Every Nearly Idempotent Plain Algebra Generates a Minimal Variety

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An algebra $\mathbf{A}$ is **plain** if it is finite, simple and has no non-trivial proper subalgebras. An element $0 \in A$ is an **idempotent element** if $\{0\}$ is a subuniverse and is a **non-idempotent element** otherwise. $\mathbf{A}$ is **idempotent** if each of its elements is idempotent. In this paper we shall say that $\mathbf{A}$ is **nearly idempotent** if $\mathbf{A}$ has at least one idempotent element and $\mathrm{Aut}(\mathbf{A})$ acts transitively on the non-idempotent elements.

In [2], Ágnes Szendrei proves that every idempotent plain algebra generates a minimal variety by showing that an idempotent plain algebra with more than two elements generates a congruence modular variety. The proof is not long, but it relies on the classification theorem in [1] for idempotent plain algebras of size > 2. The proof in [1] of this classification theorem covers several pages. The argument in [2] is completed by directly examining the congruence modular case and the 2-element case and proving for both that an idempotent plain algebra generates a minimal variety. Here we give a short proof of the result using only "$\mathsf{V} = \mathsf{HSP}$". With Theorem 4 we show how to boost the result to a proof that every nearly idempotent plain algebra generates a minimal variety.

We say that $\mathcal{V}$ satisfies condition (E) if $\mathcal{V}$ has a unary term e such that for all basic operations f the identity $f(e(x), \ldots, e(x)) = e(x)$ holds. If $\mathbf{A}$ is an idempotent plain algebra, then $\mathcal{V} = \mathsf{V}(\mathbf{A})$ satisfies condition (E) with $e(x) = x$.

If $\mathbf{A}$ is plain and $\mathsf{V}(\mathbf{A})$ is not minimal, then there is a plain algebra $\mathbf{B} \in \mathsf{V}(\mathbf{A})$ which generates a minimal subvariety. Clearly, $\mathbf{A} \not\cong \mathbf{B}$ in this case. Szendrei's result can be deduced from the following lemma, since it shows that when $\mathbf{A}$ is plain and idempotent and $\mathbf{B} \in \mathsf{V}(\mathbf{A})$ is plain (and of course idempotent), then $\mathbf{A} \cong \mathbf{B}$.

LEMMA 1 *If $\mathbf{A}$ is plain, $\mathcal{V} = \mathsf{V}(\mathbf{A})$ satisfies condition (E) and $\mathbf{B} \in \mathcal{V}$ is idempotent and plain, then $\mathbf{A} \cong \mathbf{B}$.*

Proof: Assuming the hypotheses of the lemma we can find m, a subalgebra $\mathbf{C} \le \mathbf{A}^m$ and a congruence θ on $\mathbf{C}$ such that $\mathbf{C}/\theta \cong \mathbf{B}$. Among all such situations, choose one so that $|C|$ is minimal. If η is a projection kernel restricted to $\mathbf{C}$ and $\eta \le \theta$, then $\mathbf{B} \in \mathsf{H}(\mathbf{C}/\eta) = \mathsf{HS}(\mathbf{A})$. $\mathbf{A}$ is plain and $\mathbf{B}$ is nontrivial, so this yields $\mathbf{A} \cong \mathbf{B}$ and finishes the proof. Otherwise, for each projection kernel η there is a pair $(a, b) \in \eta - \theta$. We claim that $(e(a), e(b)) \in \eta - \theta$ as well.

Of course, $(a, b) \in \eta$ implies $(e(a), e(b)) \in \eta$. Since $\mathbf{C}/\theta \cong \mathbf{B}$ is idempotent, $e(x)\ \theta\ x$ holds on $\mathbf{C}$. Hence $e(a)\ \theta\ a$ and $e(b)\ \theta\ b$ hold. Now $(a, b) \notin \theta$ implies $(e(a), e(b)) \notin \theta$ by transitivity.

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By condition (E), $e(a)$ is an idempotent element of $\mathbf{C}$. Therefore $D = e(a)/\eta$ is a subuniverse of $\mathbf{C}$ containing $e(a)$ and $e(b)$. Since θ is nontrivial on D and $\mathbf{B}$ is plain we must have $\mathbf{D}/\theta \cong \mathbf{B}$. By minimality we get $C = D$. Therefore C is a single η-class for any projection kernel η. This is impossible since the projection kernels intersect to zero. □

Now we begin the proof that every nearly idempotent plain algebra generates a minimal variety. We need two preparatory lemmas.

LEMMA 2 *If* $\mathbf{A}$ *is a nearly idempotent plain algebra, then* $\mathsf{V}(\mathbf{A})$ *satisfies condition* (E).

Proof: Let U be the set of idempotent elements of $\mathbf{A}$ and let e be a unary term of minimal range. Clearly, $e(A) \supseteq U$. If there is an element $u \in e(A) - U$, then the subalgebra generated by u equals $\mathbf{A}$ since $\mathbf{A}$ is plain. In particular, there is a unary term f such that $f(u) \in U$. But now fe has smaller range than e since f collapses two elements of $e(A)$. This contradiction proves that $e(A) = U$. This e satisfies the required identities. □

LEMMA 3 *Assume that* $\mathcal{V} = \mathsf{V}(\mathbf{A})$ *where* $\mathbf{A}$ *is plain, but not idempotent. If* $\mathrm{Aut}(\mathbf{A})$ *acts transitively on the non-idempotent elements of* $\mathbf{A}$*, then* $\mathbf{A} \cong \mathbf{F}_{\mathcal{V}}(1)$.

Proof: Let $a \in A$ be a non-idempotent element. Since $\mathbf{A} \in \mathcal{V}$ it suffices to observe that $\mathbf{A}$ satisfies the universal mapping property with respect to the set $\{a\}$ and some generating class of algebras for $\mathcal{V}$. We take $\{\mathbf{A}\}$ for this generating class. Now any function $f : \{a\} \to \mathbf{A}$ where $f(a)$ is a non-idempotent element has an extension to some homomorphism $\hat{f} : \mathbf{A} \to \mathbf{A}$. Simply take an $\hat{f} \in \mathrm{Aut}(\mathbf{A})$ such that $\hat{f}(a) = f(a)$. This extension is unique since a generates $\mathbf{A}$. If instead $f(a)$ is an idempotent element, then the constant map $\hat{f} : \mathbf{A} \to \mathbf{A} : x \mapsto f(a)$ is the unique extension of f to a homomorphism from $\mathbf{A}$ to $\mathbf{A}$. □

THEOREM 4 *If* $\mathbf{A}$ *is nearly idempotent and plain and* $\mathbf{B} \in \mathsf{V}(\mathbf{A})$ *is plain, then* $\mathbf{A} \cong \mathbf{B}$*. Hence every nearly idempotent plain algebra generates a minimal variety.*

Proof: Together, Lemmas 1 and 2 prove that if $\mathbf{B} \in \mathsf{V}(\mathbf{A})$ is plain, then $\mathbf{A} \cong \mathbf{B}$ or else $\mathbf{B}$ is not idempotent. But if the latter holds and u is a non-idempotent element of $\mathbf{B}$, then $\mathbf{B} = \mathrm{Sg}^{\mathbf{B}}(\{u\})$ is a non-trivial homomorphic image of $\mathbf{A}$ by Lemma 3, so $\mathbf{A} \cong \mathbf{B}$ holds in this case as well. The arguments in the paragraph preceding Lemma 1 explain why this conclusion proves that every nearly idempotent plain algebra generates a minimal variety. □

There is a plain algebra $\mathbf{A}$ which does not generate a minimal variety but whose automorphism group acts transitively on the non-idempotent elements. To construct one such $\mathbf{A}$, begin with the idempotent reduct of a finite, 1-dimensional vector space and add in all the translations, $x \mapsto x + a$, as new unary operations. $\mathrm{Aut}(\mathbf{A})$ contains all the translations and so acts transitively on A. If θ is the kernel of the function $A \times A \to A : (x, y) \mapsto x - y$, then $(\mathbf{A} \times \mathbf{A})/\theta$ generates a non-trivial, proper subvariety of $\mathsf{V}(\mathbf{A})$.

There are also plain algebras with idempotent elements which do not generate minimal varieties. Of course, by Theorem 4 any such example must have at least 2 non-idempotent elements. To construct an example, let A be any set which properly contains $\{0, 1\}$. Take as basic operations all those operations p on A such that $p(A^n) \neq A$ and $p(w, \ldots, w) = w$ for $w \in A - \{0, 1\}$. Then $\mathbf{A}$ is plain and has only two non-idempotent elements. $\mathsf{V}(\mathbf{A})$ is not minimal since the subvariety defined by all the identities of the form $p(\bar{x}) = p(\bar{y})$ where p is a basic operation and $\bar{x}$, $\bar{y}$ are arbitrary tuples of variables is proper and non-trivial. (A non-trivial member of this variety may be constructed as a quotient of $\mathbf{A}^{|A|}$.) This paragraph and the preceding one show that neither of the two conditions defining the phrase "nearly idempotent" can be removed if the result of Theorem 4 is to hold.

The paper [3] introduces a class of examples of nearly idempotent plain algebras with exactly one idempotent element. These algebras are used in [4] to provide examples of minimal, locally finite varieties of groupoids which are inherently non-finitely based.

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References

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