TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 5, May 1997, Pages 1749–1768
S 0002-9947(97)01883-7

A CHARACTERIZATION OF MINIMAL LOCALLY FINITE VARIETIES

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Abstract. In this paper we describe a one–variable Mal′cev–like condition satisfied by any locally finite minimal variety. We prove that a locally finite variety is minimal if and only if it satisfies this Mal′cev–like condition and it is generated by a strictly simple algebra which is nonabelian or has a trivial subalgebra. Our arguments show that the strictly simple generator of a minimal locally finite variety is unique, it is projective and it embeds into every member of the variety. We give a new proof of the structure theorem for strictly simple abelian algebras that generate minimal varieties.

1. Introduction

The compactness theorem for equational logic implies that every nontrivial variety contains a minimal subvariety. Here a **minimal** variety is a nontrivial variety whose only proper subvariety is trivial. Minimality is a strong restriction on a variety. For example, the minimal varieties of groups are the varieties of elementary abelian p–groups; the minimal varieties of unital rings are the varieties of rings of continuous functions from compact Hausdorff spaces into a discrete field of prime order; the only minimal variety of lattices is the variety of distributive lattices. There are many familiar varieties in which all minimal subvarieties are known; see [11] for a survey of results.

Minimal varieties must either satisfy an extreme finiteness condition or an extreme nonfiniteness condition: a minimal variety is either generated by a finite algebra or else it contains no nontrivial finite algebra (since a minimal variety is generated by any of its nontrivial members). In this paper we discuss only minimal varieties generated by a finite algebra. Such varieties are **locally finite**: all finitely generated members are finite. If $\mathcal{V}$ is a locally finite minimal variety, then $\mathcal{V}$ contains a finite nontrivial algebra $\mathbf{A}$. If $\mathbf{A}$ is chosen with least cardinality > 1 among all algebras in $\mathcal{V}$, then $\mathbf{A}$ must be finite simple and have no nontrivial proper subalgebras. Any algebra with these three properties is said to be **strictly simple**. We can reduce the classification of minimal varieties to the question: *Which strictly simple algebras generate minimal varieties?*

In [10], D. Scott shows that every strictly simple algebra generates a variety with at most finitely many minimal subvarieties. His arguments provide a test for

Received by the editors August 7, 1994.
1991 *Mathematics Subject Classification.* Primary 08B15; Secondary 08B30.
Research of the first author supported by a fellowship from the Alexander von Humboldt Stiftung.
Research of the second author supported by a fellowship from the Alexander von Humboldt Stiftung and by the Hungarian National Foundation for Scientific Research grant no. 1903.

whether a strictly simple algebra of finite type generates a minimal variety. We briefly describe this test. Let $\mathbf{A}$ be a strictly simple algebra. If $\mathbf{A}$ generates a minimal variety, then no strictly simple algebra $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ generates a proper subvariety of $\mathcal{V}(\mathbf{A})$. Conversely, if $\mathbf{A}$ does not generate a minimal variety, then there exists a strictly simple algebra $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ which generates a minimal subvariety of $\mathcal{V}(\mathbf{A})$ and this subvariety is clearly proper. Hence $\mathbf{A}$ generates a minimal variety if and only if $\mathcal{V}(\mathbf{B}) = \mathcal{V}(\mathbf{A})$ for every strictly simple algebra $\mathbf{B} \in \mathcal{V}(\mathbf{A})$. In order to make this into a test for whether $\mathbf{A}$ generates a minimal variety, we need to observe that each strictly simple algebra $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ is two–generated and hence is isomorphic to one of the finitely many homomorphic images of $\mathbf{F} = \mathbf{F}_{\mathcal{V}(\mathbf{A})}(2)$. $\mathbf{F}$ can be constructed as a subalgebra of $\mathbf{A}^{|A|^2}$. Indeed, one could write a computer program to locate all strictly simple $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ when $\mathbf{A}$ is of finite type. $\mathbf{A}$ will generate a minimal variety if $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{B})$ for all such $\mathbf{B}$. Since $\mathbf{A}, \mathbf{B}$ are two–generated and $\mathbf{B} \in \mathcal{V}(\mathbf{A})$, the equality $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{B})$ holds if and only if $|\mathbf{F}_{\mathcal{V}(\mathbf{B})}(2)| = |\mathbf{F}|$.

Although Scott's results allow us to determine whether a given strictly simple algebra generates a minimal variety, they do not allow us to deduce anything about the structure of strictly simple algebras which generate minimal varieties nor do they say anything about the properties of minimal varieties. In fact, these results give us no hint as to how one might begin to construct a minimal variety with specified properties. What is desired, and is posed as Problem 10 in [3], is a classification of strictly simple algebras which generate minimal varieties.

The classification problem was solved for abelian strictly simple algebras independently by K. Kearnes, E.W. Kiss and M. Valeriote in [5] and Á. Szendrei in [13] & [14]. We give a new proof in Section 4. The solution to the problem in the abelian case turns out to be quite nice; the abelian strictly simple algebras which generate minimal varieties must have a very special structure. Unfortunately, the same is not true for nonabelian strictly simple algebras. It is possible for nonabelian strictly simple algebras with very wild structure to generate minimal varieties. To illustrate this, let us consider one part of the problem that is considered to have been 'solved' for over twenty years: the congruence distributive case. Let $\mathbf{A}$ be a strictly simple algebra. When does $\mathbf{A}$ generate a minimal congruence distributive variety? The answer is "Always, as long as $\mathbf{A}$ really does generate a congruence distributive variety." That is, it is a consequence of Jónsson's Lemma that any strictly simple algebra in a congruence distributive variety generates a minimal variety. So, the only problem is determining if $\mathbf{A}$ generates a congruence distributive variety. There is an effective algorithm for determining this when $\mathbf{A}$ has finitely many fundamental operations. The algorithm amounts to checking whether $\mathcal{V}(\mathbf{A})$ has Jónsson terms or simply checking whether $\mathbf{F}_{\mathcal{V}(\mathbf{A})}(3)$ has a distributive congruence lattice. However, one cannot say that we are close to a "classification" of strictly simple algebras which generate congruence distributive varieties. (Here by "classification" we mean an exhaustive list.) The combinatorial properties of such algebras are quite complicated.

The description of strictly simple algebras generating minimal varieties which we give in this paper compares favorably in complexity with the type of description that exists for congruence distributive varieties. We describe a one–variable Mal′cev–like condition which holds in every locally finite minimal variety. We show that for a strictly simple algebra $\mathbf{A}$ the variety $\mathcal{V}(\mathbf{A})$ is minimal if and only if $\mathcal{V}(\mathbf{A})$ satisfies this Mal′cev–like condition and $\mathbf{A}$ has a trivial subalgebra provided it is abelian.

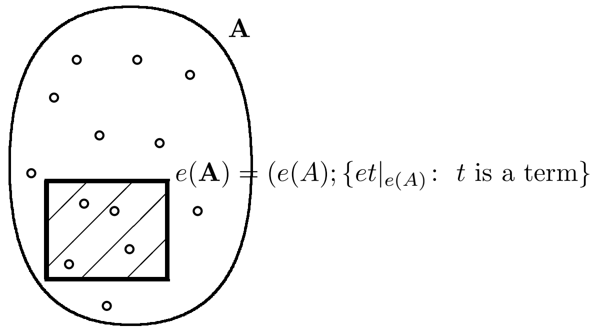

FIGURE 1. Construction of $e(\mathbf{A})$

The characterization of locally finite minimal varieties given here shows also that the strictly simple generator of a minimal variety is unique up to isomorphism. Previously, Hobby and McKenzie [3] had established that the tame congruence theoretic type label of the generator was uniquely determined, but the possibility of several nonisomorphic strictly simple algebras of type **5** in the same minimal variety was not ruled out. By slightly modifying some of the arguments used in the proof of the characterization theorem we are able to show that the strictly simple generator of a minimal variety is projective. This too was unknown before.

2. Term Minimal Algebras

In this section we introduce a construction that will play a fundamental role later on in our investigation of minimal varieties. For convenience, we use the same notation for terms and the corresponding term operations. We shall occasionally make the same identification in words. For example, we may refer to "an idempotent term e" instead of referring to "a term e whose corresponding term operation is idempotent". (Later, the reader will find that we also take liberties in our use of the phrase "term equivalent". We shall occasionally say that "$\mathbf{A}$ is term equivalent to $\mathbf{B}$" in cases where it is only true that "$\mathbf{A}$ is isomorphic to an algebra which is term equivalent to $\mathbf{B}$".)

Let $\mathbf{A}$ be an arbitrary algebra and e a fixed unary term in the language of $\mathbf{A}$. We will say that the term operation e of $\mathbf{A}$ is **idempotent** if $\mathbf{A} \models e^2 = e$. If this holds, then for every term t, et is a term of $\mathbf{A}$ of the same arity as t such that the set $e(A)$ is closed under the term operation et. We will use the symbol $e(\mathbf{A})$ to denote the following algebra. The universe of $e(\mathbf{A})$ is the set $e(A)$. The set of fundamental operation symbols will be the set

$$\{et\colon\ t \text{ is a term in the language of } \mathbf{A}\},$$

and the interpretation of et as an operation on $e(A)$ is the obvious one: the restriction $et|_{e(A)}$ of the term operation et of $\mathbf{A}$ to $e(A)$. See Figure 1.

Fixing the similarity type of $e(\mathbf{A})$ as we did allows us to consider the construction $\mathbf{A} \mapsto e(\mathbf{A})$ for any variety $\mathcal{V}$ of algebras in which the identity $e^2 = e$ holds. As a result, we get a class $\{e(\mathbf{A})\colon\ \mathbf{A} \in \mathcal{V}\}$ of similar algebras. Let $e(\mathcal{V})$ denote the variety generated by this class. There is a natural way to extend the object mapping

$$e\colon\ \mathcal{V} \to e(\mathcal{V}), \quad \mathbf{A} \mapsto e(\mathbf{A})$$

between these two varieties to a functor: to each homomorphism $\varphi\colon \mathbf{B} \to \mathbf{C}$ ($\mathbf{B}, \mathbf{C} \in \mathcal{V}$) we define the corresponding homomorphism to be

$$e(\boldsymbol{\varphi}) = \varphi|_{e(B)}\colon\ e(\mathbf{B}) \to e(\mathbf{C}).$$

Here $e(\varphi)$ is not only a handy notation for the image of φ under the functor e; it can also be interpreted as the image of the subalgebra φ of $\mathbf{B} \times \mathbf{C}$ under the term operation e. It is straightforward to check that with this latter interpretation we have $e(\varphi) = \varphi|_{e(B)}$, that $\varphi|_{e(B)}$ is indeed a homomorphism from $e(\mathbf{B})$ into $e(\mathbf{C})$ and that $e\colon \mathcal{V} \to e(\mathcal{V})$ is a functor.

We summarize some basic properties of this functor that will be needed later on in the paper.

Lemma 2.1. *Let $\mathcal{V}$ be a variety and e a unary term such that $\mathcal{V} \models e^2 = e$. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$.*

(1) *$e(\mathbf{B}^{\boldsymbol{\kappa}}) = (e(\mathbf{B}))^{\kappa}$ for any cardinal κ.*
(2) *If $\mathbf{B}$ is a subalgebra of $\mathbf{C}$, then $e(\mathbf{B})$ is a subalgebra of $e(\mathbf{C})$.*
(3) *If the homomorphism $\varphi\colon \mathbf{B} \to \mathbf{C}$ is surjective [injective], then the homomorphism $e(\boldsymbol{\varphi})\colon e(\mathbf{B}) \to e(\mathbf{C})$ is surjective [injective].*
(4) *For every algebra $\mathbf{B} \in \mathsf{SP}(\mathbf{A})$, we have $e(\mathbf{B}) \in \mathsf{SP}(e(\mathbf{A}))$.*
(5) *For every algebra $\mathbf{B} \in \mathcal{V}(\mathbf{A})$, we have $e(\mathbf{B}) \in \mathcal{V}(e(\mathbf{A}))$.*
(6) *If $\mathbf{B}$ is generated by its subset $e(B)$, then*

$$\operatorname{Con} \mathbf{B} \to \operatorname{Con} e(\mathbf{B}), \quad \theta \mapsto \theta|_{e(B)}$$

is a surjective lattice homomorphism.

Proof. The only nontrivial claim is (6). The assumption that $\mathbf{B}$ is generated by $e(B)$ ensures that, for each $n \geq 0$, every n–ary polynomial operation of $\mathbf{B}$ is of the form $t(\bar{x}, \bar{c})$ for some $k \geq 0$, some $(n+k)$–ary term operation t of $\mathbf{B}$ and some k–tuple $\bar{c} \in e(B)^k$. Thus

$$(\dagger) \qquad \operatorname{Pol} e(\mathbf{B}) = \{ef|_{e(B)}\colon\ f \in \operatorname{Pol} \mathbf{B}\},$$

and therefore the lemma of P. P. Pálfy and P. Pudlák [9] applies. (See also Lemma 2.3 in [3].) □

In view of (4) and (5), for each algebra $\mathbf{A}$ and for each unary idempotent term e of $\mathbf{A}$, we have $e(\mathcal{V}(\mathbf{A})) = \mathcal{V}(e(\mathbf{A}))$ and the functor

$$e\colon \mathcal{V}(\mathbf{A}) \to e(\mathcal{V}(\mathbf{A})) = \mathcal{V}(e(\mathbf{A}))$$

restricts to a functor

$$e\colon\ \mathsf{SP}(\mathbf{A}) \to \mathsf{SP}(e(\mathbf{A})).$$

In the investigation of minimal varieties we will use these functors for a particularly chosen term e.

Definition 2.2. A unary term e in the language of $\mathbf{A}$ will be called a **minimal idempotent** of $\mathbf{A}$ if the term operation e is idempotent, not constant, and the image $e(A)$ of e is minimal with respect to inclusion among all sets of the form $f(A)$ for nonconstant, unary, idempotent term operations f of $\mathbf{A}$.

We say that a finite algebra $\mathbf{A}$ is **term minimal** if every idempotent unary term operation f of $\mathbf{A}$ is either the identity or a constant.

For every finite algebra $\mathbf{A}$ and minimal idempotent e of $\mathbf{A}$, $e(\mathbf{A})$ is a term minimal algebra. If, in addition, $\mathbf{A}$ is strictly simple, then the construction $\mathbf{A} \mapsto e(\mathbf{A})$ has additional nice properties which we summarize in the next theorem.

Theorem 2.3. *Let $\mathbf{A}$ be a strictly simple algebra and let e be a minimal idempotent of $\mathbf{A}$. Then*

(1) *the algebra $e(\mathbf{A})$ is term minimal and strictly simple;*
(2) *for every element $u \in A$, $\{u\}$ is a trivial subalgebra of $\mathbf{A}$ if and only if $u \in e(A)$ and $\{u\}$ is a trivial subalgebra of $e(\mathbf{A})$;*
(3) *$\mathbf{A}$ and $e(\mathbf{A})$ have the same type label from $\{\mathbf{1}, \dots, \mathbf{5}\}$; in particular, $\mathbf{A}$ is abelian if and only if $e(\mathbf{A})$ is abelian.*

Proof. Since $\mathbf{A}$ has no nontrivial proper subalgebras, it is generated by its subset $e(A)$. Therefore in (1) the simplicity of $e(\mathbf{A})$ follows by applying Lemma 2.1 (6) to the algebra $\mathbf{B} = \mathbf{A}$. The remaining claims in (1) and (2) can be proved in a straightforward manner. The second claim in (3) is a consequence of the first one and the fact (cf. Theorems 5.5, 5.6 in [3]) that a finite simple algebra is abelian if and only if it is of type **1** or **2**.

Finally, to prove the first claim in (3) consider a minimal set U of $\mathbf{A}$ with $U \subseteq e(A)$. The special case $\mathbf{B} = \mathbf{A}$ of the equality (†) (which is valid under the assumptions of Lemma 2.1 (6)) means that the polynomial operations of $e(\mathbf{A})$ are exactly the restrictions $f|_{e(A)}$ of all polynomial operations f of $\mathbf{A}$ whose range is contained in $e(A)$. Thus it follows that U is a minimal set of $e(\mathbf{A})$, and the induced minimal algebras $\mathbf{A}|_U$ and $e(\mathbf{A})|_U$ coincide. (For the notions and notation used in this paragraph the reader is referred to [3].) □

Under the assumptions of Theorem 2.3, $e(A)$ is called a **term minimal set** of $\mathbf{A}$ and $e(\mathbf{A})$ the **induced term minimal algebra** on $e(A)$. This construction, which first occurred in a paper by C. Bergman and R. McKenzie [1], is analogous to that of the induced minimal algebra of a simple algebra in tame congruence theory, the only difference being that polynomial operations are replaced by term operations. By analogy, it is natural to ask whether the properties of term minimal sets in strictly simple algebras are similar to those of minimal sets in simple algebras.

It is not hard to see that for any minimal idempotents e, e' of a strictly simple algebra $\mathbf{A}$, the term minimal sets $B = e(A)$ and $C = e'(A)$ are **term isomorphic**. That is, $\mathbf{A}$ has unary term operations f, g such that $f(B) = C$, $g(C) = B$ and the restrictions of f and g to B and C respectively are inverse to each other. (A proof can be found in [12].) Consequently, the corresponding induced term minimal algebras $e(\mathbf{A})$ and $e'(\mathbf{A})$ are term equivalent. This shows that the induced term minimal algebra $e(\mathbf{A})$ of a strictly simple algebra $\mathbf{A}$ is essentially independent of the choice of the minimal idempotent e.

We note that, with respect to connectedness, the term minimal sets in strictly simple algebras behave very much differently from the minimal sets in simple algebras. For instance, every strictly simple algebra with at least two trivial subalgebras has a unique term minimal set, namely the union of its trivial subalgebras. Also, it turns out that a strictly simple algebra of type **2** or **3** with no trivial subalgebras and no constant terms is partitioned by its term minimal sets.

By the existence of trivial subalgebras and the existence of constant term operations, the class of strictly simple algebras $\mathbf{A}$ can be divided into four natural subclasses (see Figure 2):

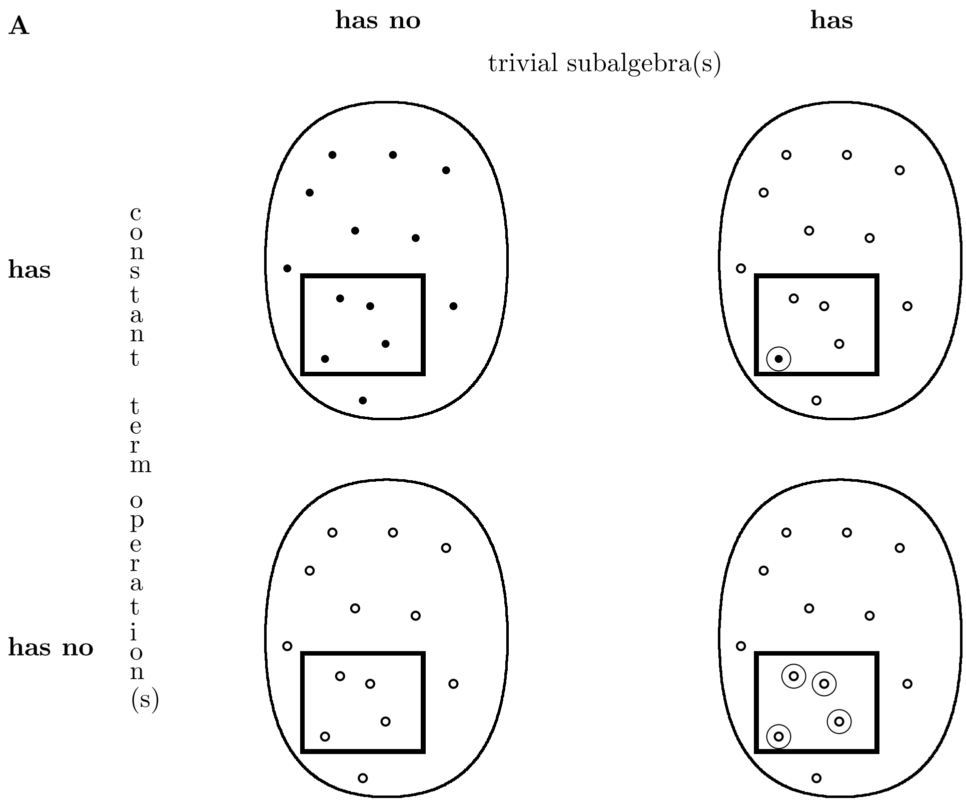

FIGURE 2. The four classes of strictly simple algebras

($\mathbf{O}^\bullet$) every element of $\mathbf{A}$ is distinguished by a constant term;
($\mathbf{O}$) $\mathbf{A}$ has no constant terms and no trivial subalgebras;
($\mathbf{I}$) $\mathbf{A}$ has a unique trivial subalgebra and this element is distinguished by a constant term;
($\mathbf{II}$) $\mathbf{A}$ has at least two trivial subalgebras.

The corresponding strictly simple term minimal algebra $\mathbf{T} = e(\mathbf{A})$ has the following properties:

($\mathbf{O}^\bullet$) every element of $\mathbf{T}$ is distinguished by a constant term and every nonconstant unary term operation of $\mathbf{T}$ is a permutation;
($\mathbf{O}$) the unary term operations of $\mathbf{T}$ form a transitive permutation group;
($\mathbf{I}$) $\mathbf{T}$ has a unique trivial subalgebra, $\{0\}$, and the unary term operations of $\mathbf{T}$ are the constant with value 0 and the members of a permutation group fixing 0 and acting transitively on $T \setminus \{0\}$;
($\mathbf{II}$) $\mathbf{T}$ is idempotent.

Here, a permutation group P acting on a set T is said to be **transitive** if for each $x, y \in T$ there exists a permutation $\pi \in P$ with $\pi(x) = y$.

Term minimal strictly simple algebras are completely classified up to term equivalence. We will list them according to the four classes above and within each class according to their types $\mathbf{1}, \dots, \mathbf{5}$. In this paper we will not need all the details of this classification; therefore in some cases we will be satisfied with stating some properties rather than providing an explicit description.

For the formulation of the classification theorem, we need some notions and notation. For a permutation group P on a set T, P is called **primitive** if the algebra $(T; P)$ is simple, and P is distinct from the one–element group when $|T| = 2$. It is easy to see that every primitive permutation group is transitive. A permutation group P acting on a set T is called **regular** if for each $x, y \in T$ there is exactly one permutation $\pi \in P$ with $\pi(x) = y$.

Now let G be an abstract group. For each element $g \in G$, multiplication by g on the left is a permutation of G, denoted l_g. Clearly, $L_G = \{l_g\colon\ g \in G\}$ is a regular permutation group on G. Replacing 'left' by 'right' we get the regular permutation group $R_G = \{r_g\colon\ g \in G\}$. It is easy to see that

(‡) a mapping $\varphi\colon G \to G$ commutes with every member of R_G if and only if $\varphi \in L_G$,

and the same holds with 'left' and 'right' interchanged.

For a group G, G^0 will denote the extension of G with a new zero element 0 and we use the same notation l_g, r_g and L_G, R_G for the extension of the permutations and permutation groups to G^0 in the obvious way: $l_g(0) = r_g(0) = 0$ for all $g \in G$. An algebra $\mathbf{T}$ with base set $T = G^0$ will be called a G^0**-algebra** if $\{0\}$ is a subalgebra of $\mathbf{T}$, the constant with value 0 (also denoted 0) and the permutations in L_G are unary operations of $\mathbf{T}$ and the permutations in R_G are automorphisms of $\mathbf{T}$. By (‡), these conditions are equivalent to requiring that the unary term operations of a G^0–algebra are exactly the constant 0 and the permutations in L_G. For each group G^0 with zero, $\wedge$ will denote the meet operation of the semilattice in which 0 is the least element and the elements of G are pairwise incomparable.

For an algebra $\mathbf{A}$, $\mathbf{A}^+$ is the algebra arising from $\mathbf{A}$ by adding constant operations for each element.

Theorem 2.4. *If $\mathbf{T}$ is a strictly simple term minimal algebra, then $\mathbf{T}$ is term equivalent to one of the following algebras:*

- **(O•)(1)** *$(T; P)^+$ for a permutation group P on T which is primitive if $|T| > 2$,*
- **(2)** *$({}_K\widehat{T})^+$ for a one–dimensional vector space ${}_K\widehat{T}$ on T,*
- **(3)** *the two–element boolean algebra,*
- **(4)** *the two–element bounded lattice,*
- **(5)** *the two–element bounded semilattice,*
- **(O)(1)** *$(T; P)$ for a primitive permutation group P on T,*
- **(2)** *the idempotent reduct of ${}_R\widehat{T}$ together with all translations added as unary operations, for a simple module ${}_R\widehat{T}$ on T,*
- **(3)** *$(G; p, L_G)$ where p is the ternary discriminator and G is a group on T,*
- **(I)(1)** *the two–element pointed set,*
- **(2)** *a one–dimensional vector space ${}_K\widehat{T}$ on T,*
- **(3–5)** *a G^0–algebra having $\wedge$ as an operation, where G is a group with $G^0 = T$ (each such algebra of type* **3** *or* **4** *generates a congruence distributive variety),*
- **(5)** *the two–element join–semilattice with 0,*
- **(II)(1)** *the two–element set,*
- **(2)** *the idempotent reduct of a simple module ${}_R\widehat{T}$ on T,*

(**3–4**) *an idempotent algebra with no nontrivial proper subalgebras, which generates a congruence distributive variety,*

(**5**) *the two–element semilattice.* □

A more explicit description of the algebras of types (**3–5**) in class (**I**) and of types (**3–4**) in class (**II**) can be found in [12].

We need to know only two facts about term minimal algebras in this paper. The first one concerns the structure of minimal varieties generated by term minimal strictly simple algebras.

Corollary 2.5. *Let* $\mathbf{T}$ *be a term minimal strictly simple algebra. The following conditions are equivalent:*

(1) $\mathcal{V}(\mathbf{T})$ *is minimal,*
(2) $\mathbf{T}$ *is nonabelian or has a trivial subalgebra,*
(3) *If* $\mathbf{S} \in \mathcal{V}(\mathbf{T})$ *is nontrivial and has no nontrivial proper subalgebras, then* $\mathbf{S} \cong \mathbf{T}$.

Proof. The implication (3) $\Rightarrow$ (1) is trivial, for if (3) holds, then $\mathbf{T}$ can be embedded into every nontrivial member of $\mathcal{V}(\mathbf{T})$.

To show that $\neg(2) \Rightarrow \neg(1)$, assume $\mathbf{T}$ is abelian and has no trivial subalgebras, and apply Theorem 2.4. It is well known how to construct in each case a congruence θ on $\mathbf{T}^2$ such that $\mathbf{T}^2/\theta$ generates a nontrivial proper subvariety in $\mathcal{V}(\mathbf{T})$. If $\mathbf{T}$ is of type **1**, let θ be the equivalence relation with blocks $\Delta = \{(t,t)\colon\ t \in T\}$ and $T^2 \setminus \Delta$; this θ is a congruence and $\mathbf{T}^2/\theta$ is a two–element unary algebra having a trivial subalgebra. If $\mathbf{T}$ is of type **2**, let θ be defined by $(x,y)\,\theta\,(u,v) \ \Leftrightarrow\ x+u=y+v$ where $+$ is the addition of the vector space or module, respectively. In this case $\mathbf{T}^2/\theta$ is the so-called linearization of $\mathbf{T}$, which is also strictly simple of type **2** and term minimal, but has a trivial subalgebra. So, whether the type is **1** or **2**, $\mathbf{T}^2/\theta$ generates a proper subvariety of $\mathcal{V}(\mathbf{T})$. This establishes the implication (1) $\Rightarrow$ (2).

It remains to show that (2) $\Rightarrow$ (3). If (2) holds and $\mathbf{T}$ is in class ($\mathbf{O}^\bullet$) or (**O**), then $\mathbf{T}$ is nonabelian. $\mathbf{T}$ is one of the two–element nonabelian algebras from class ($\mathbf{O}^\bullet$) or $\mathbf{T}$ is quasiprimal; in either situation, (3) is known to hold. Therefore (2) $\Rightarrow$ (3) for such $\mathbf{T}$. If $\mathbf{T}$ is in class (**I**), then in all cases there is a group G^0 with zero on T such that $\mathrm{Clo}_1\,\mathbf{T} = L_G \cup \{0\}$. Thus $\mathbf{T} \cong \mathbf{F}_{\mathcal{V}(\mathbf{T})}(1)$. For an arbitrary nontrivial algebra $\mathbf{S} \in \mathcal{V}(\mathbf{T})$, there is a homomorphism $\varphi\colon \mathbf{T} \cong \mathbf{F}_{\mathcal{V}(\mathbf{T})}(1) \to \mathbf{S}$ sending the free generator to an element distinct from 0. Since $\varphi(0) = 0$ and $\mathbf{T}$ is simple, φ is injective. If, moreover, $\mathbf{S}$ has no nontrivial proper subalgebras, then φ is an isomorphism, proving (3).

Finally, let $\mathbf{T}$ be in class (**II**). In all cases $\mathbf{T}$ is the only subdirectly irreducible algebra in $\mathcal{V}(\mathbf{T})$, so $\mathcal{V}(\mathbf{T}) = \mathsf{SP}(\mathbf{T})$. Clearly, if $\mathbf{S} \in \mathcal{V}(\mathbf{T})$ has no nontrivial proper subalgebras, then $\mathbf{S}$ is two–generated and hence it is finite. Thus $\mathbf{S} \in \mathsf{SP}_{fin}(\mathbf{T})$. The proof will be complete if we verify the following claim.

Claim. If $\mathbf{T}$ is a strictly simple idempotent algebra, then for every integer $n \geq 1$ and for every nontrivial subalgebra $\mathbf{S}$ of $\mathbf{T}^n$ there exist an embedding $\iota\colon \mathbf{T} \to \mathbf{S}$ and a projection $\pi_i\colon \mathbf{T}^n \to \mathbf{T}$ $(0 \leq i \leq n-1)$ such that $\pi_i|_S \circ \iota = \mathrm{id}_{\mathbf{T}}$.

Proof of Claim. In the case $n = 1$ we have $\mathbf{S} = \mathbf{T}$, as $\mathbf{T}$ has no nontrivial proper subalgebras. Let $n \geq 2$, assume the claim holds for nontrivial subalgebras of $\mathbf{T}^{n-1}$ and consider a nontrivial subalgebra $\mathbf{S}$ of $\mathbf{T}^n$. In view of $|S| > 1$, not all projections of $\mathbf{S}$ into $\mathbf{T}$ are constant, so we can assume without loss of generality that $|\pi_0(S)| >$

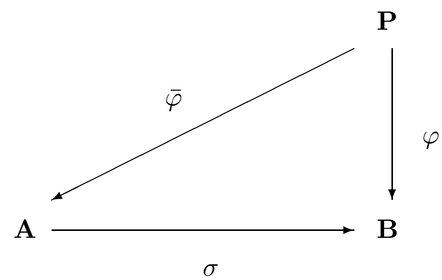

FIGURE 3. $\varphi = \sigma \circ \bar{\varphi}$

1. Since $\pi_0(\mathbf{S})$ is a subalgebra of $\mathbf{T}$, we have $\pi_0(\mathbf{S}) = \mathbf{T}$. Hence for every $t \in T$ we get an algebra $\mathbf{S}_t = \mathbf{S} \cap (\{t\} \times \mathbf{T}^{n-1})$. If $|S_t| > 1$ for some $t \in T$, then $\mathbf{S}_t$ projects isomorphically onto a nontrivial subalgebra of $\mathbf{T}^{n-1}$, whence the induction hypothesis yields the existence of an embedding $\iota_t \colon \mathbf{T} \to \mathbf{S}_t$ and an index i with $1 \leq i \leq n-1$ such that $\pi_i|_{S_t} \circ \iota_t = \mathrm{id}_{\mathbf{T}}$. Here $\pi_i|_{S_t} = \pi_i|_S \circ \iota'$ where ι' is the inclusion $\mathbf{S}_t \to \mathbf{S}$; hence for the embedding $\iota = \iota' \circ \iota_t \colon \mathbf{T} \to \mathbf{S}$ we have $\pi_i|_S \circ \iota = \pi_i|_S \circ \iota' \circ \iota_t = \pi_i|_{S_t} \circ \iota_t = \mathrm{id}_{\mathbf{T}}$, as required. In the remaining case when $|S_t| = 1$ for all $t \in T$, the projection $\pi_0 \colon \mathbf{S} \to \mathbf{T}$ is an isomorphism, so $\iota = \pi_0^{-1}$ together with π_0 has the desired property. □

Before stating the second fact on term minimal algebras, we need the following definition.

Definition 2.6. Let $\mathcal{V}$ be a variety. An algebra $\mathbf{P} \in \mathcal{V}$ is said to be **projective** in $\mathcal{V}$ if whenever

(a) $\mathbf{A}, \mathbf{B} \in \mathcal{V}$,
(b) $\sigma \colon \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism and
(c) $\varphi \colon \mathbf{P} \to \mathbf{B}$ is any homomorphism,

as in Figure 3, then there exists $\bar{\varphi} \colon \mathbf{P} \to \mathbf{A}$ as depicted such that $\sigma \circ \bar{\varphi} = \varphi$.

Recall that $\mathbf{P}$ is said to be a **retract** of an algebra $\mathbf{A} \in \mathcal{V}$ if there is a surjective homomorphism $\gamma \colon \mathbf{A} \to \mathbf{P}$ which has a right inverse. Here we say that γ' is a right inverse for γ if $\gamma' \colon \mathbf{P} \to \mathbf{A}$ and $\gamma \circ \gamma' = \mathrm{id}_{\mathbf{P}}$. For a subclass $\mathcal{C}$ of $\mathcal{V}$ we will say that $\mathbf{P}$ is an **absolute quotient retract** in $\mathcal{C}$ if $\mathbf{P} \in \mathcal{C}$ and whenever $\mathbf{A} \in \mathcal{C}$ and $\gamma \colon \mathbf{A} \to \mathbf{P}$ is surjective, then there is a right inverse γ' for γ. It is easy to see that the following conditions are equivalent:

(i) $\mathbf{P}$ is projective in $\mathcal{V}$.
(ii) $\mathbf{P}$ is an absolute quotient retract in $\mathcal{V}$.
(iii) $\mathbf{P}$ is a retract of a free algebra in $\mathcal{V}$.

Corollary 2.7. *Let $\mathbf{T}$ be a term minimal strictly simple algebra. If $\mathbf{T}$ is nonabelian or has a trivial subalgebra, then $\mathbf{T}$ is projective in $\mathcal{V}(\mathbf{T})$.*

Proof. We show that every algebra $\mathbf{T}$ satisfying the assumptions is either a free algebra or a retract of a free algebra in $\mathcal{V}(\mathbf{T})$. If $\mathbf{T}$ is in class $(\mathbf{O}^\bullet)$, then obviously $\mathbf{T} \cong \mathbf{F}_{\mathcal{V}(\mathbf{T})}(0)$. In the proof of Corollary 2.5 we established that for $\mathbf{T}$ in class $(\mathbf{I})$ we have $\mathbf{T} \cong \mathbf{F}_{\mathcal{V}(\mathbf{T})}(1)$. A similar argument applies to nonabelian algebras in class $(\mathbf{O})$ as well: since $L_G \subseteq \mathrm{Clo}_1 \mathbf{T}$ and $R_G \subseteq \mathrm{Aut}\, \mathbf{T}$ imply that $\mathrm{Clo}_1 \mathbf{T} = L_G$, we again conclude that $\mathbf{T} \cong \mathbf{F}_{\mathcal{V}(\mathbf{T})}(1)$. Finally, if $\mathbf{T}$ is idempotent, then the Claim in the proof of Corollary 2.5 shows that $\mathbf{T}$ is a retract of each nontrivial algebra in $\mathsf{SP}_{fin}(\mathbf{T})$; in particular, it is a retract of $\mathbf{F}_{\mathcal{V}(\mathbf{T})}(2)$. □

3. Minimal varieties

The next theorem is the main result of the paper. In it, we provide a useful characterization of minimal locally finite varieties. In Theorem 3.3 we prove that this characterizing condition is equivalent to a Mal′cev–like condition. In Section 4, we show that the characterizing condition has especially strong consequences for minimal varieties generated by abelian algebras.

Theorem 3.1. *Let $\mathbf{A}$ be a strictly simple algebra and e a minimal idempotent of $\mathbf{A}$. The following conditions are equivalent.*

(1) *$\mathcal{V}(\mathbf{A})$ is minimal.*
(2) *$\mathbf{A}$ is nonabelian or has a trivial subalgebra and*

$$(*) \qquad \text{if } \mathbf{B} \in \mathcal{V}(\mathbf{A}) \text{ and } \mathbf{B} \models e(x) = e(y), \text{ then } |B| = 1.$$

(3) *If $\mathbf{C} \in \mathcal{V}(\mathbf{A})$ is nontrivial and has no nontrivial proper subalgebras, then $\mathbf{C} \cong \mathbf{A}$.*
(4) *If $\mathbf{D} \in \mathcal{V}(\mathbf{A})$ is strictly simple, then $\mathbf{D} \cong \mathbf{A}$.*

Proof. The implication (3) ⇒ (4) is trivial. Since every nontrivial subvariety of $\mathcal{V}(\mathbf{A})$ contains a strictly simple algebra, it is clear that (4) ⇒ (1). The implication ¬(2) ⇒ ¬(1) is also easy to establish. For, if (2) fails because there is some nontrivial $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ such that $\mathbf{B} \models e(x) = e(y)$, then since $\mathbf{A} \not\models e(x) = e(y)$, it follows that $\mathcal{V}(\mathbf{B})$ is a nontrivial proper subvariety of $\mathcal{V}(\mathbf{A})$. Otherwise, if (2) fails because $\mathbf{A}$ is an abelian algebra without trivial subalgebras, then the term minimal algebra $e(\mathbf{A})$ inherits the same properties (cf. Theorem 2.3). Hence, by Lemma 2.1, the algebra $e(\mathbf{A})^2/\theta$ described in Corollary 2.5 which refutes the minimality of $\mathcal{V}(e(\mathbf{A}))$ can be "pulled back" along the functor e to yield a similar algebra in $\mathcal{V}(\mathbf{A})$. In more detail, let $\mathbf{S}$ be the subalgebra of $\mathbf{A}^2$ generated by $e(A)^2$. Obviously, $e(S) = e(A)^2$, so $\mathbf{S}$ has a congruence $\widehat{\theta}$ such that $\widehat{\theta}|_{e(S)} = \theta$. We have $e(\mathbf{S}/\widehat{\boldsymbol{\theta}}) = e(\mathbf{A})^2/\theta$. Thus, $e(\mathbf{A}) \notin \mathcal{V}(e(\mathbf{S}/\widehat{\boldsymbol{\theta}}))$ implies that $\mathbf{A} \notin \mathcal{V}(\mathbf{S}/\widehat{\theta})$. Hence, in both cases when (2) fails, then (1) fails.

These arguments establish the implications (3) ⇒ (4) ⇒ (1) ⇒ (2). We can finish the proof by showing that (2) ⇒ (3). The rest of our argument will be devoted to establishing this implication.

Let $\mathbf{C}$ be a nontrivial member of $\mathcal{V}(\mathbf{A})$, and assume that $\mathbf{C}$ has no nontrivial proper subalgebras. $\mathbf{C}$ is two–generated and therefore finite. This implies that for some $n < \omega$ there is a subalgebra $\mathbf{E} \leq \mathbf{A}^n$ for which there is a surjective homomorphism $\varphi\colon \mathbf{E} \to \mathbf{C}$. Fix some choice of n, $\mathbf{E}$ and φ with these properties such that

(i) n is chosen minimally and
(ii) $\mathbf{E}$ has minimal cardinality for this n.

This implies that $\mathbf{E}$ is an irredundant subdirect subalgebra of $\mathbf{A}^n$. Our goal will be to show that $n = 1$. Since $\mathbf{A}$ is strictly simple this will imply that $\mathbf{A} \cong \mathbf{E} \cong \mathbf{C}$ and will finish the proof.

Claim 1. If F is a proper subuniverse of $\mathbf{E}$, then $\varphi|_F$ is constant.

Proof of Claim. If $\varphi|_F$ is not constant, then $\varphi(F)$ is a nontrivial subuniverse of $\mathbf{C}$. The latter algebra has no proper nontrivial subalgebras, so $\varphi|_F\colon \mathbf{F} \to \mathbf{C}$ is surjective. This contradicts the minimality of $|E|$ since $\mathbf{F}$ works in place of $\mathbf{E}$ and $|F| < |E|$.

By Lemma 2.1, both $e(\mathbf{C})$ and $e(\mathbf{E})$ belong to $\mathcal{V}(e(\mathbf{A}))$. Furthermore, since $\mathbf{A}$ is nonabelian or has a trivial subalgebra, $e(\mathbf{A})$ has the same property (cf. Theorem 2.3).

Claim 2. $e(\mathbf{A}) \cong e(\mathbf{C})$.

Proof of Claim. The fact that $\mathbf{C}$ has no proper nontrivial subalgebras is inherited by $e(\mathbf{C})$ as we now explain. If $\mathbf{R}$ is a nontrivial proper subalgebra of $e(\mathbf{C})$, then the subalgebra $\mathbf{S} \leq \mathbf{C}$ generated by R is nontrivial since it contains R and proper since $e(S) = R \subset e(C)$. Now we apply Corollary 2.5 to the term minimal strictly simple algebra $e(\mathbf{A})$ to deduce that $e(\mathbf{A}) \cong e(\mathbf{C})$.

Claim 3. $e(\mathbf{C}) \cong e(\mathbf{E})$.

Proof of Claim. Since $\varphi\colon \mathbf{E} \to \mathbf{C}$ is surjective, $e(\varphi)\colon e(\mathbf{E}) \to e(\mathbf{C})$ is also surjective (cf. Lemma 2.1). By Corollary 2.7, $e(\mathbf{C}) \cong e(\mathbf{A})$ is projective in $\mathcal{V}(e(\mathbf{A}))$. Hence, there is a homomorphism $\psi\colon e(\mathbf{C}) \to e(\mathbf{E})$ such that $e(\varphi) \circ \psi = \mathrm{id}_{e(C)}$. We claim that ψ is an isomorphism of $e(\mathbf{C})$ onto $e(\mathbf{E})$. The equation $e(\varphi) \circ \psi = \mathrm{id}_{e(C)}$ proves that ψ is injective. To show that ψ is an isomorphism it is enough to show that ψ is surjective. Let F be the subuniverse of $\mathbf{E}$ generated by $\psi(e(C))$. Notice that $\varphi|_F$ is not constant since $e(C) = (e(\varphi) \circ \psi)(e(C)) = \varphi(\psi(e(C))) \subseteq \varphi(F)$ and $|e(C)| > 1$. By Claim 1, it follows that $F = E$. Hence $e(E) = e(F)$ which equals the subuniverse of $e(\mathbf{E})$ generated by $\psi(e(C))$. But $\psi(e(C))$ is a subuniverse of $e(\mathbf{E})$ since ψ is a homomorphism. Thus, $e(E) = \psi(e(C))$ as claimed. This finishes the proof of Claim 3.

Claim 4. If π is a nonconstant homomorphism with domain $\mathbf{E}$, then $\pi|_{e(E)}$ is nonconstant.

Proof of Claim. Otherwise $e(x)$ would have constant value on the nontrivial algebra $\pi(\mathbf{E})$. This violates our assumption that $(*)$ holds.

Recall that $\mathbf{E}$ is an irredundant subdirect subalgebra of $\mathbf{A}^n$ for some n. If $n > 1$, then pick two distinct projection homomorphisms π and π' from $\mathbf{E}$ onto $\mathbf{A}$. Let η and η' denote the respective kernels of π and π'. From Claim 4 it follows that $\eta|_{e(E)}$ and $\eta'|_{e(E)}$ are proper congruences of the algebra $e(\mathbf{E})$. By Claims 2 and 3 the algebra $e(\mathbf{E})$ is simple. Hence $\eta|_{e(E)} = \eta'|_{e(E)} = 0_{e(E)}$. It follows also that $\mathbf{E}$ is generated by $e(E)$, for by Claim 1, $\mathbf{E}$ is generated by any of its subsets on which φ is not constant and, in view of $\varphi(e(E)) = e(\varphi(E)) = e(C)$, $e(E)$ is such a set. Therefore, by Lemma 2.1 (6), we have

$$(\eta \vee \eta')|_{e(E)} = \eta|_{e(E)} \vee \eta'|_{e(E)} = 0_{e(E)}.$$

But η and η' are distinct coatoms in $\operatorname{Con}\mathbf{E}$ since $\mathbf{A}$ is simple and $\mathbf{E}$ is an irredundant subdirect subalgebra of $\mathbf{A}^n$. Thus $\eta \vee \eta' = 1_{\mathbf{E}}$. We get that $1_{e(E)} = (\eta \vee \eta')|_{e(E)} = 0_{e(E)}$. That is, e is constant on $\mathbf{E}$, which is impossible by property $(*)$. This is a contradiction to our assumption that $n > 1$ and so the proof is finished. □

One of the interesting consequences of Theorem 3.1 is that, in every locally finite minimal variety, the strictly simple generator can be embedded in each nontrivial member of the variety. This follows from the equivalence of Theorem 3.1 (1) $\Leftrightarrow$ (3). For congruence modular minimal varieties this fact was proved earlier by C. Bergman and R. McKenzie in [1]. Their proof is also based on the idea of reducing the problem to the term minimal case. However, at that time the classification theorem for strictly simple term minimal algebras was not available, so a different argument was needed to handle the strictly simple term minimal algebras in class (**O**) which generate congruence distributive varieties.

Another interesting property of the strictly simple generator in a locally finite minimal variety is the following.

Theorem 3.2. *If* $\mathbf{A}$ *is a strictly simple algebra which generates a minimal variety, then* $\mathbf{A}$ *is projective in* $\mathcal{V}(\mathbf{A})$.

Proof. Assume $\mathbf{A}$ is not projective in $\mathcal{V}(\mathbf{A})$. Since $\mathbf{A}$ is two–generated, $\mathbf{F} = \mathbf{F}_{\mathcal{V}(\mathbf{A})}(2)$ is an algebra in $\mathsf{SP}_{fin}(\mathbf{A})$ for which there is a surjective homomorphism $\gamma\colon \mathbf{F} \to \mathbf{A}$ which has no right inverse. Now choose some $\mathbf{E}$ of minimal cardinality for the properties

(i) $\mathbf{E} \in \mathsf{SP}_{fin}(\mathbf{A})$,
(ii) there exists a surjective homomorphism $\varphi\colon \mathbf{E} \to \mathbf{A}$ and
(iii) there is no $\tau\colon \mathbf{A} \to \mathbf{E}$ such that $\varphi \circ \tau = \mathrm{id}_{\mathbf{A}}$.

A contradiction to these conditions can be obtained by copying the proof of Theorem 3.1 from Claim 3 to the end. One needs to use the version of this proof where $\mathbf{C} = \mathbf{A}$. ☐

Now we explain how to recognize whether $\mathcal{V}(\mathbf{A})$ is minimal. We will prove that the condition $(*)$ in Theorem 3.1 (2) is equivalent to a one–variable Mal′cev–like condition.

Theorem 3.3. *Let* $\mathbf{A}$ *be an arbitrary algebra and let* e *be an idempotent term of* $\mathbf{A}$. *The following conditions are equivalent.*

$(*)$ *If* $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ *and* $\mathbf{B} \models e(x) = e(y)$, *then* $|B| = 1$.
$(**)$ *For some* $n \geq 1$, *there exist binary terms* f_i *and unary terms* g_i, h_i $(0 \leq i \leq n)$ *such that*

$$\begin{aligned} \mathcal{V}(\mathbf{A}) \quad \models \quad & x = f_0(x, eg_0(x)), \\ & f_i(x, eh_i(x)) = f_{i+1}(x, eg_{i+1}(x)) \quad (0 \leq i \leq n-1), \\ & f_n(x, eh_n(x)) = e(x). \end{aligned}$$

Proof. It is easy to see that the identities in the Mal′cev–like condition together with the identity $e(x) = e(y)$ entail $x = y$. Thus $(**) \Rightarrow (*)$. Conversely, if $(*)$ holds in $\mathcal{V}(\mathbf{A})$, then in $\mathbf{F} = \mathbf{F}_{\mathcal{V}(\mathbf{A})}(1)$ the congruence collapsing $e(F)$ has to be the full relation, so it has to contain the pair $(x, e(x))$. This yields the condition in $(**)$. ☐

Exercise. Let $\mathbf{A}$ be a strictly simple algebra which has terms 0, 1 and $x \circ y$ such that

$$\mathbf{A} \models 0 \circ x = 0, \ 1 \circ x = x.$$

If e is any nonconstant idempotent term of $\mathbf{A}$, show that e and $\mathbf{A}$ satisfy the equivalent conditions of Theorem 3.3. In particular, such an $\mathbf{A}$ generates a minimal variety.

4. Abelian versus nonabelian

As we mentioned in the Introduction, abelian strictly simple algebras generating minimal varieties are known to have a very nice structure, and thereby locally finite, minimal, abelian varieties are fully classified. In this section our aim is to give a short and elementary demonstration that the condition given in Theorem 3.1 (2) is strong enough to lead directly to this classification. This approach sheds new light on minimal abelian varieties, as our arguments hint at a plausible reason why the abelian case is simpler than the nonabelian case: when $\mathbf{A}$ is abelian, every idempotent term is invertible, which need not be true when $\mathbf{A}$ is nonabelian.

Definition 4.1. Let $\mathcal{V}$ be a variety and let e be a unary term in the language of $\mathcal{V}$. We say that e is **invertible** if for some $k \geq 1$ there exist unary terms $f_0, \dots, f_{k-1}$ and a k–ary term f such that

$$\mathcal{V} \models f(e(f_0(x)), \dots, e(f_{k-1}(x))) = x.$$

We shall only be interested in invertible terms whose corresponding term operation is idempotent. Such terms will be called **invertible idempotents**. If $\mathbf{A}$ is an algebra, then instead of saying "e is an invertible idempotent of $\mathcal{V}(\mathbf{A})$", we may say just "e is an invertible idempotent of $\mathbf{A}$". A basic result concerning invertible idempotents is Lemma 4.3. In order to state this lemma we need the following well known definition.

Definition 4.2. Let $\mathcal{C}, \mathcal{D}$ be arbitrary categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called a **categorical equivalence** if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is naturally isomorphic to the identity functor of $\mathcal{C}$ and $F \circ G$ is naturally isomorphic to the identity functor of $\mathcal{D}$.

Two categories, $\mathcal{C}$ and $\mathcal{D}$, are called **categorically equivalent** if there exists a categorical equivalence $F\colon \mathcal{C} \to \mathcal{D}$.

Let $\mathcal{V}$ be an arbitrary variety and e an idempotent term of $\mathcal{V}$. The functor $e\colon \mathcal{V} \to e(\mathcal{V})$ introduced in Section 2 is usually not a categorical equivalence. The next lemma states that the invertibility of the term e is a necessary and sufficient condition for the functor e to be a categorical equivalence.

Lemma 4.3. *For every variety $\mathcal{V}$ and every idempotent term e of $\mathcal{V}$ the following conditions are equivalent:*

(1) *each algebra $\mathbf{B}$ in $\mathcal{V}$ is generated by its subset $e(B)$;*
(2) *the one–generated free algebra $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(1)$ in $\mathcal{V}$ is generated by its subset $e(F)$;*
(3) *e is an invertible idempotent term of $\mathcal{V}$;*
(4) *$e\colon \mathcal{V} \to e(\mathcal{V})$ is a categorical equivalence.*

Proof. The implication (1)⇒(2) and the equivalence (2)⇔(3) are obvious. The implication (3)⇒(4) is established in [7]. To verify the implication (4)⇒(1) observe that if $\mathbf{B} \in \mathcal{V}$, then for the subalgebra $\mathbf{B}'$ of $\mathbf{B}$ generated by $e(B)$ we have $e(\mathbf{B}') = e(\mathbf{B})$. Let $\iota : \mathbf{B}' \to \mathbf{B}$ denote inclusion. $e(\iota)$ is an isomorphism so, if (4) holds, then ι is an isomorphism. Hence $\mathbf{B}$ must be generated by $e(B)$. □

The next theorem is the main step in deriving the structure of abelian strictly simple algebras generating minimal varieties from Theorem 3.1.

Theorem 4.4. *Let* $\mathbf{A}$ *be an abelian strictly simple algebra and* e *a minimal idempotent of* $\mathbf{A}$. *If condition* $(*)$ *in Theorem 3.1* (2) *holds, then* e *is an invertible idempotent of* $\mathbf{A}$.

Proof. By Theorem 3.3, condition $(*)$ in Theorem 3.1 (2) is equivalent to the Mal′cev–like condition $(**)$. Therefore, we assume that $\mathbf{A}$ is an abelian strictly simple algebra with minimal idempotent e such that for some binary terms f_i and unary terms g_i, h_i $(0 \le i \le n)$ the identities described in $(**)$ hold in $\mathbf{A}$. We will use the following notation: $G_i(x) = f_i(x, eg_i(x))$, $H_i(x) = f_i(x, eh_i(x))$ $(0 \le i \le n)$.

First we show that, at the cost of using higher arity terms instead of the binary terms f_i, we can get a set of one–variable identities valid in $\mathbf{A}$ that are analogous to those in $(**)$ and have the additional property that the unary terms corresponding to G_i, H_i represent idempotent term operations of $\mathbf{A}$.

Let us fix a positive integer k such that $G_i^k(x), H_i^k(x)$ are idempotent term operations of $\mathbf{A}$ for every i $(0 \le i \le n)$. Since $\mathbf{A}$ is finite, such a k is easily seen to exist. For convenience, we will abbreviate a k–tuple $(y_0, \dots, y_{k-1})$ of variables by $\overline{y}$ and a k–tuple $(g_0(x), \dots, g_{k-1}(x))$ of unary terms by $\overline{g(x)}$. If k–tuples of variables or terms appear in place of each variable of a term, then this is to be understood so that the term is applied coordinatewise.

Now we define some terms of $\mathbf{A}$ as follows: let

$$e_0(x) = x, \quad e_{i+1}(x) = H_i^k(x) \qquad (0 \le i \le n),$$

and for $0 \le i \le n$ let

$$\begin{aligned} F_i(x, \overline{y}) &= f_i(f_i(\dots f_i(f_i(x, y_0), y_1) \dots, y_{k-2}), y_{k-1}), \\ \overline{g_i(x)} &= (g_i(x), g_iG_i(x), \dots, g_iG_i^{k-2}(x), g_iG_i^{k-1}(x)), \\ \overline{h_i(x)} &= (h_i(x), h_iH_i(x), \dots, h_iH_i^{k-2}(x), h_iH_i^{k-1}(x)). \end{aligned}$$

It is straightforward to check that

$$\begin{aligned} \mathbf{A} \ \models \ & e_0(x) = x = F_0(x, e(\overline{g_0(x)})), \\ & e_{i+1}(x) = F_i(x, e(\overline{h_i(x)})) = F_{i+1}(x, e(\overline{g_{i+1}(x)})) \quad (0 \le i \le n-1), \\ & e_{n+1}(x) = F_n(x, e(\overline{h_n(x)})) = e(x); \end{aligned}$$

moreover,

$$\mathbf{A} \ \models \ e_i^2(x) = e_i(x) \quad (0 \le i \le n+1).$$

From now on the proof splits according to the type of $\mathbf{A}$.

Case 1. $\mathbf{A}$ is of type $\mathbf{1}$.

For every i $(0 \le i \le n)$, the identities for $e_{i+1}(x)$ imply that

$$\begin{aligned} \mathbf{A} \ \models \ F_i(x, e(\overline{h_i(x)})) &= e_{i+1}(x) = e_{i+1}(e_{i+1}(x)) \\ &= F_i(e_{i+1}(x), e(\overline{h_i(e_{i+1}(x))})). \end{aligned}$$

Since $\mathbf{A}$ is strongly abelian, it follows that

$$\mathbf{A} \ \models \ F_i(x, \overline{y}) = F_i(e_{i+1}(x), \overline{y}).$$

This identity and the identities in the Mal′cev–like condition yield that

$$(\bullet) \qquad \mathbf{A} \ \models \ e_i(x) = F_i(x, e(\overline{g_i(x)})) = F_i(e_{i+1}(x), e(\overline{g_i(x)})).$$

Now it is easy to see that for each term e_i $(0 \le i \le n+1)$ there exist an m–ary term t (for some m) and unary terms $t_0, \dots, t_{m-1}$ such that

$$\mathbf{A} \ \models \ e_i(x) = t(et_0(x), \dots, et_{m-1}(x)).$$

Indeed, this claim is trivial for e_{n+1} and the identity proved in ($\bullet$) implies that this claim is true for e_i whenever it is true for e_{i+1}. Thus, the claim holds for $e_0(x) = x$ as well, showing that e is invertible.

Case 2. $\mathbf{A}$ is of type $\mathbf{2}$.

In this case, the term minimal algebra $e(\mathbf{A})$ has a Mal′cev term. Hence, $\mathbf{A}$ has a ternary term p such that

$$\mathbf{A} \models ep(e(x), e(y), e(y)) = e(x) = ep(e(y), e(y), e(x)).$$

For every i ($0 \le i \le n$) consider the terms

$$F_i'(x, \overline{y}, \overline{z}) = F_i(x, ep(e(\overline{h_i(x)}), e(\overline{y}), e(\overline{z}))).$$

Using the identities for p and e_{i+1} we get that

$$\begin{aligned} \mathbf{A} \models F_i'(x, \overline{y}, \overline{y}) &= F_i(x, e(\overline{h_i(x)})) \\ &= e_{i+1}(x) = e_{i+1}(e_{i+1}(x)) \\ &= F_i'(e_{i+1}(x), \overline{y}, \overline{y}). \end{aligned}$$

Since $\mathbf{A}$ is abelian, it follows that

$$\mathbf{A} \models F_i'(x, \overline{y}, \overline{z}) = F_i'(e_{i+1}(x), \overline{y}, \overline{z}).$$

This identity and the identities in the Mal′cev–like condition yield that

$$\begin{aligned} \mathbf{A} \models e_i(x) &= F_i(x, e(\overline{g_i(x)})) \\ &= F_i(x, ep(e(\overline{h_i(x)}), e(\overline{h_i(x)}), e(\overline{g_i(x)}))) \\ &= F_i'(x, e(\overline{h_i(x)}), e(\overline{g_i(x)})) \\ &= F_i'(e_{i+1}(x), e(\overline{h_i(x)}), e(\overline{g_i(x)})). \end{aligned}$$

The same argument as in Case 1 shows that e is invertible. This completes the proof. □

Now we explain how to put the invertibility of e to use. We need the following fact, a variation of Remark 2 in [7], the proof of which is straightforward. This lemma refers to the matrix power construction. See [15] for a description of this construction and its properties.

Lemma 4.5. *Let $\mathbf{A}$ be an algebra and e an invertible idempotent of $\mathbf{A}$. Assume that the invertibility of e is witnessed by the unary terms $f_0, \dots, f_{k-1}$ and the k–ary term f. Then*

$$\varepsilon(y) = (e(f_0(f(y))), \dots, e(f_{k-1}(f(y))))$$

is an invertible idempotent term of the matrix power $(e(\mathbf{A}))^{[k]}$ and $\mathbf{A}$ is term equivalent to $\varepsilon((e(\mathbf{A}))^{[k]})$. □

This lemma proves that $\mathbf{A}$ can be recovered from $e(\mathbf{A})$, up to term equivalence, when e is an invertible idempotent. This fact, combined with Theorem 4.4, leads to a description of those abelian strictly simple algebras which satisfy condition ($*$) of Theorem 3.1 (2) for a minimal idempotent.

Theorem 4.6. *Let $\mathbf{A}$ be an abelian strictly simple algebra and let e be a minimal idempotent of $\mathbf{A}$. Then $\mathbf{A}$ and e satisfy condition ($*$) of Theorem 3.1 (2) if and only if either $\mathbf{A}$ is term equivalent to a matrix power of a strictly simple term minimal algebra of type $\mathbf{1}$, or $\mathbf{A}$ is affine.*

Proof. Let $e(\mathbf{A}) = \mathbf{T}$. We know from Theorem 2.3 that $\mathbf{T}$ is a term minimal strictly simple algebra of the same type as $\mathbf{A}$. Furthermore, $\mathbf{T}$ is described up to term equivalence by Theorem 2.4. Assume first that **A** satisfies condition $(*)$ of Theorem 3.1 (2). By Theorem 4.4 and Lemma 4.5 the algebra **A** is term equivalent to $\varepsilon(\mathbf{T}^{[k]})$ for some positive integer k and some invertible idempotent ε of $\mathbf{T}^{[k]}$.

In case **A** is of type **1** the forward implication of the theorem is a consequence of the following fact.

Claim. Let **T** be a strictly simple term minimal algebra of type **1**. For every positive integer k and every nonconstant idempotent term ε of $\mathbf{T}^{[k]}$ the algebra $\varepsilon(\mathbf{T}^{[k]})$ is term equivalent to $\mathbf{T}^{[m]}$ for some positive integer $m \leq k$.

Proof of Claim. For any k–tuple $b \in T^k$, the coordinates of b will be denoted by b^l $(0 \leq l < k)$. For any collection of elements u_l $(0 \leq l < k)$, the k–tuple with these coordinates will be denoted by $(u_l)_{l<k}$.

The algebra **T** is essentially unary, so for every integer $n \geq 1$ the n–ary term operations of $\mathbf{T}^{[k]}$ have the following form:

$$(\triangleright) \qquad x_0, \dots, x_{n-1} \overset{h}{\longmapsto} \Big(g_l(x_{\mu(l)}^{\sigma(l)})\Big)_{l<k} \quad \text{for all } x_0, \dots, x_{n-1} \in T^k,$$

where $\mu\colon \{0, \dots, k-1\} \to \{0, \dots, n-1\}$, $\sigma\colon \{0, \dots, k-1\} \to \{0, \dots, k-1\}$ are arbitrary mappings and $g_0, \dots, g_{k-1}$ are arbitrary unary term operations of **T**.

Let ε be a nonconstant idempotent unary term operation of $\mathbf{T}^{[k]}$, say $\varepsilon(x) = \big(f_l(x^{\alpha(l)})\big)_{l<k}$. Since **T** is a term minimal algebra, each f_l is either constant or a permutation. Let J denote the set of indices l such that f_l is a permutation and let $I = \alpha(J)$. We have $J \neq \emptyset$, because ε is not constant, so $I \neq \emptyset$. A straightforward analysis of what it means for ε to be idempotent shows that $I \subseteq J$, $\alpha|_I = \mathrm{id}$ and $f_l = \mathrm{id}$ for all $l \in I$. If $l \notin J$ then f_l is constant, so we may change $\alpha(l)$ to any value we wish without changing ε. Therefore we may assume without loss of generality that $\alpha(l) \in I$ for all l $(0 \leq l < k)$.

It follows from these observations that the projection mapping

$$\pi_I\colon\ \varepsilon(T^k) \to T^I, \quad x = (x^l)_{l<k} \mapsto (x^l)_{l \in I},$$

is a bijection and its inverse is

$$\pi_I^{-1}\colon\ T^I \to \varepsilon(T^k), \quad y = (y^l)_{l \in I} \mapsto \Big(f_l(y^{\alpha(l)})\Big)_{l<k}.$$

Let n be a positive integer and let h in $(\triangleright)$ be an arbitrary n–ary term operation of $\mathbf{T}^{[k]}$. The operation corresponding under the bijection π_I to the operation $\varepsilon h|_{\varepsilon(T^k)}$ of $\varepsilon(\mathbf{T}^{[k]})$ is shown by the dashed arrow in the commutative diagram below:

$$\begin{array}{ccc}
y_0, \dots, y_{n-1}\ (\in T^I) & \dashrightarrow & \Big(g_l f_{\sigma(l)}(y_{\mu(l)}^{\alpha\sigma(l)})\Big)_{l \in I} \\
\pi_I^{-1} \big\downarrow & & \big\uparrow \pi_I \\
\Big(f_l(y_0^{\alpha(l)})\Big)_{l<k}, \dots, \Big(f_l(y_{n-1}^{\alpha(l)})\Big)_{l<k} & \xrightarrow[\varepsilon h|_{\varepsilon(T^k)}]{} & \Big(g_l f_{\sigma(l)}(y_{\mu(l)}^{\alpha\sigma(l)})\Big)_{l<k}
\end{array}$$

This proves that the algebra on T^I isomorphic to $\varepsilon(\mathbf{T}^{[k]})$ via π_I is a reduct of $\mathbf{T}^{[m]}$ where $m = |I|$. However this reduct is in fact $\mathbf{T}^{[m]}$ itself, since all the operations of $\mathbf{T}^{[m]}$ are induced by operations h for which $\sigma(I) \subseteq I$. Thus $\varepsilon(\mathbf{T}^{[k]})$ is term equivalent to $\mathbf{T}^{[m]}$, as claimed.

Although it does not affect the present course of our arguments, we point out that the proof of the claim does not require that $\mathbf{T}$ be strictly simple. The conclusion of the claim is valid if $\mathbf{T}$ is any essentially unary algebra such that every nonconstant unary operation of $\mathbf{T}$ is a permutation.

In the case when $\mathbf{A}$ is of type $\mathbf{2}$ it would be possible to proceed along the same lines as before by analyzing the invertible idempotents of $\mathbf{T}^{[k]}$. We would get slightly different results: this time it is *not* always true that $\varepsilon(\mathbf{T}^{[k]})$ is term equivalent to $\mathbf{T}^{[m]}$ for some $m \leq k$. However, we shall avoid this analysis by using the facts that

- (i) $\mathbf{T}$ is affine. (See Theorem 2.4.)
- (ii) Matrix powers of affine algebras are affine. (A Mal′cev term for $\mathbf{T}$ acting coordinatewise is a Mal′cev term of $\mathbf{T}^{[k]}$.)
- (iii) If an algebra $\mathbf{B}$ is affine and ε is an invertible idempotent of $\mathbf{B}$, then $\varepsilon(\mathbf{B})$ is affine. (If $p(x, y, z)$ is a Mal′cev term of $\mathbf{B}$, then $\varepsilon p(x, y, z)$ is a Mal′cev term of $\varepsilon(\mathbf{B})$.)

Clearly, items (i)–(iii) imply that $\varepsilon(\mathbf{T}^{[k]})$, and hence $\mathbf{A}$ is affine. This proves the forward direction of the theorem.

Now we prove the backward implication. In the case where $\mathbf{A}$ has type $\mathbf{1}$ it suffices to verify $(*)$ for $\mathbf{A} = \mathbf{T}^{[k]}$ where $\mathbf{T}$ is a term minimal strictly simple algebra of type $\mathbf{1}$ and for e a specifically chosen minimal idempotent (since all term minimal sets of a strictly simple algebra are term isomorphic). We choose $e(x) = (x^0, \dots , x^0)$. This e is minimal because $|e(T^k)| = |T|$ and $|f(T^k)| \geq |T|$ for any nonconstant unary term f of $\mathbf{T}^{[k]}$.

Any member of $\mathcal{V}(\mathbf{T}^{[k]})$ is isomorphic to an algebra of the form $\mathbf{S}^{[k]}$ for some $\mathbf{S} \in \mathcal{V}(\mathbf{T})$. If $\mathbf{S}^{[k]} \models e(x) = e(y)$, then $\mathbf{S} \models x^0 = y^0$. In this case, $\mathbf{S}$ is trivial. Consequently, $\mathbf{S}^{[k]}$ is trivial. This proves that condition $(*)$ holds.

Now if $\mathbf{A}$ is a strictly simple affine algebra, then $(*)$ follows from Theorem 12.4 of [2] where it is shown that $\mathcal{V}(\mathbf{A})$ has a unique minimal subvariety which is generated by the linearization, $\mathbf{A}_\triangledown$, of $\mathbf{A}$. But if e is not constant on $\mathbf{A}$, then e is not constant on $\mathbf{A}_\triangledown$. Hence, the equation $e(x) = e(y)$ defines the trivial subvariety of $\mathcal{V}(\mathbf{A})$, which implies that $(*)$ holds. □

By Theorem 3.1 an abelian strictly simple algebra $\mathbf{A}$ generates a minimal variety if and only if $\mathbf{A}$ is one of the algebras described in Theorem 4.6 and $\mathbf{A}$ has a trivial subalgebra. In the case where $\mathbf{A}$ is of type $\mathbf{1}$ the latter condition holds exactly when the corresponding strictly simple term minimal algebra has a trivial subalgebra. Up to term equivalence, there are only two such algebras (cf. Theorem 2.4). Namely, there is the two–element unary algebra in class (**I**) which has a single constant operation: $(2; 0)$. The other is the two–element unary algebra in class (**II**) which may be viewed as an algebra with no operations: $(2; \emptyset)$. Thus we obtain the following corollary.

Corollary 4.7. *The abelian strictly simple algebras which generate minimal varieties are precisely the algebras that are term equivalent to a matrix power of* $(2; \emptyset)$ *or* $(2; 0)$, *and the affine strictly simple algebras which have a trivial subalgebra.* □

If $\mathbf{A}$ is a nonabelian strictly simple algebra generating a minimal variety and e is a minimal idempotent of $\mathbf{A}$, then e need not be invertible. As a consequence, the functor $e\colon \mathcal{V}(\mathbf{A}) \to \mathcal{V}(e(\mathbf{A}))$ is not necessarily a categorical equivalence. In this case, the structure of the variety $\mathcal{V}(\mathbf{A})$ can be much more complicated than the structure of $\mathcal{V}(e(\mathbf{A}))$ as the examples below show.

The first example is the algebra $\mathbf{A} = (\{0,1,2\}; +, \cdot, e, f, 0)$ where 0 denotes the unary constant operation with value 0 and the remaining four operations are defined as follows:

$$x + y = \begin{cases} 0 & \text{if } x = y = 0, \\ 2 & \text{otherwise,} \end{cases} \qquad xy = \begin{cases} 0 & \text{if } x = 0, \\ y & \text{otherwise,} \end{cases}$$

and

$$e(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2 & \text{otherwise,} \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$$

It is easy to check that $\mathbf{A}$ is a strictly simple algebra in class (**I**) and e is a minimal idempotent of $\mathbf{A}$ with $e(A) = \{0, 2\}$. Furthermore, the identities $e(0) = 0$, $e(x) \cdot x = x$ and $0 \cdot x = 0$ hold in $\mathbf{A}$. Clearly, these identities together with $e(x) = e(y)$ entail $e(x) = 0$ and hence $x = 0$. Thus, by Theorem 3.1, $\mathbf{A}$ generates a minimal variety. (Or, if we want to use Theorem 3.3, we can take $f_0(x, y) = y \cdot x$, $f_1(x, y) = y$, $g_0(x) = x = h_1(x)$ and $h_0(x) = 0 = g_1(x)$ to see that $\mathbf{A}$ generates a minimal variety.)

The natural order $0 \leq 1 \leq 2$ is a compatible order of $\mathbf{A}$ and $e(\mathbf{A})$ is term equivalent to the two–element lower bounded lattice; therefore $\mathbf{A}$ is of type **4**. Let $\mathbf{S}$ be the subalgebra of $\mathbf{A}^2$ whose underlying set is the relation $\leq$. It can be verified that $\operatorname{Con}\mathbf{S}$ is isomorphic to N_5 and all prime quotients are of type **4**, except for the critical quotient $\langle \delta, \theta \rangle$, which is of type **1**. Here θ is the kernel of the projection to the first coordinate and δ is the congruence smaller than θ, in which the two–element block $\{(1,1),(1,2)\}$ of θ is split into two blocks. Since the variety $\mathcal{V}(e(\mathbf{A}))$ is congruence distributive, while the variety $\mathcal{V}(\mathbf{A})$ fails to have this property, it follows that e is not a categorical equivalence.

The second example is the algebra $\mathbf{A} = (\{0,1,2\}; \circ, e, f, 0, p_1, p_2)$ where $e, f, 0$ are the same as in the previous example and the remaining operations are defined as follows:

$$x \circ y = \begin{cases} 1 & \text{if } x = y = 1, \\ 2 & \text{if } x \neq 2 = y, \\ 0 & \text{otherwise,} \end{cases}$$

$$p_1(x, y, z) = \begin{cases} x & \text{if } y = z \text{ or } x = 2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$p_2(x, y, z) = \begin{cases} x & \text{if } x = y = z, \\ 2 & \text{if } x = 2 \neq y, z \quad \text{or} \quad z = 2 \neq x, y, \\ 0 & \text{otherwise.} \end{cases}$$

Again, $\mathbf{A}$ is a strictly simple algebra in class (**I**) and e is a minimal idempotent of $\mathbf{A}$ with $e(A) = \{0, 2\}$. Furthermore, the identities $f(e(x)) \circ x = x$, $f(0) \circ x = e(f(x))$ and $e(0) = 0$ hold in $\mathbf{A}$. These identities, together with $e(x) = e(y)$ entail $e(x) = 0$ and hence $x = 0$. Therefore, by Theorem 3.1, $\mathbf{A}$ generates a minimal variety. It easy to verify that, for p_1, p_2 and $p_3(x, y, z) = p_1(z, y, x)$, the identities

$$x = p_1(x, y, y), \ p_1(x, x, y) = p_2(x, y, y), \ p_2(x, x, y) = p_3(x, y, y), \ p_3(x, x, y) = y$$

hold in $\mathbf{A}$, witnessing that the variety $\mathcal{V}(\mathbf{A})$ is congruence four–permutable.

The minimal algebra $e(\mathbf{A})$ is a two–element quasiprimal algebra; hence $\mathbf{A}$ is of type **3**. One can check that $S = A^2 \setminus \{(2,1)\}$ is a subuniverse of $\mathbf{A}^2$, the congruence lattice of the corresponding algebra $\mathbf{S}$ is again isomorphic to N_5 and now all prime quotients are of type **3**. In the critical quotient $\langle\delta,\theta\rangle$, θ is again the kernel of the projection to the first coordinate and δ is the congruence smaller than θ, in which the three–element block $\{(1,0),(1,1),(1,2)\}$ of θ is split into two blocks: $\{(1,0),(1,1)\}$ and $\{(1,2)\}$. In this example, $\mathcal{V}(\mathbf{A})$ is a minimal congruence four–permutable variety which is not congruence distributive, although $\mathcal{V}(e(\mathbf{A}))$ is a minimal discriminator variety.

In [6], it is proved that a minimal variety generated by a strictly simple algebra of type **3** or **4** is residually small if and only if it is congruence distributive. Therefore, in either of the above examples, $\mathbf{A}$ generates a residually large variety while $e(\mathbf{A})$ generates a residually small variety.

Even if $\mathbf{A}$ is a strictly simple algebra generating a congruence distributive variety, the minimal variety $\mathcal{V}(\mathbf{A})$ is usually not categorically equivalent to $\mathcal{V}(e(\mathbf{A}))$. For example, let $\mathbf{A}=(\{0,1,2\};m,\oplus,e,f,0)$ where $e,f,0$ are the same as in the previous examples, m is the majority operation with $m(x,y,z)=0$ whenever $\{x,y,z\}=\{0,1,2\}$ and

$$x\oplus y=\begin{cases}2 & \text{if } \{x,y\}=\{0,2\},\\ 0 & \text{otherwise.}\end{cases}$$

As before, $\mathbf{A}$ is a strictly simple algebra in class (**I**) and e is a minimal idempotent of $\mathbf{A}$ with $e(A)=\{0,2\}$. It is easy to check that $e(\mathbf{A})$ is a two–element quasiprimal algebra; hence $\mathcal{V}(e(\mathbf{A}))$ is a minimal discriminator variety. Since $\mathbf{A}$ has a majority operation, $\mathcal{V}(\mathbf{A})$ is a minimal congruence distributive variety. However, $\mathcal{V}(\mathbf{A})$ is not congruence permutable because $A^2\setminus\{(1,2),(2,1)\}$ is a subuniverse of $\mathbf{A}^2$ and hence $\mathbf{A}$ cannot have a Mal′cev term. Thus $\mathcal{V}(\mathbf{A})$ and $\mathcal{V}(e(\mathbf{A}))$ are not categorically equivalent, although both of them are congruence distributive.

Contrary to the congruence distributive case discussed in the previous paragraph, it turns out that when $\mathbf{A}$ is a strictly simple algebra generating a congruence permutable variety, then the variety $\mathcal{V}(\mathbf{A})$ must be categorically equivalent to $\mathcal{V}(e(\mathbf{A}))$. The easiest way to show this is to establish that condition (2) in Lemma 4.3 holds for $\mathcal{V}(\mathbf{A})$. The free algebra $\mathbf{F}=\mathbf{F}_{\mathcal{V}(\mathbf{A})}(1)$ in $\mathcal{V}(\mathbf{A})$ is finite; therefore $\mathbf{F}\cong\mathbf{A}^k$ for some $k<\omega$ since $\mathcal{V}(\mathbf{A})$ is congruence permutable and $\mathbf{A}$ is strictly simple. It follows that $e(\mathbf{F})\cong e(\mathbf{A})^k$. The subalgebra $\mathbf{F}'$ of $\mathbf{F}$ generated by $e(F)$ is isomorphic to $\mathbf{A}^m$ for some m. We have $m=k$ since

$$e(\mathbf{A})^k\cong e(\mathbf{F})=e(\mathbf{F}')\cong e(\mathbf{A})^m.$$

It follows that $\mathbf{F}'=\mathbf{F}$, as required.

We conclude this section with a general remark. Let $\mathbf{A}$ be an arbitrary algebra, let e be an idempotent term of $\mathbf{A}$ and let $\mathbf{F}=\mathbf{F}_{\mathcal{V}(\mathbf{A})}(1)$. In this section we have compared two different conditions: condition $(*)$ of Theorem 3.1 (2) which, in light of Theorem 3.3, may be formulated as

$$(\diamond)\qquad \mathrm{Cg}^{\mathbf{F}}(e(F)\times e(F))=1_{\mathbf{F}}$$

and the invertibility of e which, for comparison, may be formulated as

$$(\diamond\diamond)\qquad \mathrm{Sg}^{\mathbf{F}}(e(F))=\mathbf{F}.$$

The invertibility equation for e from Definition 4.1 shows that $(\diamond)\Leftarrow(\diamond\diamond)$. Theorem 4.4 proves that $(\diamond)\Rightarrow(\diamond\diamond)$ holds when $\mathbf{A}$ is an abelian strictly simple algebra and e is a minimal idempotent. In fact, the implication $(\diamond)\Rightarrow(\diamond\diamond)$ holds for a wider class of algebras and idempotents:

Let $\mathbf{A}$ *be a finite left nilpotent algebra and* e *an idempotent term of* $\mathbf{A}$*. If condition* $(*)$ *in Theorem 3.1* (2) *holds for* $\mathbf{A}$ *and* e*, then* e *is invertible.*

This fact is a consequence of Theorem 3.9 of [4] which may be rephrased slightly to say:

Theorem 4.8. *If* $\mathbf{A}$ *is a finite left nilpotent algebra, then for every algebra* $\mathbf{F} \in \mathcal{V}(\mathbf{A})$ *it is the case that every maximal subuniverse of* $\mathbf{F}$ *is a congruence block.*

The proof of this theorem requires special machinery. The arguments rely heavily on techniques and results developed in [3] and further papers on commutator theory and abelian congruences of finite algebras.

To prove the claim from Theorem 4.8 assume that $\mathbf{A}$ is a finite left nilpotent algebra and $\mathbf{F} = \mathbf{F}_{\mathcal{V}(\mathbf{A})}(1)$. If e is any idempotent term operation and $\mathbf{G} = \mathrm{Sg}^{\mathbf{F}}(e(F)) \neq \mathbf{F}$, then, by Theorem 4.8, G is contained in a block of a proper congruence θ of $\mathbf{F}$. In this case,

$$\mathrm{Cg}^{\mathbf{F}}(e(F) \times e(F)) \subseteq \theta < 1_{\mathbf{A}}.$$

This proves the contrapositive of $(\diamond) \Rightarrow (\diamond\diamond)$.

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