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Residual bounds for varieties of modules

KEITH A. KEARNES

In [3] and also in [4], Freese and McKenzie prove that a finitely generated congruence modular variety is residually small if and only if it is residually less than n for some $n < \omega$. In fact, they showed that if $\mathbf{A}$ generates a residually small congruence modular variety and $|A| = m$, then $\mathcal{V}(\mathbf{A})$ is residually less than the bound $l! + m$ where $l = m^{m^m+3}$. They obtained this result by associating to $\mathcal{V}(\mathbf{A})$ a finite ring with unit, $\mathbf{R} = \mathbf{R}(\mathcal{V}(\mathbf{A}), m)$, and relating the size of the subdirectly irreducible algebras in $\mathcal{V}(\mathbf{A})$ to the size of subdirectly irreducible $\mathbf{R}$-modules. Their final bound is based on two estimates. First they estimate that $|R| \le l = m^{m^m+3}$. Then they show that if $\mathbf{R}$ is a ring of cardinality l, the subdirectly irreducible $\mathbf{R}$-modules are bounded by $l!$. Using these estimates and applying modular commutator theory leads them to their result.

In [1], we asked whether there is a function f such that for any residually small variety $\mathcal{V}$ with the CEP that is generated by an algebra of cardinality m, one has that $\mathcal{V}$ is residually less than $f(m)$. This question was answered affimatively by E. Kiss in [2]. His proof does not produce a suitable function and no choice for f is known.

We will show that, if $\mathbf{R}$ is finite, no subdirectly irreducible $\mathbf{R}$-module can have larger cardinality than $|R|$. This has the effect of removing the factorial sign in Freese and McKenzie's result. It has a second consequence of providing a tight bound on the size of subdirectly irreducibles in finitely generated, congruence modular varieties with the CEP.

THEOREM 1. *Let* $\mathbf{R}$ *be a ring with unit. If* $\mathbf{R}$ *is left noetherian and left artinian, they any (unital) subdirectly irreducible left* $\mathbf{R}$*-module* $\mathbf{A}$ *has cardinality* $\le |R|$. *If* $\mathbf{R}$ *is finite, then* $|A|$ *divides* $|R|$.

Proof. We can find a descending chain of left ideals:

$$\mathbf{R} = \mathbf{L}_0 \supseteq \mathbf{L}_1 \supseteq \cdots \supseteq \mathbf{L}_n = (0)$$

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such that $\mathbf{L}_{i+1}$ is maximal in $\mathbf{L}_i$. Let $\mathbf{A}$ be a subdirectly irreducible left $\mathbf{R}$-module of maximum cardinality. We also assume that $\mathbf{A}$ is maximal with respect to essential extensions. This means that $\mathbf{A}$ is an injective $\mathbf{R}$-module. It will suffice to prove that $|A|$ is less than $|R|$ and that if $\mathbf{R}$ is finite then $|A|$ divides $|R|$.

Let $\mathbf{A}_i = \{x \in A \mid \mathbf{L}_i \cdot x = (0)\}$. We have:

$$(0) = \mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \cdots \subseteq \mathbf{A}_n = \mathbf{A}.$$

Since

$$0 \to \mathbf{L}_k/\mathbf{L}_{k+1} \to \mathbf{R}/\mathbf{L}_{k+1} \to \mathbf{R}/\mathbf{L}_k \to 0$$

in an exact sequence of $\mathbf{R}$-modules, and $\mathbf{A}$ is an injective $\mathbf{R}$-module, we get an exact sequence of abelian groups:

$$0 \to (\mathbf{R}/\mathbf{L}_k)^* \to (\mathbf{R}/\mathbf{L}_{k+1})^* \to (\mathbf{L}_k/\mathbf{L}_{k+1})^* \to 0.$$

In the previous line, the notation $\mathbf{X}^*$ means $\mathbf{Hom_R}(\mathbf{X}, \mathbf{A})$. Now,

$$\alpha : (\mathbf{R}/\mathbf{L}_i)^* \xrightarrow{\sim} \mathbf{A}_i : f \mapsto f(1)$$

is an isomorphism of abelian groups. Thus, the last exact sequence yields an isomorphism of abelian groups: $\mathbf{A}_{k+1}/\mathbf{A}_k \cong \mathbf{Hom_R}(\mathbf{L}_k/\mathbf{L}_{k+1}, \mathbf{A})$. $\mathbf{L}_k/\mathbf{L}_{k+1}$ is simple, so $\mathbf{Hom_R}(\mathbf{L}_k/\mathbf{L}_{k+1}, \mathbf{A}) = \mathbf{Hom_R}(\mathbf{L}_k/\mathbf{L}_{k+1}, \mathit{Soc}\ \mathbf{A})$ where $\mathit{Soc}\ \mathbf{A}$ denotes the minimum submodule (the "socle") of $\mathbf{A}$.

If $\mathbf{L}_k/\mathbf{L}_{k+1} \cong \mathit{Soc}\ \mathbf{A}$, then we can find an isomorphism $\beta : \mathit{Soc}\ \mathbf{A} \xrightarrow{\sim} \mathbf{L}_k/\mathbf{L}_{k+1}$. We now compose each map $f \in \mathbf{Hom_R}(\mathbf{L}_k/\mathbf{L}_{k+1}, \mathit{Soc}\ \mathbf{A})$ with β and get an isomorphism between the abelian group $\mathbf{Hom_R}(\mathbf{L}_k/\mathbf{L}_{k+1}, \mathit{Soc}\ \mathbf{A})$ (which we already know to be isomorphic to $\mathbf{A}_{k+1}/\mathbf{A}_k$) and the additive group of $\mathbf{End_R}(\mathbf{L}_k/\mathbf{L}_{k+1})$. Notice also that $\mathbf{End_R}(\mathbf{L}_k/\mathbf{L}_{k+1}) = \mathbf{D}$, where $\mathbf{D}$ is a division ring over which $\mathbf{L}_k/\mathbf{L}_{k+1}$ is a vector space. Now, if $\mathbf{L}_k/\mathbf{L}_{k+1} \not\cong \mathit{Soc}\ \mathbf{A}$, then $|\mathbf{A}_{k+1}/\mathbf{A}_k| = |\mathbf{Hom_R}(\mathbf{L}_k/\mathbf{L}_{k+1}, \mathit{Soc}\ \mathbf{A})| = 1$. In either case, $|\mathbf{A}_{k+1}/\mathbf{A}_k|$ ($=$ or $|\mathbf{D}|$) is less than or equal to $|\mathbf{L}_k/\mathbf{L}_{k+1}|$. If $\mathbf{L}_k/\mathbf{L}_{k+1}$ is finite, then $|\mathbf{A}_{k+1}/\mathbf{A}_k|$ divides $|\mathbf{L}_k/\mathbf{L}_{k+1}|$.

To finish the proof, we simply notice that

$$|\mathbf{A}| = \prod_{i=1}^{n} |\mathbf{A}_i/\mathbf{A}_{i-1}| \leq \prod_{i=1}^{n} |\mathbf{L}_{i-1}/\mathbf{L}_i| = |\mathbf{R}|,$$

and that the left side divides the right when $\mathbf{R}$ is finite. □

After this theorem was proved G. Bergman supplied a different proof of the part of the theorem that concerns finite rings. His proof actually classifies the isomorphism types of finite subdirectly irreducible modules over an aribitrary ring in terms of algebraic information about the ring. Then R. McKenzie pointed out that the arguments in [4], when applied to infinite rings instead of finite rings, actually proved the part of Theorem 1 that deals with infinite rings under the hypothesis that **R** is only left noetherian. This observation along with either Theorem 1 or Bergman's theorem, to cover the cases when **R** is finite, implies that if **R** is left noetherian, then the variety of **R**-modules is residually $\leq |R|$. The converse of this is false. The variety of **S**-modules can be residually $\leq |S|$ even if **S** is not noetherian as the following example shows.

EXAMPLE 2. Take **R** to be a finite non-zero ring with unit and X to be an infinite set. Let **T** be the infinite direct power $\mathbf{R}^X$. **T** is clearly not noetherian; any infinite properly descending chain of subsets of X yields an infinite properly ascending chain of ideals corresponding to projection congruences. Now suppose that **M** is a subdirectly irreducible **T**-module. For any partition of X into two subsets Y and Z we have a decomposition $\mathbf{T} \cong \mathbf{I} \times \mathbf{J}$ where **I** and **J** are the ideals $\mathbf{R}^Y \times 0$ and $0 \times \mathbf{R}^Z$. $\mathbf{M} = \mathbf{M}/\mathbf{IM} \oplus \mathbf{M}/\mathbf{JM}$, so either $\mathbf{IM} = 0$ or $\mathbf{JM} = 0$. Let **K** be the join of all ideals **I** of this form which have the property that $\mathbf{IM} = 0$. **K** is an ideal associated with an ultraproduct congruence on $\mathbf{R}^X$ and $\mathbf{KM} = 0$. **M** is a subdirectly irreducible **T/K**-module and, since **T/K** is an ultrapower of the finite ring **R**, this means that $\mathbf{T}/\mathbf{K} \cong \mathbf{R}$ and **M** is a subdirectly irreducible module over a finite ring. Theorem 1 applies and shows that $|M| \leq |R| \leq |T|$.

It is well known that any variety of modules is residually small so, by the results in [5], no subdirectly irreducible **S**-module can be larger than $2^{|S|}$ for any ring S. However, without some restrictions on **S**, it is possible to have subdirectly irreducibles larger than $|S|$.

EXAMPLE 3. If λ is an infinite cardinal and **S** is the ring of all $\lambda \times \lambda$ matrices over the rational numbers which differ from a scalar matrix in at most finitely many entries, then $|S| = \lambda$ but the space of all λ-dimensional column vectors over the rationals forms a subdirectly irreducible module of size 2^λ.

In Example 3, **S** has a maximal chain of left ideals that is inversely well ordered. Just take

$$\mathbf{S} = \mathbf{L}_0 \supseteq \mathbf{L}_1 \supseteq \cdots$$

where $\mathbf{L}_i$ is the left ideal of all matrices in $\mathbf{S}$ whose first i columns consist of zeros. The condition of being left noetherian is equivalent to the condition that *every* maximal chain of left ideals is inversely well ordered. This example seems to show that for infinite cardinals McKenzie's version is, in some sense, tight.

Next we address the problem of finding a function f such that every residually small, congruence modular variety with the CEP generated by a finite algebra of size n is residually less than $f(n)$. The notation and results of commutator theory that we shall use can be found in [4]. By a result of Kiss, every modular variety with the CEP satisfies the commutator conditions C2 and $\mathbf{R}$. If $\mathbf{A}$ is of size $\leq n$ and generates a variety with C2, then every nonabelian subdirectly irreducible algebra in $\mathscr{V} = \mathscr{V}(\mathbf{A})$ is a member of $\mathbf{HS}(\mathbf{A})$ and so is no bigger than n. Hence, the subdirectly irreducibles that are bigger than n must be abelian. If $\mathscr{V}$ satisfies $\mathbf{R}$, then every abelian algebra in $\mathscr{V}$ lies in $\mathscr{V}(\mathbf{A}/[1, 1])$. (The proof of this last claim does not appear in [4], but it appears in [1] and, in any case, is very easy to discover.) Now, by the relationship between modular abelian varieties and varieties of modules, we see that if we can find a function g such that $g(n)$ bounds the size of the subdirectly irreducible modules in any variety generated by a module of size $\leq n$, then a suitable choice for f is the function $f(n) = \max\{g(n), n\}$.

If a module $\mathbf{N}$ of cardinality n generates a variety $\mathscr{V}$, then we can assume that the ring $\mathbf{R}$ of the variety acts faithfully on $\mathbf{N}$ (if we are only concerned with the residual character of $\mathscr{V}$). Thus, $\mathbf{R} \subseteq \mathbf{End}(\mathbf{N})$. But every endomorphism of $\mathbf{N}$ is determined by its values on a minimal set of generators $G = \{g_1, \ldots, g_t\}$. Hence,

$$|R| \leq |\mathbf{End}(\mathbf{N})| \leq |N|^{|G|} = n^t.$$

An abelian group of cardinality n can be generated by $\lg n = \log_2 n$ of its elements, so $|R| \leq n^{\lg n}$. By Theorem 1, $\mathscr{V}$ is residually $\leq n^{\lg n}$. We record this as:

PROPOSITION 4. *Every residually small modular variety with the* CEP *that is generated by an algebra of size n is residually $\leq n^{\lg n}$.* □

In this estimate, equality can hold only if $|N| \leq 2$. Thus, $\mathscr{V}$ is residually less than $n^{\lg n}$ whenever $|N| > 2$. To show that equality holds only when $|N| \leq 2$, observe that if $\mathscr{V}$ has a subdirectly irreducible module of cardinality equal to $n^{\lg n}$ then every minimal set of generators for $\mathbf{N}$ has $\lg n$ elements and $\mathbf{N}$ is isomorphic to $\mathbf{Z}_2^t$. $\mathbf{R}$ must be the full additive endomorphism ring of $\mathbf{N}$ which is the $t \times t$ matrix ring over the two element field. This implies that N is the only subdirectly irreducible module in $\mathscr{V}$ which forces $n = |N| = n^{\lg n}$ or $n \leq 2$. Notice that every congruence modular variety generated by a 2-element algebra is residually small and therefore all nontrivial subdirectly irreducible members have 2 elements.

The next example shows that $n^{\lg n}$ is a fairly good estimate for the function that we are looking for, at least for modular varieties.

EXAMPLE 5. Let $\mathbf{F} = \mathbf{F}_2 \subseteq \mathbf{F}_{2^m} = \mathbf{K}$ be the 2-element and 2^m-element fields, respectively. Let $\mathbf{R}$ be the subring of the $m \times m$ upper triangular matrices over $\mathbf{K}$ whose entries below the first row are restricted to lie in $\mathbf{F}$. $\mathbf{R}$ has a faithful module $\mathbf{N}$ consisting of all the column vectors whose first component lies in $\mathbf{K}$, but whose other entries lie in $\mathbf{F}$. $|N| = n = 2^{2m-1}$ and, since $\mathbf{N}$ is faithful, it generates the variety of all $\mathbf{R}$-modules. The $\mathbf{R}$-module consisting of all column vectors whose components lie in $\mathbf{K}$ has size 2^{m^2} and is subdirectly irreducible with minimal submodule equal to those vectors whose coordinates are zero except for the first one. This module is a member of $\mathcal{V}(\mathbf{N})$ so $\mathcal{V}(\mathbf{N})$ is not residually $< 2^{(\frac{1}{2}\lg n)^2} = n^{\frac{1}{4}\lg n}$. This is an example of a residually small, congruence modular variety with the CEP that is generated by an n-element algebra (where $n = 2^{2m-1}$) which is *not* residually less than $n^{\frac{1}{4}\lg n}$.

REFERENCES

[1] KEARNES, K., *Topics in Algebra: Injective Completeness, Fine Spectra and Relative Presentability*, Ph.D. Thesis, U.C. Berkeley, 1988.
[2] KISS, E. and PROHLE, P., *Problems and results in tame congruence theory*, Preprint No. 60, Math. Inst. Hung. Acad. Sci., 1988.
[3] FREESE, R. and MCKENZIE, R., *Residually small varieties with modular congruence lattices*. Trans. Amer. Math. Soc. *264* (1981), 419–430.
[4] FREESE, R. and MCKENZIE, R., *Commutator Theory for Congruence Modular Varieties*, LMS Lecture Notes Series No. 125, Cambridge University Press, 1987.
[5] TAYLOR, W., *Residually small varieties*, Algebra Universalis *2* (1972), 33–53.

The University of Hawaii at Manoa
Department of Mathematics
Honolulu