a, b \in U$ are coprime. If*

$$f, g \in U[X,Y]/(aX+bY-1),$$

g divides f, and $f \in U \setminus \{0\}$, then $g \in U$. In particular (when $f = 1$), any unit of $U[X,Y]/(aX+bY-1)$ lies in U. Moreover, if $f_1, f_2 \in U$ are coprime in U, then they remain coprime in the extension $U[X,Y]/(aX+bY-1)$,

Proof. Every element of $U[X,Y]/(aX+bY-1) = U[X, \frac{1-aX}{b}]$ $(\leq U_b[X])$ is a polynomial in X over the localization U_b. If $f \in U \setminus \{0\}$, then f has degree zero with respect to X, hence any divisor of f must have degree zero with respect to X. This forces $g \in U_b$. A similar argument using the representation $U[X,Y]/(aX+bY-1) = U[\frac{1-bY}{a}, Y] \leq U_a[Y]$ shows that $g \in U_a$. Therefore $g \in U_a \cap U_b = U$, where the last equality follows from the facts that U is a UFD and a and b are coprime.

The last two assertions of the lemma follow from the first. □

Theorem 2.3. *If U is a UFD, then U has an extension P that is a PID such that U and P have exactly the same set of units. Moreover, P can be chosen so that the field of fractions $\widehat{P}$ is a pure transcendental extension of $\widehat{U}$ of degree at most $|U|$.*

Proof. In this first paragraph we describe the strategy of the proof. If U is already a PID, then there is nothing to do. Otherwise U is infinite and contains elements a and b such that the ideal (a,b) is not principal. If $c = \gcd(a,b)$, then $a = a'c$ and $b = b'c$ for some coprime elements a' and b' such that the ideal (a',b') is not principal. The proof consists of a construction designed to kill off all such "bad pairs" of coprime elements (i.e., pairs of coprime elements that generate nonprincipal ideals).

The proof begins now. Assume that U is an infinite UFD. Let $\kappa = |U|$ and enumerate with κ a set of coprime pairs of elements of U which includes all bad pairs of U (that is, all pairs of coprime elements generating nonprincipal ideals). Here $(1,1)$ is a coprime pair, and pairs are allowed to be reused in the enumeration, so this kind of enumeration is possible.

If the enumeration function is $\beta\colon \kappa \to U^2$, then define rings V_i, $i < \kappa$, as follows.

(1) $V_0 = U$.
(2) $V_{i+1} = V_i[X_i, Y_i]/(a_iX_i + b_iY_i - 1)$ if $\beta(i) = (a_i, b_i)$.
(3) If $\lambda \leq \kappa$ is limit, then $V_\lambda = \bigcup_{i<\lambda} V_i$.

The statement "V_μ is a UFD and the pairs enumerated by β remain coprime in V_μ" can be established for all $\mu \leq \kappa$ by transfinite induction using Lemmas 2.1 and 2.2. When $\mu = 0$ the statement holds by our initial hypothesis that U is a UFD and by the definition of β. When $\mu = i+1$ is a successor ordinal, Lemma 2.1 proves that V_μ is a UFD while Lemma 2.2 proves that the pairs enumerated by β remain coprime in V_μ. If μ is a limit ordinal, any element $f \in V_\mu = \bigcup_{i<\mu} V_i$ occurs first at some successor stage V_{i+1} or else in V_0, and when it first occurs all divisors of f that lie in V_μ already exist in V_i or V_0 respectively. Thus, unique factorization of elements and coprimeness of β-enumerated pairs in V_μ is inherited from V_{i+1} or V_0.

Thus $U_1 := V_\kappa$ is a UFD containing $U_0 := U$ as a subring. Since new divisors of elements are not introduced during the construction, no new units are introduced. Hence the UFD U_1 is an extension of U_0 that has the same unit group, but all bad pairs in U_0 have been "killed" in U_1.

To keep track of the transcendence degree of the field of fractions as the construction progresses we note the following:

Claim 2.4. *Let κ be a cardinal and let F_i, $i < \kappa$, be a sequence of fields such that*

(1) *F_{i+1}/F_i is a pure transcendental extension with transcendence base T_i for all $i < \kappa$, and*
(2) *$F_\lambda := \bigcup_{i<\lambda} F_i$ when $\lambda \leq \kappa$ is limit.*

Then F_κ/F_0 is a pure transcendental extension with transcendence base $\bigcup_{i<\kappa} T_i$.

To prove the claim, one argues by transfinite induction on λ that $\bigcup_{i<\lambda} T_i$ is algebraically independent and, together with F_0, generates F_λ as a field. □

Applying Claim 2.4 to the situation where $F_i = \widehat{V_i}$, $i < \kappa$, we obtain that $\widehat{U_1}/\widehat{U_0}$ is a pure transcendental extension of transcendence degree κ. (In particular, $|U_1| = \kappa = |U|$.)

We may iterate the construction from above to produce a chain $U = U_0 \leq U_1 \leq \cdots$ where each U_i is a UFD with the same group of units as U, in each U_{i+1} all bad pairs from U_i have been killed, any divisor of an element that first appears at the ith stage also exists at the ith stage, and each $\widehat{U_{i+1}}$ is a pure transcendental extension of $\widehat{U_i}$ of degree $\kappa = |U|$. The union $P = \bigcup_{i<\omega} U_i$ is therefore a UFD with no bad pairs. Such a ring is necessarily a PID, as the following argument shows. No pair of coprime elements in P can generate a nonprincipal ideal, so P is a Bezout domain. To show that P is a PID, it suffices to show that it is Noetherian. If this is not the case, then there is a strictly increasing chain of ideals $I_0 \subsetneq I_1 \subsetneq \cdots$ in P. This can be adjusted to a strictly increasing chain of principal ideals, as follows. Choose $d_{i+1} \in I_{i+1} \setminus I_i$ for all i. Now choose f_i so that $(f_i) = (d_1, \ldots, d_i)$ in P for all i. This is possible since P is Bezout. The chain $(f_1) \subsetneq (f_2) \subsetneq \cdots$ of principal ideals in P has been constructed so that it is strictly increasing. This is impossible, since P is a UFD.

Applying Claim 2.4 to the chain $\widehat{U} = \widehat{U_0} \leq \widehat{U_1} \leq \cdots \leq \bigcup_{i<\omega} \widehat{U_i} = \widehat{P}$ we obtain that $\widehat{P}$ is a pure transcendental extension of $\widehat{U}$ of transcendence degree $\omega \cdot \kappa = \kappa = |U|$. □

3. Problems and discussion. If $\mathcal{D}$ is a subcategory of a category $\mathcal{C}$, one may ask if each $\mathcal{C}$-object has a morphism to some $\mathcal{D}$-object. If the inclusion functor $\mathcal{D} \to \mathcal{C}$ has a left adjoint, then indeed each $\mathcal{C}$-object has a *universal* morphism to a $\mathcal{D}$-object given by the unit of the adjunction. This is the case, for example, when $\mathcal{C}$ is the category of integral domains equipped with embeddings and $\mathcal{D}$ is the full subcategory of fields. The universal embedding of an integral domain into a field is its embedding into its field of fractions.

It may happen that each $\mathcal{C}$-object has a morphism to a $\mathcal{D}$-object, but not a universal such morphism, such as when $\mathcal{C}$ is the category of fields and $\mathcal{D}$ is the full subcategory of algebraically closed fields. The author does not know a conventional term for this situation, so (borrowing a term from order theory) let's call $\mathcal{D}$ a *cofinal* subcategory if each $\mathcal{C}$-object has a morphism to a $\mathcal{D}$-object.

Every ring homomorphism $\varphi\colon R \to S$ preserves units in the sense that $u \in R^\times$ implies $\varphi(u) \in S^\times$. Say that φ *reflects* units if $v \in S^\times$ implies that $\varphi^{-1}(v)$ is a nonempty subset of $R^\times$. Thus a unit-reflecting embedding $\varphi : R \to S$ restricts to an isomorphism between unit groups.

The theorem of this paper may be expressed as follows: if $\mathcal{C}$ is the category of UFD's equipped with unit-reflecting embeddings, then the full subcategory of PID's is cofinal. This paper does not resolve the more general question:

Question 3.1. Does the inclusion functor from the category of PID's (with unit-reflecting embeddings) into the category of UFD's (with unit-reflecting embeddings) have a left adjoint? Is there a *universal* unit-reflecting embedding of a UFD into a PID?

Now we turn to another observation and question. Recall that Romanowska's original question was whether there is a subring $P \leq \mathbb{R}$ of the field of real numbers such that

(1) P properly contains $\mathbb{Z}$,
(2) P is a PID, and
(3) the only units of P are $+1$ and -1.

Claim: If P is such a ring, then any algebraic number in P must be a rational integer. To see this, choose any algebraic number $\alpha \in P$. The rings $K := \mathbb{Q}[\alpha]$ (a field) and P (a PID) are integrally closed, so the intersection $I := K \cap P$ is integrally closed and lies between $\mathbb{Z}[\alpha]$ and P. It follows that I contains the integral closure of $\mathbb{Z}$ in K, which is the ring $\mathcal{O}_K$ of algebraic integers in K. By Dirichlet's Unit Theorem, the group of units in $\mathcal{O}_K$ is the product of a finite group of roots of unity and a free abelian group of rank $r + s - 1$ where r is the number of real embeddings of K and s is the number of pairs of conjugate complex embeddings. Since $\mathcal{O}_K \leq I \leq P$ has $+1$ and -1 as its only units, it follows that $r + s - 1 = 0$. Since K is real, it follows that $r \geq 1$, while of course $s \geq 0$, hence $r = 1$, $s = 0$, and K has inclusion as its unique embedding into $\mathbb{C}$. If d is the degree of the minimal polynomial of α over $\mathbb{Q}$, then there are at least d embeddings of K into $\mathbb{C}$, so $d = 1$ and α is rational. If $\alpha = p/q$ where $\gcd(p, q) = 1$, then choose $u, v \in \mathbb{Z}$ such that $pu + qv = 1$, or $\alpha u + v = 1/q$. Since $\alpha, u, v, q \in S$, we get $q, 1/q \in P$, hence $q = \pm 1$, hence $\alpha \in \mathbb{Z}$.

Thus if P is any PID answering Romanowska's question, then any number in $P - \mathbb{Z}$ is transcendental. This suggests:

Question 3.2. Given a UFD U, what is the minimum transcendence degree of the extension $\widehat{P}/\widehat{U}$ where P is a PID that contains U and satisfies $P^\times = U^\times$? Is it always possible to find a PID P such that the transcendence degree of $\widehat{P}/\widehat{U}$ is finite? Is it always possible to find a PID P such that the transcendence degree of $\widehat{P}/\widehat{U}$ is 1?

The strategy used in this paper to construct a PID satisfying $\mathbb{Z} \subsetneq P \subsetneq \mathbb{R}$ and $P^\times = \{\pm 1\}$ is to first adjoin a transcendental number to $\mathbb{Z}$ (forming, say, $\mathbb{Z}[\pi]$), and then to eliminate all occurrences of nonprincipal ideals via a sequence of extensions. But observe that this must be done carefully. If, for example, at some point of the construction we have a ring containing the transcendentals $\pi, \pi^{\frac{1}{2}}, \pi^{\frac{1}{4}}, \pi^{\frac{1}{8}}, \ldots$, then the ring cannot be further extended to a PID without adding units. More generally, if at some point of the construction we have a domain containing any strictly increasing sequence of principal ideals $(d_1) \subsetneq (d_2) \subsetneq \cdots$, then in any larger domain with no additional units

this chain remains a properly increasing chain. This can't happen in a PID. A stronger statement is true: If $P \leq \mathbb{R}$ answers Romanowska's question, then in any subring of P any principal ideal is contained in only finitely many other principal ideals.

Let's change the question. Rather than extending a UFD to a PID without adding units, can we extend a UFD to a Euclidean domain without adding units? Interestingly, this is not always possible: there exist UFD's that cannot be extended to Euclidean domains without adding units.

For example, it is well known that $P = \mathbb{Z}[(1+\sqrt{-19})/2]$ is a PID that is not a Euclidean domain (see pages 277 and 282 of **[3]**). This ring P cannot even be extended to a Euclidean domain without adding units. This can be established by a slight modification (described below) of the argument used on page 277 of **[3]** to show that P is not a Euclidean domain.

Suppose that $\varphi\colon P \to E$ is a unit-reflecting embedding of P into a Euclidean domain E. E has nonunits, so E is not a field. Therefore E has an element u of least Euclidean norm among elements in $E - (E^\times \cup \{0\})$. Such an element $u \in E$ is a *universal side divisor* for E, which means that every nonzero coset of the ideal (u) contains a unit. In particular, $E/(u)$ has cardinality at most $|E^\times \cup \{0\}| = |E^\times| + 1$. $E/(u)$ has cardinality at least 2, since u is not a unit, so

$$2 \leq [E : (u)] \leq |E^\times| + 1.$$

The units of $P = \mathbb{Z}[(1+\sqrt{-19})/2]$ are only ± 1, as one can show with a norm argument. If $\varphi\colon P \to E$ is a unit-reflecting embedding, then $E^\times = \{\pm 1\}$. The previous displayed line becomes

$$2 \leq [E : (u)] \leq 3,$$

so E must have an ideal (u) of index 2 or 3. Restricting (u) to P we obtain an ideal $(u)|_P = \varphi^{-1}((u))$ of index 2 or 3 in P. But there is no such ideal in $Z[(1+\sqrt{-19})/2]$, as can be shown by a norm argument (page 277 of **[3]**). Thus there is no unit-reflecting embedding of the PID $\mathbb{Z}[(1+\sqrt{-19})/2]$ into a Euclidean domain.

Question 3.3. What conditions on a PID P are necessary for there to exist a unit-reflecting embedding from P into a Euclidean domain?

Problem 3.4. Let $\mathcal{ID}$ be the category of integral domains equipped with unit-reflecting embeddings. Discover interesting instances $(\mathcal{C}, \mathcal{D})$ of pairs of full subcategories where $\mathcal{C} \supsetneq \mathcal{D}$ and $\mathcal{D}$ is cofinal in $\mathcal{C}$.

For example, this paper shows that $(\mathcal{C}, \mathcal{D}) = (\text{UFD's}, \text{PID's})$ is an instance, while $(\text{PID's}, \text{Euclidean domains})$ is not an instance.

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