CATEGORICAL QUASIVARIETIES VIA MORITA EQUIVALENCE

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Abstract. We give a new proof of the classification of $\aleph_0$-categorical quasivarieties by using Morita equivalence to reduce to term minimal quasivarieties.

1. Introduction

When T is a first–order theory and λ is a cardinal, we write $\mathrm{Spec}_T(\lambda)$ to denote the number of isomorphism classes of models of T which have power λ. T, or the class of models of T, is said to be λ*–categorical* if $\mathrm{Spec}_T(\lambda) = 1$. Baldwin and Lachlan proved in [1] that for a universal Horn class $\mathcal{Q}$ the following conditions are equivalent:

- $\mathcal{Q}$ is λ–categorical for some infinite λ and $\mathcal{Q}$ has a finite member.
- $\mathcal{Q}$ is $\aleph_0$–categorical.
- $\mathcal{Q}$ is λ–categorical for all infinite λ.

This raises the question of which quasivarieties of algebras satisfy these conditions. Independently and at about the same time, Givant ([3], [4]) and Palyutin ([15], [16]) answered this question. Later, in [11], McKenzie gave a new proof using the newly developed techniques of tame congruence theory. In this paper we give another proof. Our proof, which is shorter than previous proofs, is based on the easily proved but crucial fact established in Lemma 5.3. Therefore, it is worthwhile now to explain the significance of this lemma.

Let $\mathbf{A}$ be an algebra and let e be a unary term in the language of $\mathbf{A}$. If $\mathbf{A} \models e(e(x)) = e(x)$, then we call e an *idempotent* of $\mathbf{A}$. If e is an idempotent of $\mathbf{A}$, then we call its range, $e(A)$, a *neighborhood* of $\mathbf{A}$. There is a way, described in Section 2, of restricting the structure of $\mathbf{A}$ to a neighborhood $e(A)$. This restriction, denoted by $e(\mathbf{A})$, is called the *localization of* $\mathbf{A}$ *to the neighborhood* $e(A)$. If $\mathcal{Q}$ is a quasivariety and e is a unary term in the language such that $\mathcal{Q} \models e(e(x)) = e(x)$,

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then the class, $e(\mathcal{Q})$, of localizations of members of $\mathcal{Q}$ to the range of e is also a quasivariety and the assignment $\mathbf{A} \mapsto e(\mathbf{A})$ is the object part of a functor. Localization simplifies the situation, usually at the price of a loss of some information. The significance of Lemma 5.3 is that it shows that the localization functor is a Morita equivalence for $\aleph_0$–categorical quasivarieties. This means that one loses no information by localizing, although one does simplify things. Since $\aleph_0$–categoricity is Morita invariant, this allows us to reduce the classification problem for categorical quasivarieties to the special case where all members of the quasivariety are minimal with respect to localization. In this setting the problem is easily solved.

To read the paper one must know what a Boolean power is (see [6]), and a little bit about tame congruence theory (see [5] or [10]). In this paper a 'type' will always refer to a type in the sense of tame congruence theory. To avoid conflict, the similarity type of an algebra will be referred to as its *signature*. We have no need to refer to types in the sense of model theory. There will be conflicting uses of the words 'idempotent' and 'affine', and here is how we will resolve them. An operation h of arity greater than one is *idempotent* if the equation $h(x, x, \ldots, x) = x$ holds. A class of algebras is idempotent if each member has only idempotent term operations. As described above, a unary operation e is idempotent if $e(e(x)) = e(x)$ holds. An algebra is *affine* if it is polynomially equivalent to a module, and a class is affine if it consists of affine algebras. An *affine operation* is an operation representable as a module polynomial. However, an *affine module* is the reduct of a module to its idempotent operations.

In order to keep the paper as self-contained as possible and to emphasize the algebraic flavor of the proof, I have supplied short algebraic arguments for some basic facts about $\aleph_0$-categoricity which are known to all model theorists although perhaps not to all algebraists (for example, Theorems 5.1, 5.2 and Lemma 5.5). One model-theoretic fact which I did not see how to replace with a short algebraic argument is the fact that an $\aleph_0$-categorical theory is definitionally equivalent to a theory in a countable language (see Theorem 12.2.2 of [6]). To avoid this issue I assume throughout the paper that the language is countable.

2. Morita Equivalence

In the classical theory of Morita equivalence one describes certain concrete functors between full module categories and proves that those functors are categorical equivalences. Then one shows that all categorical equivalences are of the concrete type. In [12] McKenzie extends this scheme of ideas so that it encompasses not only module categories but to fairly general categories of algebras of any signature. For the purposes of this paper it suffices to know that the results of [12] apply to all quasivarieties. In this section we describe a type of concrete functor called 'Morita equivalence' which [12] proves to represent a general categorical equivalence between quasivarieties.

'A class of algebras' will always mean 'a class of algebras of the same signature'. When a class of algebras is considered as a category, the morphisms are taken to be all homomorphisms. We consider only algebras with no constant symbols. Instead, the role of constants will be played by constant unary operations. The only situation where constant unary operations are inadequate substitutes for zeroary constants are when one must deal with the empty algebra, which supports constant unary operations but no constant zeroary operations. We leave it to the interested reader to interpret the results herein as they apply to the empty algebra.

Let $\mathcal{K}$ be a class of algebras of signature Σ. A *definitional expansion* of $\mathcal{K}$ is a class $\mathcal{K}^+$ of algebras obtained from $\mathcal{K}$ by expanding Σ to a signature $\Sigma^+ \supseteq \Sigma$ where for each symbol $F \in \Sigma^+ - \Sigma$ there is a Σ–term f of the same arity as F such that F and f have the same interpretation in $\mathcal{K}^+$. Classes $\mathcal{K}$ and $\mathcal{L}$ are *definitionally equivalent*, written $\mathcal{K} \equiv \mathcal{L}$, if they have equal definitional expansions. We write $\mathbf{A} \equiv \mathbf{B}$ to mean that $\{\mathbf{A}\} \equiv \{\mathbf{B}\}$. Changing from a class $\mathcal{K}$ of algebras to a definitionally equivalent class $\mathcal{L}$ is nothing more than changing the language. This 'change of language' is a categorical equivalence from $\mathcal{K}$ to $\mathcal{L}$ which assigns to each member of $\mathcal{K}$ 'itself' considered as a member of $\mathcal{L}$, and assigns to each morphism itself.

An algebra $\mathbf{A}$ is *weakly isomorphic* to an algebra $\mathbf{B}$ (written $\mathbf{A} \cong_w \mathbf{B}$) if $\mathbf{A}$ is isomorphic to an algebra $\mathbf{A}'$ where $\mathbf{A}' \equiv \mathbf{B}$. This means exactly that there is a bijection $\varphi : A \to B$ which induces an isomorphism $(\varphi) : \mathrm{Clo}\,(\mathbf{A}) \to \mathrm{Clo}\,(\mathbf{B})$ between the corresponding clones of term operations according to the following rule:

$$(\varphi)_n(h(x_1, \ldots, x_n)) = \varphi(h(\varphi^{-1}(x_1), \ldots, \varphi^{-1}(x_n)).$$

(The *clone* of an algebra is the multisorted algebra whose elements are the term operations of the algebra and whose operations are composition and projection.)

If $\mathbf{A}$ is an algebra, then the *$[k]$–th power* of $\mathbf{A}$ is the algebra with universe A^k whose n–ary operation symbols are the k–tuples $F = \langle f_1, \ldots, f_k \rangle$ where each f_i is an nk–ary term in the language of $\mathbf{A}$.

Here is how we interpret F as an n–ary operation on A^k. First, if $\mathbf{a}_1, \dots, \mathbf{a}_n \in A^k$, then write these n vectors as the columns of a $k \times n$ matrix M. Let $\vec{M}$ denote the nk–vector consisting of the concatenation $\mathbf{a}_1^t{}^\frown \cdots {}^\frown \mathbf{a}_n^t$. (As a row vector, $\vec{M} = \langle a_1^1, a_1^2, \cdots, a_1^k, a_2^1, \cdots, a_n^k \rangle$.) Now, F acts on M according to the rule

$$F(\mathbf{a}_1, \dots, \mathbf{a}_n) = F\left(\begin{bmatrix} a_1^1 \\ \vdots \\ a_1^k \end{bmatrix}, \cdots, \begin{bmatrix} a_n^1 \\ \vdots \\ a_n^k \end{bmatrix} \right) := \begin{bmatrix} f_1(\vec{M}) \\ \vdots \\ f_k(\vec{M}) \end{bmatrix} \in A^k.$$

The $[k]$–th power of $\mathbf{A}$ is denoted $\mathbf{A}^{[k]}$. If $\mathcal{K}$ is a class of algebras, then the algebras isomorphic to $[k]$–th powers of algebras in $\mathcal{K}$ have the same signature and the collection of them is denoted $\mathcal{K}^{[k]}$. The $[k]$–th power construction is the object part of a categorical equivalence from $\mathcal{K}$ to $\mathcal{K}^{[k]}$ whose morphism part assigns to a homomorphism its k–fold power acting coordinatewise.

Let $\mathcal{K}$ be a class of algebras and let e be a unary term in the language of $\mathcal{K}$. We describe a construction on classes $\mathcal{K} \mapsto e(\mathcal{K})$ by first describing the construction on individual algebras. We write $e(\mathbf{A})$ to denote the algebra with universe $e(A)$ and operation symbols $\{et \mid t \text{ a term of } \mathbf{A}\}$. We interpret et as an operation on $e(A)$ by interpreting et as an operation on A and then restricting the domain to $e(A)$. If $\mathcal{K}$ is a class of algebras and e is an idempotent of $\mathcal{K}$, then we write $e(\mathcal{K})$ for the class $\{e(\mathbf{A}) \mid \mathbf{A} \in \mathcal{K}\}$. The class $e(\mathcal{K})$ consists of algebras of the same signature. The assignment $\mathbf{A} \mapsto e(\mathbf{A})$ is the object part of a functor from $\mathcal{K}$ to $e(\mathcal{K})$. The morphism part of the functor assigns to a homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$ its restriction $\varphi|_{e(A)} : e(\mathbf{A}) \to e(\mathbf{B})$.

Regarding the construction described in the previous paragraph, a special case of interest is the case where e is an invertible unary term. A unary term e (not necessarily idempotent) is said to be *invertible* with respect to the class $\mathcal{K}$ if there exists, for some n, an n–ary term σ and n unary terms τ_i such that

$$\mathcal{K} \models \sigma(e\tau_1(x), \dots, e\tau_n(x)) = x. \tag{2.1}$$

Invertibility of an idempotent term e implies that $\mathcal{K} \mapsto e(\mathcal{K})$ is a categorical equivalence.

Definition 2.1. A class $\mathcal{K}$ of algebras is *Morita equivalent* to a class $\mathcal{L}$ if there is a $k < \aleph_0$ and an invertible unary term e of $\mathcal{K}^{[k]}$ such that $e(\mathcal{K}^{[k]}) \equiv \mathcal{L}$.

Morita equivalence is an equivalence relation on classes of algebras. If two classes of algebras are Morita equivalent, then they are equivalent as categories. The converse holds for quasivarieties (see [12] for this and for everything else claimed up to this point concerning $[k]$–th powers, invertible idempotents and Morita equivalence). Morita equivalence preserves finiteness and cardinalities of infinite algebras. (This is obvious for definitional equivalence and $[k]$–th powers and follows from equation (2.1) for the construction $\mathcal{K} \mapsto e(\mathcal{K})$.) Hence, λ–categoricity is a Morita invariant for infinite λ.

3. Term Minimal and Permutational Algebras

An algebra is *term minimal* if its only nonconstant idempotent unary term operation is the identity. An algebra is *permutational* if its nonconstant unary term operations form a group under composition. A class of algebras is said to be term minimal or permutational if each member is. Clearly "permutational $\Longrightarrow$ term minimal", since the only idempotent element of a group is the identity element, but the converse implication fails. For example, any p-group is term minimal, but it is not permutational if its exponent is greater than p.

Our plan is to show that every $\aleph_0$–categorical quasivariety is Morita equivalent to a minimal, locally finite, permutational $\aleph_0$–categorical quasivariety. Such a quasivariety is generated by a finite permutational algebra with no proper nontrivial subalgebras. Understanding the structure of such algebras is a fundamental problem in the model theory of locally finite algebras. See the citations following Theorem 3.4 to learn what is known about the solution to this problem.

Definition 3.1. Let G be a group. An algebra is a *G–algebra* if it is weakly isomorphic to an algebra with universe G which has

- (i) each left multiplication $\lambda_g(x) := gx$, $g \in G$, as a term operation and
- (ii) each right multiplication $\rho_g(x) := xg$, $g \in G$, as an automorphism.

An algebra is a *G^0–algebra* if it is weakly isomorphic to an algebra with universe $G \cup \{0\}$ (we assume $0 \notin G$) which has

- (i) the constant $0(x) \equiv 0$ and each left multiplication $\lambda_g(x) := gx$, $g, x \in G$, $\lambda_g(0) = 0$, as term operations and
- (ii) each right multiplication $\rho_g(x) := xg$, $g, x \in G$, $\rho_g(0) = 0$, as an automorphism.

A G–algebra is nothing more than an algebra which is freely generated by each one of its elements. A similar statement, with the obvious modification, holds for G^0–algebras.

In order to state the next result we introduce some standard terminology: a finite algebra whose expansion by constants is term minimal is called *E-minimal*.

Theorem 3.2. *Let* $\mathbf{A}$ *be a finite term minimal algebra with no nontrivial proper subalgebras. Then* $\mathbf{A}$ *is:*

(i) *an idempotent simple algebra,*
(ii) *the constant expansion of an E-minimal algebra,*
(iii) *a quotient of a G–algebra, or*
(iv) *a quotient of a* G^0*–algebra.*

Proof. An observation that we will use repeatedly, which depends on the finiteness of $\mathbf{A}$, is that if $f(x)$ is a unary term operation then some iterate $f^k(x)$ is idempotent. Therefore if f is not a permutation then the term minimality of $\mathbf{A}$ implies that for some k the term operation $f^k(x)$ is constant.

Claim 3.3. *If there are elements* $0, a \in A$ *such that* $\{0\}$ *is a subuniverse and* $\{a\}$ *is not, then* $\{0\}$ *is the range of a constant term operation and is the only 1-element subuniverse of* $\mathbf{A}$.

Proof. Since $\{a\}$ is not a subuniverse it is a generating set. There is a unary term f such that $f(a) = 0$. This f is not a permutation, since $f(a) = f(0)$, so there is some k such that $f^k(x)$ is constant. Since $f^k(0) = 0$, the range of f^k is $\{0\}$. No element other than 0 is preserved by f^k, so $\{0\}$ is the only 1-element subuniverse.

Let's continue with the proof of the theorem. Assume that $\mathbf{A}$ has at least two 1-element subuniverses. Claim 3.3 implies that every element of $\mathbf{A}$ is a 1-element subuniverse. This is equivalent to the statement that $\mathbf{A}$ is idempotent. Any congruence class of an idempotent algebra is a subuniverse, so any idempotent algebra with no proper nontrivial subalgebras is simple. We are in Case (i).

The set of images of constant term operations is a subuniverse. If this subuniverse has more than one element, then it must be all of $\mathbf{A}$. In this case every element of $\mathbf{A}$ is the image of a constant term. Now we are in Case (ii).

If we are not in Cases (i) or (ii), then $\mathbf{A}$ has at most one element which is the image of a constant term (and that element is necessarily a subuniverse), and it has at most one 1-element subuniverse (which, by Claim 3.3, is the image of a constant term). Thus, $\mathbf{A}$ has exactly one element, 0, which is the image of a constant term operation and this element represents the unique 1-element subuniverse of $\mathbf{A}$, or else $\mathbf{A}$ has no proper subalgebras and no constant term operations. We claim that in either case $\mathbf{A}$ is permutational.

If $\mathbf{A}$ has no constant term operations, then for any unary f its idempotent iterate f^k must be the identity function on $\mathbf{A}$. Hence all unary term operations are permutations of finite order, which implies that $\mathbf{A}$ is permutational.

Now suppose that $\mathbf{A}$ has exactly one element 0 such that $\{0\}$ is a subuniverse and the range of a constant operation. Assume that f is a unary term operation which is not a permutation. Then for some k we have $f^k(x) = 0$. Assume that k is minimal for this and that $k > 1$. Choose $u \in f^{k-1}(A) - \{0\}$ and $v \in f^{-1}(u)$. Since $\{u\}$ is not a subuniverse it is a generating set, so there is a term g such that $g(u) = v$. Now $h := fg$ is a unary term satisfying $h(A) \subseteq f(A) \neq A$ and $h(u) = u \neq 0 = h(0)$. It follows that any idempotent iterate of h is neither the identity nor a constant. This contradicts our assumption that $\mathbf{A}$ is term minimal, so $k > 1$ is not possible.

We continue under the assumption that $\mathbf{A}$ is not in Cases (i) or (ii), and therefore that $\mathbf{A}$ is permutational. Let $\mathcal{Q}$ be the quasivariety generated by $\mathbf{A}$. The fact that the unary term operations of $\mathbf{A}$ form a group with perhaps one constant can be expressed equationally, so the the unary term operations of $\mathbf{F} = \mathbf{F}_{\mathcal{Q}}(1)$ have the same properties as those of $\mathbf{A}$. This and the freeness of $\mathbf{F}$ imply that $\mathbf{F}$ is either a G–algebra or a G^0–algebra. Since $\mathbf{A}$ is generated by any element not equal to zero it is a quotient of $\mathbf{F}$. Hence we are in either Case (iii) or (iv) depending on whether or not we have a constant term. □

The algebras in Cases $(i), (iii)$ and (iv) of Theorem 3.2 are permutational. There exist 'Case (ii)'–algebras which are not permutational (p-groups), but they do not appear in this paper. Therefore we restate the previous theorem for permutational algebras. We introduce another standard term: If the constant expansion of an algebra is permutational, then the algebra is called *minimal.*

Theorem 3.4. *Let $\mathbf{A}$ be a finite term minimal algebra with no nontrivial proper subalgebras. $\mathbf{A}$ is permutational iff every nonconstant unary term is invertible. Such an algebra is:*

- (i) *an idempotent simple algebra,*
- (ii) *the constant expansion of a minimal algebra,*
- (iii) *a quotient of a G–algebra, or*
- (iv) *a quotient of a G^0–algebra.*

Proof. It is only the first statement of this theorem that has not yet been proved. For the trivial direction of that statement: If $\mathbf{A}$ is permutational then it is term minimal and every nonconstant unary term operation is invertible in the sense of equation 2.1 since this kind of invertibility for a term f is obviously a weakening of the property that f^{-1} is a term operation.

Assume conversely that $\mathbf{A}$ is term minimal and that every nonconstant unary term is invertible in the sense of equation 2.1. In order to obtain a contradiction assume that $\mathbf{A}$ is not permutational. Then we must be in Case (*ii*) of Theorem 3.2 and $\mathbf{A}$ must have a nonconstant f which is not a permutation. Since the nonconstant unary terms of $\mathbf{A}$ are invertible in the sense of equation 2.1 there are terms σ and τ_i such that $\mathbf{A} \models \sigma(f\tau_1(x), \ldots, f\tau_k(x)) = x$. Choose a pair of congruences $\alpha \prec \beta$ in $\mathbf{A}$. By Lemma 4.28 of [5] the fact that $\mathbf{A}$ is E-minimal and f is not a permutation implies that $f(\beta) \subseteq \alpha$. Thus, if $(a, b) \in \beta - \alpha$ we get that $(f\tau_i(a), f\tau_i(b)) \in \alpha$ for all i. Hence

$$a = \sigma(f\tau_1(a), \ldots, f\tau_k(a)) \equiv_\alpha \sigma(f\tau_1(b), \ldots, f\tau_k(b)) = b,$$

which is false. This contradiction completes the proof. □

All finite algebras that fall into Case (*i*) of Theorems 3.2/3.4 are known. See [18] for details. The finite algebras described in Theorem 3.2 (*ii*) can be obtained by combining results of [5] and [10]. The finite algebras from Theorem 3.4 (*ii*) are described in [13]. The finite simple algebras described in Cases (*iii*) and (*iv*) of Theorems 3.2/3.4 are described in [19] and [20].

'Unfortunately' for our paper, the finite nonsimple algebras in Cases (*iii*) and (*iv*) are not yet known. One reason that we are still unfamiliar with G– and G^0–algebras is that for a nontrivial group G the interval in the lattice of clones on the set $G \cup \{0\}$ which corresponds to the G^0–algebras is big.[1]

Having alerted the reader to the mystery surrounding the structure of G– and G^0–algebras we now confess that these complexities will not intrude upon us, for to describe categorical quasivarieties it will be enough to understand certain abelian G– and G^0–algebras. We describe these algebras in the next section.

We close this section with a result (needed later) concerning Morita equivalence and the simplest kinds of permutational quasivarieties.

Lemma 3.5. (See [9].) *If $\mathcal{Q}$ is an essentially unary permutational quasivariety, then a quasivariety is Morita equivalent to $\mathcal{Q}$ if and only if it is definitionally equivalent to $\mathcal{Q}^{[k]}$ for some k.*

[1]It is shown in [20] that this interval contains $2^{\aleph_0}$ inequivalent G^0–algebras for any finite nontrivial group G. The interval in the lattice of clones on G which corresponds to the G–algebras can be shown to be infinite for 'most' groups G. The only known exceptions are the finite simple groups. See [14] for a proof that there are only finitely many inequivalent G–algebras when G is finite and simple.

4. Affine and Abelian Algebras

An algebra $\mathbf{A}$ is *abelian* if the diagonal of $A \times A$ is a congruence class. There is a commutator of congruences described in Chapter 3 of [5] that generalizes the group commutator, and with respect to this commutator an algebra is abelian in the sense just defined iff it satisfies $[1, 1] = 0$.

An algebra $\mathbf{A}$ is *affine* if it is polynomially equivalent to a module. This statement means that there is a unital ring $\mathbf{R}$ and a (left, unital) $\mathbf{R}$-module $\mathbf{M}$ such that $\mathbf{M}$ has the same universe and polynomial operations as $\mathbf{A}$. Thus any polynomial operation of $\mathbf{A}$ is expressible in the form $p(x_1, \dots, x_n) = r_1x_1 + \cdots + r_nx_n + m$ with $r_i \in R$ and $m \in M$. To see that affine algebras are abelian it is enough to note that the definition of the abelian property depends only on polynomial structure, and that modules satisfy the definition.

It is not hard to show that if $\mathbf{A}$ is polynomially equivalent to $\mathbf{M}$, then a module polynomial

$$p(x_1, \dots, x_n) = r_1x_1 + \cdots + r_nx_n + m$$

of $\mathbf{M}$ belongs to the clone of $\mathbf{A}$ if and only if the derived unary polynomial $p(x, \dots, x) = (r_1 + \cdots + r_n)x + m$ belongs to the clone of $\mathbf{A}$. Therefore, an algebra $\mathbf{A}$ polynomially equivalent to an $\mathbf{R}$-module $\mathbf{M}$ is specified up to definitional equivalence by the data: $\mathbf{R}$, $\mathbf{M}$, and the set of unary module polynomials which are unary term operations of $\mathbf{A}$. To make the transition from "knowing $\mathbf{A}$ up to polynomial equivalence" to "knowing $\mathbf{A}$ up to definitional equivalence" we have to determine the possible sets of unary functions on M which could be the unary part of the clone of $\mathbf{A}$. This is worked out in an elegant way in Proposition 2.6 of [17] which we reproduce in a reworded form.

Theorem 4.1. *Let $\mathbf{A}$ be an algebra which is polynomially equivalent to a faithful $\mathbf{R}$-module $\mathbf{M}$. The set $U_{\mathbf{A}}$ of all pairs of the form $(1-r, m) \in R \times M$, where $r(x)+m$ is a unary term operation of $\mathbf{A}$, is a submodule of the $\mathbf{R}$-module $\mathbf{R} \times \mathbf{M}$.*

Conversely, given any $\mathbf{R}$-submodule U of $\mathbf{R} \times \mathbf{M}$, the algebra $\mathbf{A}_U$ whose universe is $\mathbf{M}$ and whose operations are all $r_1x_1+\cdots+r_nx_n+m$ with $(1 - (\sum r_i), m) \in U$ is an algebra polynomially equivalent to $\mathbf{M}$.

The mappings $\mathbf{A} \mapsto U_{\mathbf{A}}$ and $U \mapsto \mathbf{A}_U$ are inverse bijections between the definitional equivalence classes of algebras polynomially equivalent to $\mathbf{M}$ and the submodules of $\mathbf{R} \times \mathbf{M}$.

The proof of Theorem 4.1 follows immediately from its statement.

For the next theorem, which determines affine G–algebras, $J(\mathbf{R})$ denotes the Jacobson radical of $\mathbf{R}$.

Theorem 4.2. *An affine algebra* $\mathbf{A}$ *is a* G*–algebra iff there is a ring* $\mathbf{R}$*, a faithful* $\mathbf{R}$*–module* $\mathbf{M}$ *and a homomorphism* $\varphi : \mathbf{M} \to J(\mathbf{R})$ *of (left)* $\mathbf{R}$*–modules such that* $\mathbf{A}$ *is definitionally equivalent to the algebra whose universe is* M *and whose term operations are all module polynomials* $(\sum_{i=1}^{n} r_i x_i) + m$ *such that* $(\sum_{i=1}^{n} r_i) + \varphi(m) = 1$*.*

Proof. This is a straightforward application of Theorem 4.1, and we give only the following hint: If $\mathbf{A}$ is polynomially equivalent to the $\mathbf{R}$-module $\mathbf{M}$, then the structure of $\mathbf{A}$ is determined by $\mathbf{R}$, $\mathbf{M}$, and an $\mathbf{R}$-submodule U of $\mathbf{R} \times \mathbf{M}$. Show that $\mathbf{A}$ is a G–algebra if and only if the converse of U is the graph of a function $\varphi : \mathbf{M} \to J(\mathbf{R})$. □

In the previous theorem, the group of unary operations of $\mathbf{A}$ are those polynomials of $\mathbf{M}$ of the form $(1 - \varphi(a))x + a$, $a \in A$. When $\mathbf{R}$ is Jacobson semisimple then $\varphi(M) = \{0\}$, so the unary operations of $\mathbf{A}$ must be the set of translations: $\{x + a \mid a \in A\}$. Hence, $\mathbf{A}$ is definitionally equivalent to the expansion by translations of the idempotent reduct of $\mathbf{M}$.

Theorem 4.3. *A finite* G^0*–algebra is abelian iff it is it is essentially unary or definitionally equivalent to a one–dimensional vector space over a finite field.*

Proof. Essentially unary algebras and vector spaces are abelian, so we need to prove the converse.

If $\mathbf{A}$ is an abelian G^0–algebra and $p(x) = t(x, a_1, \ldots, a_n)$ is a unary polynomial of $\mathbf{A}$, then by the abelian property $p(x)$ has the same kernel as $p'(x) = t(x, 0, \ldots, 0)$ (see 3.1.1 of [5]). As p' is a unary term it is a permutation or a constant; the same must be true for p. Thus, $\mathbf{A}$ is an abelian minimal algebra, and by Pálfy's Theorem it is essentially unary or polynomially equivalent to a vector space over a finite field. Theorem 4.1 implies that an algebra that has the same polynomial operations as a vector space is definitionally to that vector space if it has a unique 1-element subalgebra. Since G^0–algebras are one–generated, $\mathbf{A}$ must be a one–dimensional vector space. □

We close this section by remarking that the property of being affine is a Morita invariant for quasivarieties. This can be proved by checking that each of the constituent functors in a Morita equivalence preserve affineness.

5. $\aleph_0$–Categorical Quasivarieties

This section houses our proof of the main result of the paper. As stated in the introduction of the paper, we deal only with quasivarieties in a countable language.

Theorem 5.1. *An $\aleph_0$–categorical quasivariety is a minimal, locally finite quasivariety.*

Proof. The free algebra $\mathbf{F}$ in the quasivariety $\mathcal{Q}$ generated by a countably infinite set is a countably infinite member of $\mathcal{Q}$. Since this algebra is not finitely generated and each finitely generated free algebra $\mathbf{F}_{\mathcal{Q}}(n)$ is countable, it follows by $\aleph_0$–categoricity that each $\mathbf{F}_{\mathcal{Q}}(n)$ is finite. Thus $\mathcal{Q}$ is locally finite.

If $\mathbf{A}$ is any finitely generated (hence countable) member of $\mathcal{Q}$ and $\mathbf{B}$ is a countably infinite Boolean algebra, then the Boolean power $\mathbf{A}[\mathbf{B}]^* \cong \mathbf{F}$ by $\aleph_0$–categoricity. But $\mathbf{A}$ and $\mathbf{A}[\mathbf{B}]^*$ generate the same quasivariety. Hence all finitely generated members of $\mathcal{Q}$ generate the same subquasivariety; this proves that $\mathcal{Q}$ is minimal. $\square$

We now fix notation we will use through the rest of this paper. $\mathcal{Q}$ denotes an $\aleph_0$–categorical quasivariety. $\mathbf{F}$ denotes a free algebra of $\mathcal{Q}$ generated by a countably infinite set. $\mathbf{A}$ denotes a nontrivial member of $\mathcal{Q}$ which has no nontrivial proper subalgebras. The choices of $\mathbf{F}$ and $\mathbf{A}$ are unique up to isomorphism — for $\mathbf{F}$, by the categoricity of $\mathcal{Q}$; for $\mathbf{A}$, by the minimality and local finiteness of $\mathcal{Q}$. In particular, $\mathbf{A}$ must be the smallest nontrivial algebra in $\mathcal{Q}$.

The following is the only fragment of the Ryll-Nardzewski Theorem that we will need.

Theorem 5.2. *The permutation group $\langle F; \mathrm{Aut}\,(\mathbf{F})\rangle$ has finitely many orbits.*

Proof. Let $\mathbf{B}$ be the countable atomless Boolean algebra. It is well known and easy to prove that any isomorphism between finite subalgebras of $\mathbf{B}$ extends to an automorphism of $\mathbf{B}$. (See Example 4 of Section 3.2 of [6].) This implies that if $f, g \in \mathbf{A}[\mathbf{B}]^*$ are continuous functions from the Stone space of $\mathbf{B}$ to $\mathbf{A}$ which have the same range, then there is an automorphism of $\mathbf{A}[\mathbf{B}]^*$ which maps f to g. But there are only finitely many possible ranges of functions into $\mathbf{A}$ since $\mathbf{A}$ is finite. It follows that $\mathbf{A}[\mathbf{B}]^*$ has finitely many orbits under its automorphism group. Since $\mathbf{F} \cong \mathbf{A}[\mathbf{B}]^*$ by categoricity, we are done. $\square$

Lemma 5.3. *Every nonconstant unary term of $\mathcal{Q}$ is invertible.*

Proof. Choose a unary term f and let $X = \{x_1, x_2, \dots\}$ be a set of free generators for $\mathbf{F}$. If the term operation associated to f is not an injective function from X into F, then $f(x_i) = f(x_j)$ for some $i \neq j$. This is an equation of $\mathcal{Q}$ which forces f to be constant. Therefore, if f is not constant, then $f(F)$ is infinite.

Assume that $f(F)$ is infinite and let $\mathbf{F}'$ be the subalgebra of $\mathbf{F}$ generated by $f(F)$. The algebra $\mathbf{F}'$ is a characteristic subalgebra of $\mathbf{F}$ which, by categoricity, is isomorphic to $\mathbf{F}$. As a first case we assume that $\mathbf{F}' \neq \mathbf{F}$. If so, then since $\mathbf{F}' \cong \mathbf{F}$ we get that $\mathbf{F}'$ also has a proper characteristic subalgebra $\mathbf{F}''$. In fact we get a proper descending sequence of characteristic subalgebras $\mathbf{F} \supset \mathbf{F}' \supset \mathbf{F}'' \supset \cdots$. Since this is a descending chain of characteristic subalgebras, each subset $F^{(i)} - F^{(i+1)}$ is a nonempty union of $\mathrm{Aut}\,(\mathbf{F})$–orbits. However, there are only finitely many $\mathrm{Aut}\,(\mathbf{F})$–orbits altogether by Theorem 5.2, so the case where $\mathbf{F}' \neq \mathbf{F}$ cannot occur.

It must be that $\mathbf{F}$ is generated by $f(F)$. Since $x_1 \in F = \mathrm{Sg}^{\mathbf{F}}(f(F))$, there is a term σ of arity m and m terms t_i of arity n, for some m and n, such that

$$\sigma(ft_1(\bar{x}), \dots, ft_m(\bar{x})) = x_1.$$

Applying the endomorphism of $\mathbf{F}$ determined by mapping all free generators to x_1 we obtain

$$\sigma(f\tau_1(x_1), \dots, f\tau_m(x_1)) = x_1.$$

where $\tau_i(x) := t_i(x, x, \dots, x)$. This equality in $\mathbf{F}$ implies that

$$\mathcal{Q} \models \sigma(f\tau_1(x), \dots, f\tau_m(x)) = x.$$

Hence f is invertible. □

Choose a nonconstant (invertible) idempotent e of $\mathcal{Q}$ which minimizes $|e(A)|$. This minimality condition implies that $e(\mathbf{A})$ is term minimal. Lemma 5.3 applied to $\mathcal{Q}$ shows that $e : \mathcal{Q} \to e(\mathcal{Q})$ is a Morita equivalence, so $e(\mathcal{Q}) = \mathsf{SP}(e(\mathbf{A}))$ is $\aleph_0$–categorical. Now we may apply Lemma 5.3 to $e(\mathcal{Q})$ to deduce that all nonconstant unary terms of this quasivariety are invertible in the sense of equation 2.1. By the first remark in Theorem 3.4, $e(\mathbf{A})$ (and hence $e(\mathcal{Q})$) is permutational. Replacing $\mathcal{Q}$, $\mathbf{A}$ and $\mathbf{F}$ with $e(\mathcal{Q})$, $e(\mathbf{A})$ and $e(\mathbf{F})$ we henceforth assume that $\mathcal{Q}$ is a permutational quasivariety.

Lemma 5.4. $\mathbf{A}$ *is*

(*i*) *an idempotent simple algebra,*
(*ii*) *the constant expansion of a minimal algebra,*
(*iii*) *a G–algebra, or*
(*iv*) *a G^0–algebra.*

Proof. This statement follows from Theorem 3.4, provided that we can show in (*iii*) (and (*iv*)) that $\mathbf{A}$ is a G–algebra (G^0–algebra), rather than just a quotient of a G–algebra (G^0–algebra).

In both of the cases $\mathbf{F}_{\mathcal{Q}}(1)$ is a G–algebra (G^0–algebra) with no proper nontrivial subalgebras, and so $\mathbf{F}_{\mathcal{Q}}(1)$ satisfies the defining property for $\mathbf{A}$. Hence, $\mathbf{A}$ ($\cong \mathbf{F}_{\mathcal{Q}}(1)$) is a G–algebra (G^0–algebra). □

We need to rule out the possibility that $\mathbf{A}$ is a nonabelian simple algebra in Cases (*i*) and (*ii*) of Lemma 5.4. Let $\mathrm{Spec}_{\mathbf{C}}(\lambda)$ denote the function $\mathrm{Spec}_T(\lambda)$ where T is the theory of $\mathsf{SP}(\mathbf{C})$.

Lemma 5.5. *If* $\mathbf{C}$ *is finite, simple and nonabelian, then* $\mathrm{Spec}_{\mathbf{C}}(\lambda) = 2^{\lambda}$ *for all infinite* λ.

Proof. Since $\mathbf{C}$ is finite $\mathrm{Spec}_{\mathbf{C}}(\lambda) \leq 2^{\lambda}$, so we only need to prove the reverse inequality. Also, it suffices to prove this result for the expansion of $\mathbf{C}$ by constants. To see that this is so, there are two things to note: first, expanding $\mathbf{C}$ doesn't affect finiteness, simplicity or nonabelianness. Second, the reduct functor from $\mathsf{SP}(\mathbf{C}_C)$ to $\mathsf{SP}(\mathbf{C})$ is cardinality preserving on objects and is at most a λ–to–1 mapping among the algebras of power $\lambda \geq \aleph_0$. Hence, $\mathrm{Spec}_{\mathbf{C}}(\lambda)+\lambda = \mathrm{Spec}_{\mathbf{C}_C}(\lambda)+\lambda$ for infinite λ. It follows that $\mathrm{Spec}_{\mathbf{C}}(\lambda) = 2^{\lambda}$ iff $\mathrm{Spec}_{\mathbf{C}_C}(\lambda) = 2^{\lambda}$. We assume now that every polynomial of $\mathbf{C}$ is a term.

Our argument will be to prove that $\mathbf{C}$ is cancellable in Boolean powers (or *B-seperating*). Since $\mathbf{C}$ is nonabelian, its tame congruence theoretic type label is **3** , **4** , or **5** . There is an idempotent unary polynomial (= term) e such that $e(\mathbf{C})$ is weakly isomorphic to the two–element Boolean algebra, the two–element bounded lattice or the two–element bounded semilattice. In the type **3** case (Boolean type), we can see that

$$\mathbf{C}[\mathbf{B}]^* \cong \mathbf{C}[\mathbf{B}']^* \Longrightarrow \mathbf{B} \cong \mathbf{B}'$$

as follows. If $\mathbf{C}[\mathbf{B}]^* \cong \mathbf{C}[\mathbf{B}']^*$, then

$$\mathbf{B} \cong_w e(\mathbf{C})[\mathbf{B}]^* \cong_w e(\mathbf{C}[\mathbf{B}]^*) \cong e(\mathbf{C}[\mathbf{B}']^*) \cong_w e(\mathbf{C})[\mathbf{B}']^* \cong_w \mathbf{B}'.$$

But the only weak isomorphisms between Boolean algebras are isomorphisms or dual isomorphisms. Hence, $\mathbf{B} \cong \mathbf{B}'$ and $\mathbf{C}$ is cancellable in Boolean powers. It follows that $\mathrm{Spec}_{\mathbf{C}}(\lambda) \geq 2^{\lambda}$, since the latter function is the spectrum function for Boolean algebras.

The type **4** and **5** cases are handled in the same way, since the Boolean operations are definable from the operations of a (Boolean–ordered) bounded lattice or semilattice. □

A more complicated argument is given in [2] to prove the stronger result that $\mathbf{C}$ is cancellable in Boolean powers if it is nonabelian and subdirectly irreducible. Some of the arguments that follow can be shortened if one uses this result.

Theorem 5.6. *The algebra* $\mathbf{A}$ *is an idempotent simple algebra if and only if it is weakly isomorphic to*

(i) *a two–element set (considered as an algebra with no operations) or*

(ii) *a simple affine module over a simple ring.*

Proof. Assume that $\mathbf{A}$ is an idempotent simple algebra. By Lemma 5.5, $\mathbf{A}$ is abelian. But it follows from [18] or [7] that an idempotent simple algebra which is both finite and abelian is weakly isomorphic to a two–element set or to a simple affine module over a simple ring.

To prove the converse, we must show that a quasivariety generated by a two–element set or an affine module over a simple ring is $\aleph_0$–categorical. The argument is trivial for the two–element set. Assume that $\mathbf{B}$ is a simple affine module over the simple ring, $\mathbf{R}$. Let $\mathbf{H}$ and $\mathbf{H}'$ be any two members of $\mathsf{SP}(\mathbf{B})$ which have the same cardinality. If we choose arbitrary $0 \in H$ and $0' \in H'$ and form the one–point expansions $\mathbf{H}^*$ and $\mathbf{H}'^*$, we get a pair of algebras which are definitionally equivalent to $\mathbf{R}$–modules and have the same cardinality. Note that (a) the variety of $\mathbf{R}$–modules is categorical in every power for which there is a model (because $\mathbf{R}$ is Morita equivalent, in the classical sense, to a finite field) and (b) the definitions used to convert $\mathbf{H}^*$ and $\mathbf{H}'^*$ into $\mathbf{R}$–modules are the same. From (a) and (b), we conclude that $\mathbf{H}^* \cong \mathbf{H}'^*$. This proves that $\mathbf{H} \cong \mathbf{H}'$. □

Theorem 5.7. *The algebra* $\mathbf{A}$ *is the constant expansion of a minimal algebra if and only if it is weakly isomorphic to*

(i) *the constant expansion of a nontrivial finite* G*–set where every* $g \in G - \{1\}$ *has at most one fixed point or*

(ii) *the constant expansion of a nontrivial finite vector space.*

Proof. Assume that $\mathbf{A}$ is a permutational algebra which is the constant expansion of a minimal algebra. By Lemma 5.5, $\mathbf{A}$ is not a simple nonabelian algebra. By the main result of [13] (or by Corollary 4.11 of [5]) $\mathbf{A}$ is weakly isomorphic to the constant expansion of a finite G–set or to a finite vector space. To prove the forward direction of the theorem, we only need to consider the case where $\mathbf{A}$ is the constant expansion of a finite G–set, $\langle G; A\rangle$, where G acts faithfully. In this case we must show that every $g \in G - \{1\}$ has at most one fixed point.

For a subset $X \subseteq A$ we write $\mathrm{Stab}\,(X)$ to denote the subgroup of G which fixes X pointwise. The statement that G acts faithfully on A is equivalent to $\mathrm{Stab}\,(A) = \{1\}$. When κ is an ordinal, we define a κ–*termed sequence* of elements of A to be a function from κ to A. A *sequence* is an ω–termed sequence. We may write a sequence f as $a_0a_1a_2\cdots$ to mean that $f(i) = a_i$.

Choose $a \neq b \in A$. Let $\mathbf{C}$ be a subalgebra of $\mathbf{A}^\omega$ generated by countably infinitely many $\{a, b\}$–valued sequences. Since $\mathbf{A}$ is a constant expansion of a G–set, to describe $\mathbf{C}$ it will suffice to say how many orbits of each kind $\mathbf{C}$ has and how the constants are to be interpreted. $\mathbf{C}$ will have a subalgebra consisting of the interpretations of constant terms, and this subalgebra will be isomorphic to $\mathbf{A}$. This is the subalgebra of constant sequences. Any orbit not contained in this subalgebra is the orbit of some $\{a, b\}$–valued sequence which is not a constant sequence. Such an orbit is a G–set isomorphic to the G–set of left translations of cosets of $\mathrm{Stab}\,(\{a, b\})$ in G. This orbit contains no element which is the image of a constant term. Thus, $\mathbf{C}$ is (isomorphic to) the disjoint union of $\mathbf{A}$ and countably infinitely many copies of $G/\mathrm{Stab}\,(\{a, b\})$ considered as a left G–set. Let $\mathbf{D}$ be a subalgebra of $\mathbf{A}^\omega$ generated by countably infinitely many sequences which, considered as functions, are surjective or constant. Arguing as with $\mathbf{C}$, $\mathbf{D}$ is the disjoint union of $\mathbf{A}$ and countably infinitely many copies of $G/\mathrm{Stab}\,(A)$ ($\cong G$) as a left G–set. By categoricity, orbits in $\mathbf{C}$ must match orbits in $\mathbf{D}$, so $\mathrm{Stab}\,(\{a, b\}) = \mathrm{Stab}\,(A) = \{1\}$ whenever $a \neq b$ in A. This is equivalent to the assertion that each $g \in G - \{1\}$ has at most one fixed point.

To prove categoricity for Case (i), assume that $\mathbf{B}$ is as described in (i), $0 < \kappa$ and $\mathbf{C} \leq \mathbf{B}^\kappa$ is nontrivial. Any κ–termed sequence $f \in C$ will generate a subalgebra isomorphic to $\mathbf{B}$ if f is constant, otherwise it will generate an orbit which is isomorphic to the G–set $G/\mathrm{Stab}\,(f(\kappa))$ ($\cong G$) under left translations. All constant κ–termed sequences have range contained in the orbit isomorphic to $\mathbf{B}$. Hence, any member of $\mathsf{SP}(\mathbf{B})$ is isomorphic to the disjoint union of $\mathbf{B}$ and some number of copies of G considered as a left G–set. $\mathrm{Spec}_{\mathbf{B}}(\lambda) \leq 1$ for all λ in this situation.

If $\mathbf{B}$ is as in Case (ii), let $\mathcal{V}$ be the subvariety of $\mathsf{HSP}(\mathbf{B})$ defined by equations that assert that all constants are equal. $\mathcal{V}$ is definitionally equivalent to the variety of vector spaces over the same field as $\mathbf{B}$, so $\mathcal{V}$ is totally categorical. Any nontrivial $\mathbf{C} \leq \mathbf{B}^\kappa$ contains the constant κ–termed sequences, hence $\mathbf{C}$ has a subuniverse U consisting of these constants which generates a subalgebra isomorphic to $\mathbf{B}$. U is closed under the vector space operations, so U is a class of some congruence θ. For any vector space congruence θ' which complements θ in $\mathrm{Con}\,(\mathbf{C})$, $\mathbf{C}/\theta'$ is isomorphic to $\mathbf{B}$ since U is a transversal for θ'. For any $\mathbf{C} \in \mathsf{SP}(\mathbf{B})$ we have $\mathbf{C} \cong \mathbf{B} \times (\mathbf{C}/\theta)$ where $\mathbf{C}/\theta \in \mathcal{V}$. Since $\mathcal{V}$ is totally categorical, $\mathbf{C}$ is determined up to isomorphism by $|C|$, so $\mathrm{Spec}_{\mathbf{B}}(\lambda) \leq 1$ for all λ. □

Theorem 5.8. *The algebra* $\mathbf{A}$ *is a* G*–algebra if and only if it is weakly isomorphic to*

(*i*) $\langle G; \lambda_g, g \in G\rangle$ *(i.e., to the left–regular representation of* G*) or*
(*ii*) *the expansion by translations of a finite affine module over a simple ring.*

Proof. Our strategy will be to reduce this case to Case (*ii*) of Lemma 5.4. The following fact is the basis for the reduction.

Claim 5.9. $\mathrm{Aut}\,(\mathbf{F})$ *acts transitively on* F.

Proof. A sequence $f : \omega \to S$ is *almost constant* if for some $s \in S$ we have $f(x) = s$ for all but finitely many $x \in \omega$. Assume that $\mathbf{A}$ is a G–algebra. Let $\mathbf{C}$ be the subalgebra of $\mathbf{A}^{\omega}$ whose universe is the set of almost constant sequences. Since $\mathbf{C}$ is countably infinite, $\mathbf{C} \cong \mathbf{F}$. Any almost constant sequence of elements of $\mathrm{Aut}\,(\mathbf{A})$, acting coordinatewise, is an automorphism of $\mathbf{C}$. The collection of such automorphisms acts transitively on C, since $\mathrm{Aut}\,(\mathbf{A})$ acts transitively on A. □

Let $\mathbf{A}^*$ be the expansion of $\mathbf{A}$ by a single constant and let $\mathcal{Q}^* = \mathsf{SP}(\mathbf{A}^*)$. The countably infinite members of $\mathcal{Q}^*$ can all be obtained from countably infinite members of $\mathcal{Q}$ by adjoining one constant. Any two such algebras are isomorphic since $\mathcal{Q}$ is $\aleph_0$-categorical and, according to Claim 5.9, the countably infinite member of $\mathcal{Q}$ has a transitive automorphism group. Therefore $\mathcal{Q}^*$ is $\aleph_0$-categorical. Choose an idempotent unary term e^* which minimizes $|e^*(A^*)|$. Then, as argued before Lemma 5.4, $e^*(\mathbf{A}^*)$ is a permutational algebra which generates the $\aleph_0$-categorical quasivariety $e^*(\mathcal{Q}^*)$. Moreover, $e^*(\mathbf{A}^*)$ falls into Case (*ii*) of Lemma 5.4 since we obtained $\mathbf{A}^*$ by adding a constant to $\mathbf{A}$ and the transitive action of the unary terms of $\mathbf{A}$ guarantee that every element of $\mathbf{A}$ is named by a constant in $\mathbf{A}^*$. Therefore we are in a position to use Theorem 5.7: we know that $e^*(\mathbf{A}^*)$ is polynomially equivalent to a G-set or it is affine.

Claim 5.10. *If* $e^*(\mathbf{A}^*)$ *is polynomially equivalent to a* G*–set, then* $\mathbf{A}$ *is polynomially equivalent to a* G*–set. If* $e^*(\mathbf{A}^*)$ *is affine, then* $\mathbf{A}$ *is affine.*

Proof. Lemma 3.5 implies that $\mathbf{A}^* \cong_w e^*(\mathbf{A}^*)^{[k]}$ for some k. All quasi-identities of the form

$$s(\bar{x}, \bar{u}) = s(\bar{y}, \bar{v}) \Longrightarrow s(\bar{z}, \bar{u}) = s(\bar{z}, \bar{v})$$

are true of unary algebras and the satisfaction of all such quasi-identities is preserved by the formation of $[k]$–th powers, weak isomorphic images and reducts. It follows that $\mathbf{A}$ satisfies all such quasi-identities. (An algebra satisfying all quasi-identities of this form is said to be *strongly abelian.*) The fact that $\mathbf{A}$ is strongly abelian implies that every surjective polynomial of $\mathbf{A}$, $p : A^m \to A$, equals a term operation. For if $p(\bar{x}) = s(\bar{x}, \bar{a})$ with $\bar{a} \in A^n$, and $s(\bar{c}, \bar{a}) \neq s(\bar{c}, \bar{b})$ for some $\bar{c} \in A^m$, $\bar{b} \in A^n$, then from the surjectivity of $s(\bar{x}, \bar{a})$ we can find $\bar{d} \in A^m$ such that $s(\bar{d}, \bar{a}) = s(\bar{c}, \bar{b})$. Applying the displayed quasi-identity, we get $s(\bar{z}, \bar{a}) = s(\bar{z}, \bar{b})$ for any $\bar{z} \in A^m$. This is a contradiction for $\bar{z} = \bar{c}$. We conclude that $s(\bar{x}, \bar{u})$ is independent of $\bar{u}$ when $s(\bar{x}, \bar{a})$ is surjective, so the polynomial $p(x_1, \ldots, x_m)$ equals the term operation $s(x_1, \ldots, x_m, x_1, \ldots, x_1)$.

$\mathbf{A}^* \cong_w e^*(\mathbf{A}^*)^{[k]}$, so the algebra $\mathbf{A}^*$ (and hence $\mathbf{A}$) has a unary polynomial $b(x)$ whose $[k]$–th power representation is $\langle p_2, p_3, \ldots, p_k, p_1\rangle$ where $p_i(\bar{x}) \in \mathrm{Clo}_k(e^*(\mathbf{A}^*))$ is projection onto the i–th variable. The operation $b(x)$ cyclically permutes the coordinates of any k–tuple. $\mathbf{A}^*$ (hence $\mathbf{A}$) has a k–ary polynomial $d(\bar{x})$ whose $[k]$–th power representation is $\langle p_{11}, p_{22}, \ldots, p_{kk}\rangle$ where $p_{ii}(\bar{x}) \in \mathrm{Clo}_{k^2}(e^*(\mathbf{A}^*))$ is projection onto variable ii, viewing the k^2 variables as arranged in a $k \times k$ array. Since b and d are both surjective, they are term operations of $\mathbf{A}$. Hence, $E(x) := d(x, b^{k-1}(x), \ldots, b(x))$ is a unary term operation of $\mathbf{A}$, and it maps a k–tuple $(a_1, \ldots, a_k) \in A^k$ to $(a_1, \ldots, a_1)$. This term operation must be a permutation or a constant, since $\mathbf{A}$ is permutational, so we conclude that $|A| = 1$ (in which case the claim holds trivially) or else $k = 1$. In the latter case, $\mathbf{A}^* \equiv e^*(\mathbf{A}^*)$ and we get that $\mathbf{A}^*$ (hence $\mathbf{A}$) is polynomially equivalent to a G–set.

The last remark of the claim follows from the fact that affineness is preserved by Morita equivalence and by polynomial equivalence. □

Now we continue the proof of the theorem. Claim 5.10 implies that $\mathbf{A}$ is polynomially equivalent to a G–set or affine. In the first case $\mathbf{A}$ is an essentially unary G–algebra, so it can only be $\langle G; \lambda_g, g \in G\rangle$. In the second case, $\mathbf{A}$ is affine over a ring Morita equivalent (in the classical sense) to a finite field. Since $\mathbf{A}$ is a G–algebra, Theorem 4.2 proves that $\mathbf{A}$ is weakly isomorphic to the expansion by translations of a finite affine $\mathbf{R}$–module where $\mathbf{R}$ is a finite simple ring.

Using an argument similar to that used in the proof of Theorem 5.7 it is easy to see that the quasivariety generated by $\mathbf{B} = \langle G; \lambda_g, g \in G\rangle$ is categorical in every power in which it has a model; the models are disjoint unions of copies of $\mathbf{B}$. So what remains to be proven is that the algebras described in Case (ii) generate categorical quasivarieties.

Let $\mathbf{C}$ be a nontrivial algebra in $\mathcal{P} = \mathsf{SP}(\mathbf{B})$ where $\mathbf{B}$ is the expansion by translations of a finite affine module over a finite simple ring $\mathbf{R}$. Choose $0 \in C$ arbitrarily and let $\widehat{\mathbf{C}}$ denote the $\mathbf{R}$–module which shares the universe and idempotent operations of $\mathbf{C}$, but has 0 as a constant operation. Let G denote the group of translations of $\mathbf{B}$. Since $\mathbf{B}$ is a G–algebra, every nontrivial one–generated algebra in $\mathcal{P}$ is isomorphic to $\mathbf{B}$. In particular, the subalgebra of $\mathbf{C}$ generated by 0 is isomorphic to $\mathbf{B}$; this subalgebra corresponds to a submodule of $\widehat{\mathbf{C}}$ which we denote $\mathbf{M}$. The fact that $\mathcal{P} \models x - y + (g(y)) = g(x)$ for each $g \in G$ implies that the G–orbits in $\mathbf{C}$ are precisely the cosets of $\mathbf{M}$. $\mathbf{R}$ is a finite simple ring, so $\widehat{\mathbf{C}}$ has a submodule $\mathbf{N}$ which complements $\mathbf{M}$. If θ and θ' are the congruences of $\widehat{\mathbf{C}}$ corresponding to $\mathbf{M}$ and $\mathbf{N}$, then $\mathbf{C}/\theta$ is idempotent (since θ–classes equal G–orbits) and $\mathbf{C}/\theta' \cong \mathbf{B}$ (since the subalgebra generated by $0 \in C$ is isomorphic to $\mathbf{B}$ and is a transversal for θ'). We conclude that $\mathbf{C} \cong \mathbf{B} \times (\mathbf{C}/\theta)$ where $(\mathbf{C}/\theta)$ is an affine $\mathbf{R}$–module. Now (total) categoricity of $\mathcal{P}$ is a corollary of the (total) categoricity of any quasivariety of affine modules over a finite simple ring, which we proved in Theorem 5.6 (*ii*). □

Theorem 5.11. *The algebra $\mathbf{A}$ is a G^0–algebra if and only if it is weakly isomorphic to*

(*i*) $\langle G \cup \{0\}; 0, \lambda_g, g \in G\rangle$ *or*
(*ii*) *a finite one–dimensional vector space.*

Proof. By simplifying arguments of the previous proof, one sees that the algebras of types (*i*) and (*ii*) generate totally categorical quasivarieties. To prove the other direction we first argue that $[1, 1] = 0$ in $\mathrm{Con}\,(\mathbf{A})$. Since $[1, 1]$ is a fully invariant congruence, it is worth saying a few words now about fully invariant congruences on G^0–algebras.

If H is a subgroup of G, then the relation θ_H defined by $(a, b) \in \theta_H$ iff $a = b = 0$ or a and b lie in the same left coset of H is an equivalence relation on $\mathbf{A}$. Any congruence θ on $\mathbf{A}$ is compatible with all λ_g, and this forces either $\theta = A \times A$ or else $\theta = \theta_H$ for some $H < G$. Any fully invariant congruence θ on $\mathbf{A}$ is compatible with all ρ_g, and this forces either $\theta = A \times A$ or else $\theta = \theta_M$ for some normal subgroup $M \lhd G$. An algebra of the form $\mathbf{A}/\theta_M$ with $M \lhd G$ is a G^0–algebra with group equal to G/M. Since $\{0\}$ is a congruence class of every θ_H, it follows that there is a largest proper congruence θ_N on any G^0–algebra. Since any nonconstant endomorphism of a G^0–algebra is an automorphism, the largest proper congruence θ_N is fully invariant (so $N \lhd G$).

Claim 5.12. $[1, 1] < 1$.

Proof. Fix $N \lhd G$ such that θ_N is the largest proper congruence on $\mathbf{A}$. If the claim is false, then $\mathbf{A}/\theta_N$ is a nonabelian simple G^0–algebra with respect to the group G/N. According to the structure theorem for nonabelian simple G^0–algebras (in [20]) there is a binary term $x \wedge y$ which is a meet semilattice operation on $\mathbf{A}/\theta_N$. The semilattice order is that of a height–one semilattice with bottom element $0^{A/\theta_N}$. Choose $1 \in A/\theta_N - \{0^{A/\theta_N}\}$; the polynomial $E(x) = 1 \wedge x$ is an idempotent polynomial with range $\{0^{A/\theta_N}, 1\}$, and $x \wedge y$ restricts to this set to be a meet semilattice operation with respect to the order $0^{A/\theta_N} < 1$.

$\mathbf{A}$ has an idempotent polynomial $e(x)$ such that $e(x)/\theta_N = E(x)$ and $e(A) = U$ is a $\langle \theta_N, 1\rangle$–minimal set. U has two θ_N classes O and I such that $O/\theta_N = 0^{G/N}, I/\theta_N = 1$. According to Lemmas 4.15 and 4.17 of [5], I is a one–element set. Since O is a $\theta_N|_U$–class that contains 0^A, and $\{0^A\}$ is a θ_N–class, we also have that O has size one. Thus $|U| = 2$.

Now we are in a position to repeat the argument from the proof of Lemma 5.5: we have an algebra $\mathbf{A}$ which has a unary polynomial $e(x)$ such that $U = e(A)$ has size two, and $\mathbf{A}$ has a polynomial that restricts to U to be a semilattice operation. The conclusion is that $\mathrm{Spec}_{\mathbf{A}}(\lambda) = 2^\lambda$ for all infinite λ, which contradicts categoricity. □

Claim 5.13. *If* $\mathbf{A}/[1,1]$ *is not essentually unary, then* $[1,1] = 0$.

Proof. Since $[1,1]$ is a proper fully invariant congruence it has the form θ_N for some $N \lhd G$, and $\mathbf{A}/[1,1]$ is a G^0–algebra with respect to G/N. By Theorem 4.3 we must have that $\mathbf{A}/[1,1]$ is equivalent to a vector space. Therefore there is a binary term s' such that $s'(x,y) \equiv x - y \pmod{\theta_N}$. If $\ell(x) = s'(x,0) \equiv x - 0 = x \pmod{\theta_N}$, then $\ell(x)$ is not constant. Since $\mathbf{A}$ is a G^0–algebra, it must be that the term ℓ has an inverse. If $s(x,y) = s'(\ell^{-1}(x), y)$, then $\mathbf{A} \models s(x,0) = x$ and $s(x,y) \equiv x - y \pmod{\theta_N}$. Observe that if $a \equiv b \pmod{\theta_N}$ we have that $s(a,b) \equiv s(a,a) \equiv 0 \pmod{\theta_N}$. But since $\{0\}$ is a θ_N–class, this means that $s(a,b) = 0$ in $\mathbf{A}$.

Now we construct two nonisomorphic countable algebras in $\mathsf{SP}(\mathbf{A})$. Let $\mathbf{B}$ countably infinite Boolean algebra. Let $\mathbf{C} = \mathbf{A}[\mathbf{B}]^*$ viewed as the algebra of continuous functions from the Stone space of $\mathbf{B}$ to A given the discrete topology. Let θ denote the congruence on $\mathbf{C}$ which relates two functions in $\mathbf{C}$ if they are $[1,1]$-related pointwise. Let $\mathbf{D}$ be the subalgebra of $\mathbf{C}$ consisting of functions which are θ–related to a constant function. If $[1,1] > 0$, then $\mathbf{D}$ is infinite. Since $\mathbf{C}$ is countably infinite and $\mathsf{SP}(\mathbf{A})$ is $\aleph_0$–categorical, to prove the claim it is enough to prove that $\mathbf{C} \ncong \mathbf{D}$.

Since $\mathbf{C}/\theta = (\mathbf{A}/[1,1])[\mathbf{B}]^*$ is an infinite dimensional vector space, it follows that the universal congruence $C \times C$ on $\mathbf{C}$ is not compact.

However, the universal congruence on $\mathbf{D}$ is principal! To see this, choose any $a \in A - \{0\}$. Let $\widehat{a} \in D$ be the function with range $\{a\}$ and let $\widehat{0}$ be the function with range $\{0\}$. To finish the proof of the claim we show that for any $d \in D$ we have $(\widehat{0}, d) \in \mathrm{Cg}\,(\widehat{0}, \widehat{a})$. Choose a constant function $\widehat{b}$ such that $d \equiv \widehat{b} \pmod{\theta}$. Without loss of generality $d \neq \widehat{0}$, so $b \neq 0$ and there is a λ_g such that $\lambda_g(a) = b$. Now

$$d = s(d, \widehat{0}) = s(d, \lambda_g(\widehat{0}))\ \mathrm{Cg}\,(\widehat{0}, \widehat{a})\ s(d, \lambda_g(\widehat{a})) = s(d, \widehat{b}) = \widehat{0}.$$

This proves that $\mathrm{Cg}\,(\widehat{0}, \widehat{a}) = D \times D$, which completes the proof of the claim. □

Claim 5.14. *If* $\mathbf{A}/[1, 1]$ *is essentually unary, then* $[1, 1] = 0$.

Proof. Assume that $[1, 1] < 1$ and that $\mathbf{A}/[1, 1]$ is essentially unary. Choose $1 \in A - \{0\}$. Let $\mathbf{B}$ be a countable atomless Boolean algebra. The algebra $\mathbf{X} = \mathbf{A}[\mathbf{B}]^*$ is countably infinite and has a wealth of automorphisms. Let $G_0 = (\mathrm{Aut}(\mathbf{A}))[\mathbf{B}]^*$ acting pointwise on $\mathbf{X} = \mathbf{A}[\mathbf{B}]^*$. Elements of G_0 suffice to show that any element of X is in the same $\mathrm{Aut}(\mathbf{X})$–orbit as a 0,1–sequence. Let G_1 be the group consisting of automorphisms of $\mathbf{X}$ induced by automorphisms of $\mathbf{B}$ acting on the points of its Stone space. Elements of G_1 suffice to show that any two 0,1–sequences in which both 0 and 1 appear belong to the same $\mathrm{Aut}(\mathbf{X})$–orbit. (This uses the fact that $\{\text{bottom element}\} \cup \{\text{middle elements}\} \cup \{\text{top element}\}$ is a partition of B into $\mathrm{Aut}(\mathbf{B})$–orbits.) So, if we let $S_0 = \{0^X\}$, $S_1 =$ the set of functions which do not have 0 in their range, and $S_{0,1} =$ the rest of $X =$ the set of nonconstant functions that have 0 in their range, then we see that $X = S_1 \cup S_{0,1} \cup S_1$ is a partition of X into $G_0 \vee G_1$–orbits.

If $f(x_1, \ldots, x_n)$ depends only on its first variable modulo $[1, 1]$ in $\mathbf{A}$ and we apply it to elements $\mathbf{a}_i \in X = A[\mathbf{B}]^*$, then $\mathbf{a} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a function in $\mathbf{A}[\mathbf{B}]^*$ which is zero wherever $\mathbf{a}_1$ is zero. (This depends on the fact that $\{0\}$ is a $[1, 1]$–class in $\mathbf{A}$.) Consequently the set $Y = S_0 \cup S_{0,1}$ is a (countably infinite) subuniverse of $\mathbf{X}$. Since all elements of $G_0 \vee G_1$ can be restricted to automorphisms of $\mathbf{Y}$, there are only two $\mathrm{Aut}(\mathbf{Y})$–orbits: $\{0\}$ and $Y - \{0\}$. This property is shared by $\mathbf{F} \cong \mathbf{Y}$. Now we are in a position to imitate the proof of Theorem 5.8.

If we expand $\mathbf{A}$ to $\mathbf{A}^*$ by adding one nonzero constant, then $\mathsf{SP}(\mathbf{A}^*)$ is categorical and it is Morita equivalent to a quasivariety of the type classified in Theorem 5.7. From this it follows that $\mathbf{A}$ is abelian, so $[1, 1] = 0$. □

We know that $[1, 1] = 0$, or that $\mathbf{A}$ is abelian, so Theorem 4.3 completes the proof. □

If one is satisfied with a description of $\aleph_0$–categorical quasivarieties up to Morita equivalence, then the combination of Theorems 5.6, 5.7, 5.8, 5.11 provides such a description. (More specifically, the theorem statements describe the minimal generator of the quasivariety while the theorem proofs describe the other models.) We now reformulate those results in ways that avoid mentioning Morita equivalence.

Theorem 5.15. *A quasivariety is $\aleph_0$–categorical if and only if it is minimal, locally finite and*

(i) *definitionally equivalent to some $\mathcal{Q}^{[k]}$ where $\mathcal{Q}$ is a quasivariety of G–sets, possibly with constants adjoined, where every non-identity permutation has at most one fixed point, or*

(ii) *an affine quasivariety over a simple ring.*

Proof. Case (i) follows directly from Theorems 5.6, 5.7, 5.8 and 5.11, using Corollary 3.5 to understand Morita equivalence.

In Case (ii) of this theorem, we have already determined that the Case (ii) quasivarieties in Theorems 5.6, 5.7, 5.8 and 5.11 are minimal, locally finite and affine over a finite simple ring. These properties for a quasivariety are Morita invariants. The only thing we have to show, therefore, is that any minimal, locally finite, affine quasivariety over a simple ring is $\aleph_0$–categorical. In fact, we can finish the proof by doing less; it will suffice to prove that each nonconstant unary term of such a quasivariety is invertible in the sense of equation 2.1. For then we can argue, as we did before Lemma 5.4, that any such quasivariety is Morita equivalent to a permutational quasivariety. The permutational, minimal, locally finite, affine quasivarieties over simple rings are precisely the Case (ii) quasivarieties that appear in Theorems 5.6, 5.7, 5.8 and 5.11, and in all four theorems we proved these quasivarieties to be categorical.

Let $\mathcal{Q}$ be a minimal, locally finite, affine quasivariety over a simple ring $\mathbf{R}$. Let $\mathbf{P} = \mathbf{F}_{\mathcal{Q}}(x)$ be the free algebra on $\{x\}$. Assume that $\mathbf{P}$ is polynomially equivalent to the module $\mathbf{M}$. If $f(x)$ is a nonconstant unary term of $\mathcal{Q}$, then f is nonconstant on $\mathbf{P}$. We may choose an affine representation for $\mathbf{P}$ where $0 \in f(P)$, and with respect to this representation f is representable in the form $f(x) = r(x) + m$ for some nonzero $r \in R$. If $f(P)$ is a generating set for $\mathbf{P}$, then by the argument used in the last paragraph of Lemma 5.3 f is invertible and we are done. So assume that $f(P)$ generates a proper subalgebra $\mathbf{P}'$ of $\mathbf{P}$. We arranged the representation so that $0 \in P'$, so P' is a proper submodule of $\mathbf{M}$. Now the quotient module $\mathbf{M}/P'$ has the property that it is a nontrivial $\mathbf{R}$-module for which the polynomial $r(x) + \bar{m}$ is identically zero. But this is impossible if $\mathbf{R}$ is simple. For if $r(x) + \bar{m} \equiv 0$, then

$\bar{m} = r(0) + \bar{m} = 0$ and so $r(x) = r(x) + \bar{m} \equiv 0$. That is, the nonzero element $r \in R$ acts like zero on the nontrivial module $\mathbf{M}/P'$. The same holds for all ring elements r' in the ideal generated by r, which is all of $\mathbf{R}$ by simplicity. But this is impossible, since $1(x) = x$ on $\mathbf{M}/P'$. This contradiction proves that nonconstant terms are invertible, and hence finishes the proof. ☐

Now we re-express this theorem in a way that makes use of all the information we gathered in Theorems 5.6, 5.7, 5.8, 5.11.

Theorem 5.16. *$\mathcal{Q}$ is an $\aleph_0$–categorical quasivariety if and only if $\mathcal{Q}$ is definitionally equivalent to a locally finite quasivariety of one of the following types.*

- (i) *A $[k]$–th power of*
  - (a) *the quasivariety of sets,*
  - (b) *the quasivariety generated by the expansion by constants of a faithful G–set where every $g \in G - \{1\}$ has at most one fixed point,*
  - (c) *the quasivariety of regular G–sets for some group G,*
  - (d) *the quasivariety of regular G–sets with zero for some group G, or*
- (ii) *An affine quasivariety of the following form:*
  - (a) *a quasivariety generated by a finite simple module under all affine operations which pointwise fix a subspace,*
  - (b) *a quasivariety generated by an expansion by constants of a finite module over a simple ring,*
  - (c) *a quasivariety generated by a finite module over a simple ring under all affine operations which commute with the translations by elements of a fixed subspace, or*
  - (d) *the quasivariety of modules over a simple ring.*

Proof. The description of these cases follows from Theorems 5.6, 5.7, 5.8 and 5.11, using Theorem 4.1 and Corollary 3.5 to understand Morita equivalence. ☐

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