Idempotent Simple Algebras *

Keith A. Kearnes †

Abstract

We show that every idempotent simple algebra which has a skew congruence in a power either has an absorbing element or is a subreduct of an affine module. We refine this result for idempotent algebras which have no nontrivial proper subalgebras. One corollary we obtain is a new proof of Á. Szendrei's classification theorem for minimal locally finite idempotent varieties. We partially extend this classification to non-locally finite varieties. Another corollary is a complete classification of all minimal varieties of modes.

1 Introduction

An operation $f(x_1, \ldots, x_n)$ on a set U is said to be **idempotent** if $f(u, u, \ldots, u) = u$ is true of any $u \in U$. An algebra is idempotent if each of its fundamental operations is idempotent. Equivalently, $\mathbf{A}$ is idempotent if every constant function $c : A \to A$ is an endomorphism. As is usual, we say that $\mathbf{A}$ is **simple** if it has exactly two congruences. In our paper we prove the following theorem concerning idempotent simple algebras.

THEOREM 1.1 *If* $\mathbf{A}$ *is an idempotent simple algebra, then exactly one of the following conditions is true.*

- (a) $\mathbf{A}$ *has a unique absorbing element.*
- (b) $\mathbf{A}$ *is a subalgebra of a simple reduct of a module.*
- (c) *Every finite power of* $\mathbf{A}$ *is skew-free.*

We call $0 \in A$ an **absorbing element** for $\mathbf{A}$ if whenever $t(x, \bar{y})$ is an $(n + 1)$-ary term operation of $\mathbf{A}$ such that $t^{\mathbf{A}}$ depends on x and $\bar{a} \in A^n$, then $t^{\mathbf{A}}(0, \bar{a}) = 0$. We say that $\mathbf{A}^n$ is **skew-free** if the only congruences on $\mathbf{A}^n$ are the product congruences. Since $\mathbf{A}$ is simple in the above theorem, the product congruences on $\mathbf{A}^n$ are just the canonical factor congruences. Thus, the statement that "every finite power of $\mathbf{A}$ is skew-free" is exactly the claim that $\mathbf{Con}\mathbf{A}^n \cong \mathbf{2}^n$ for each $n < \omega$.

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The class of simple algebras of type (c) includes the class of all idempotent functionally complete algebras and this class is interdefinable with the class of all idempotent algebras. Hence the simple algebras of type (c) resist classification. A similar statement may be made about the simple algebras of type (a). Therefore, the theorem is properly viewed as a criterion for an idempotent simple algebra to be a subreduct of a module. For, if an idempotent simple algebra has a skew congruence in a power but has no absorbing element, then it is a subreduct of a module.

It is plausible that our theorem will lead to a complete classification of all minimal varieties of idempotent algebras. By Magari's Theorem, any variety contains a simple algebra (see [8]); hence a minimal idempotent variety is generated by an algebra of type (a), (b) or (c). It is not hard to prove that any minimal variety containing a simple algebra of type (a) is equivalent to the variety of semilattices. Any minimal variety containing a simple algebra of type (b) is equivalent to the variety of sets or to a variety of affine modules over a simple ring. (In this paper, all rings have a unit element, an **affine R-module** is the idempotent reduct of an **R**-module and an **affine vector space** is the idempotent reduct of a vector space.) The minimal varieties containing only simple algebras of type (c) are difficult to analyze. The only examples that we know are congruence distributive. If such a variety is not congruence distributive, it cannot be locally finite.

The **entropic law** for an algebra $\mathbf{A}$ is the statement that for $f \in \mathrm{Clo}_m\mathbf{A}$ and $g \in \mathrm{Clo}_n\mathbf{A}$ and an $m \times n$ array of elements of A,

$$\begin{bmatrix} u_1^1 & u_1^2 & \cdots & u_1^n \\ u_2^1 & u_2^2 & \cdots & \\ \vdots & \vdots & \ddots & \\ u_m^1 & u_m^2 & & u_m^n \end{bmatrix},$$

we have $f(\overline{g(\bar{u}_i)}) = g(\overline{f(\bar{u}^j)})$. Idempotent entropic algebras are called **modes** (see [11]). We prove that there is no simple mode of type (c) and that the only one of type (a), up to equivalence, is the 2-element semilattice. This leads to a good description of simple modes and a classification of all minimal varieties of modes. Up to equivalence, the minimal varieties are: the variety of semilattices, the variety of sets and any variety of affine vector spaces.

Our notation for congruences in powers shall roughly follow [2]. The projection homomorphism from $\mathbf{A}^\kappa$ onto a sequence of coordinates σ will be denoted π_σ. For example, in $\mathbf{A}^3$ we have $\pi_{01}((a,b,c)) = (a,b)$ while $\pi_{10}((a,b,c)) = (b,a)$. We will write η_σ for the kernel of π_σ and write α_σ for $\pi_\sigma^{-1}(\alpha)$ where α is a congruence on $\mathbf{A}^\sigma$. We write η'_σ to denote the canonical factor congruence which complements η_σ. We prefer to write 0 in place of $\eta_{01\cdots(n-1)}$ and 1 in place of $\eta_\emptyset$. A congruence on $\mathbf{A}^n$ of the form

$$\alpha_0 \wedge \beta_1 \wedge \cdots \wedge \nu_{n-1}$$

will be called a **product congruence**. A congruence which is not a product congruence is a **skew congruence**. Hence a power of $\mathbf{A}$ is skew-free iff it has no skew congruences.

2 Skew Congruences in Powers

In this section we show that if $\mathbf{A}$ is a simple idempotent algebra which has a skew congruence in a finite power, then $\mathbf{A}$ has a unique absorbing element or else $\mathbf{A}$ is abelian. We shall find it convenient to restrict our arguments to algebras that have at least three elements. The reason for this is that the 2-element set with no operations is an algebra which is abelian, but has an absorbing element. In fact, each of the two elements of the 2-element set is an absorbing element, so this algebra does not contradict Theorem 1.1; it belongs in class (b). But the fact that it has an absorbing element and is at the same time abelian makes it difficult to separate classes (a) and (b) of Theorem 1.1 when $|A| = 2$. So, we now observe simply that Theorem 1.1 can be established by hand in the 2-element case and we leave it to the reader to do this. Alternately, one can refer to Post's description of 2-element algebras in [10] or to J. Berman's simplification of Post's argument in [1]. From these sources one finds that a 2-element idempotent algebra must be, up to term equivalence, one of the following:

(i) a semilattice,

(ii) a set,

(iii) an affine vector space or

(iv) a member of a congruence distributive variety.

A semilattice belongs only to class (a) of Theorem 1.1, a set or an affine vector space belongs only to class (b) of Theorem 1.1 and a member of a congruence distributive variety belongs only to class (c) of Theorem 1.1.

Here is how we shall use our hypothesis that $|A| > 2$. We shall need to know that any idempotent simple algebra of cardinality greater than two (i) has at most one absorbing element and (ii) is not essentially unary. We prove this now.

LEMMA 2.1 *If $\mathbf{A}$ is an idempotent simple algebra, then the following are equivalent.*

(i) *$\mathbf{A}$ has more than one absorbing element.*

(ii) *$\mathbf{A}$ is essentially unary.*

(iii) *$\mathbf{A}$ is term equivalent to the 2-element set.*

Proof: We shall prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. Assume (i) and deny (ii). Then $\mathbf{A}$ has distinct absorbing elements 0 and 1 and a term $t(x, y, \bar{z})$ which depends on at least x and y in $\mathbf{A}$. Since 0 and 1 are absorbing we have

$$0 = t^{\mathbf{A}}(0, 1, 1, \ldots, 1) = 1,$$

a contradiction.

Now assume (ii). If $s(x, \bar{y})$ is a term that depends on x in $\mathbf{A}$, then since $\mathbf{A}$ is essentially unary and idempotent we get

$$s^{\mathbf{A}}(x, y_0, \ldots, y_k) = s^{\mathbf{A}}(x, x, \ldots, x) = x.$$

Hence any term depending on a variable is projection onto that variable. It is impossible for $\mathbf{A}$ to have a term depending on no variables, since $\mathbf{A}$ is idempotent. Hence every term of $\mathbf{A}$ is projection onto some variable. (iii) follows from this and the simplicity of $\mathbf{A}$.

If (iii) holds, then each of the elements of $\mathbf{A}$ is an absorbing element. Hence (i) holds. This completes the lemma. □

Henceforth we assume that $|A| > 2$.

Definition 2.2 If α and β are congruences on $\mathbf{A}$, then α **centralizes** β if whenever $p(x, \bar{y}) \in \mathrm{Pol_{n+1}}\mathbf{A}$ and $(u, v), \in \alpha$, $(a_i, b_i) \in \beta$ the implication

$$p(u, \bar{a}) = p(u, \bar{b}) \Rightarrow p(v, \bar{a}) = p(v, \bar{b})$$

holds. We denote the fact that α centralizes β by writing $C(\alpha, \beta)$. An algebra $\mathbf{A}$ is said to be **abelian** if $C(1_{\mathbf{A}}, 1_{\mathbf{A}})$.

We remark that the implication described in this definition is called the α, β-**term condition**. When $\alpha = \beta = 1$, then it is simply called the **term condition**. "$\mathbf{A}$ satisfies the term condition" is synonymous with "$\mathbf{A}$ is abelian." For additional properties of abelian congruences and algebras, see Chapter 3 of [4]. Here are a few facts we shall need.

- (i) An algebra $\mathbf{A}$ is abelian if and only if the diagonal of $\mathbf{A}^2$ is a congruence class.
- (ii) There is always a largest congruence α which centralizes a given β. This congruence is called the **centralizer** of β and it is denoted $(0 : \beta)$.
- (iii) It is the case that $\alpha \wedge \beta = 0 \Rightarrow C(\alpha, \beta) \Leftrightarrow \alpha \leq (0 : \beta)$.
- (iv) If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and α and β are congruences on $\mathbf{A}$, then $C(\alpha, \beta) \Rightarrow C(\alpha|_B, \beta|_B)$.

All of the properties of $C(x, y)$ mentioned in this paragraph have easy proofs.

We begin with a first approximation to the main theorem.

THEOREM 2.3 *If $\mathbf{A}$ is an idempotent simple algebra, then exactly one of the following is true.*

- (a) *$\mathbf{A}$ has a unique absorbing element.*
- (b) *$\mathbf{A}$ is abelian.*
- (c) *Every finite power of $\mathbf{A}$ is skew-free.*

Proof: The theorem is true when $|A| = 2$, so we focus only on the case where $|A| > 2$.

First we argue that $\mathbf{A}$ belongs to at least one of the classes $(a), (b)$, or (c) described in the theorem. Later we explain why $\mathbf{A}$ cannot belong to two different classes. Assume that $\mathbf{A}$ is not in class (c). Then for some $m > 1$ there is a skew-congruence on $\mathbf{A}^m$. We assume that m is chosen minimally for this. Let θ be a skew congruence of $\mathbf{A}^m$. We may choose i so that $\eta_i \not\geq \theta$ since $\theta \neq 0$. By permuting coordinates if necessary, we may assume that $i = 0$. If $\theta \geq \eta_0'$, then θ/η_0' is a skew congruence on $\mathbf{A}^m/\eta_0' \cong \mathbf{A}^{m-1}$. This contradicts the minimality of m. Hence $\theta \not\geq \eta_0'$. We shall split our argument into two cases. Case 1: $\mathbf{A}^m$ has a congruence δ such that $0 < \delta < \eta_0'$. Case 2: $\mathbf{A}^m$ has no congruence δ such that $0 < \delta < \eta_0'$.

Proof for Case 1: Assume that $\mathbf{A}^m$ has a congruence δ such that $0 < \delta < \eta_0'$. We shall argue that $\mathbf{A}$ has an absorbing element. Each η_0'-class of $\mathbf{A}^m$ is a subalgebra of $\mathbf{A}^m$ since $\mathbf{A}$ is idempotent. Furthermore, each of these subalgebras is isomorphic to $\mathbf{A}$ via the restriction of the first coordinate projection. If $\mathbf{B} = \mathbf{A} \times \{u\}$, $u \in A^{m-1}$, is such a subalgebra, then $\delta|_{\mathbf{B}}$ is a congruence on $\mathbf{B}$. But, as $\mathbf{B} \cong \mathbf{A}$, $\mathbf{B}$ is simple. Hence $\delta|_{\mathbf{B}} = 0_{\mathbf{B}}$ or $1_{\mathbf{B}}$. Define

$$U = \{u \in A^{m-1} \mid \delta|_{A\times\{u\}} = 1_{A\times\{u\}}\}.$$

We cannot have $U = \emptyset$, for this is equivalent to $\delta = 0$. We cannot have $U = A^{m-1}$, for this is equivalent to $\delta = \eta_i'$. Hence there is a $u \in U$ and a $v \in A - U$. If $u = (b_2, \ldots, b_{m-1})$ and $v = (c_2, \ldots, c_{m-1})$, then by sequentially changing each b_j to the value c_j in these $(m-1)$-tuples we find that for some i we must have

$$(c_2, \ldots, c_{i-1}, b_i, b_{i+1}, \ldots, b_{m-1}) \in U \quad \text{and} \quad (c_2, \ldots, c_{i-1}, c_i, b_{i+1}, \ldots, b_{m-1}) \notin U.$$

Consider the subalgebra of $\mathbf{C} \leq \mathbf{A}^n$ which has universe

$$A \times \{c_2\} \times \cdots \times \{c_{i-1}\} \times A \times \{b_{i+1}\} \times \cdots \times \{b_{m-1}\}.$$

Note that $\mathbf{C} \cong \mathbf{A}^2$ via projection onto coordinates 0 and i. Furthermore, $\delta|_{\mathbf{C}}$ is a congruence which has

$$A \times \{c_2\} \times \cdots \times \{c_{i-1}\} \times \{b_i\} \times \{b_{i+1}\} \times \cdots \times \{b_{m-1}\}$$

as a congruence class, but not

$$A \times \{c_2\} \times \cdots \times \{c_{i-1}\} \times \{c_i\} \times \{b_{i+1}\} \times \cdots \times \{b_{m-1}\}.$$

It follows that

$$0_{\mathbf{C}} < \delta|_{\mathbf{C}} < \eta_0'|_{\mathbf{C}}.$$

From the isomorphism $\mathbf{C} \cong \mathbf{A}^2$ we deduce that $\mathbf{A}^2$ has a skew congruence which lies below a projection kernel. By the minimality assumption on m, it must be that $m = 2$. We proceed with the knowledge that $m = 2$ retaining the definitions of δ and U from above. Note that $U \subseteq A^{m-1} = A$ now.

Claim. If $t(x, \bar{y})$ is an $(n+1)$-ary term which depends on x in $\mathbf{A}$ and $\bar{s} \in A^n$, then $t^{\mathbf{A}}(U, \bar{s}) \subseteq U$.

Proof of Claim: Assume otherwise that $t(x, \bar{y})$ depends on x in $\mathbf{A}$ and for some $u \in U$ we have $t^{\mathbf{A}}(u, \bar{s}) = v \notin U$. Since $t^{\mathbf{A}}(x, \bar{y})$ depends on x, there is some $\bar{r} \in A^n$ such that

$t^{\mathbf{A}}(x, \bar{r})$ is non-constant. Recall that $u \in U$ means precisely that $A \times \{u\}$ is a nontrivial δ-class. Applying the polynomial $t^{\mathbf{A}\times\mathbf{A}}(x, \overline{(r_i, s_i)})$ to this class, we obtain that the set

$$t^{\mathbf{A}\times\mathbf{A}}(A \times \{u\}, \overline{(r_i, s_i)}) = t^{\mathbf{A}}(A, \bar{r}) \times \{v\}$$

is a nontrivial subset contained in $A \times \{v\}$. Since this subset is a polynomial image of a δ-class, δ restricts nontrivially to $A \times \{v\}$. As argued above, this implies that $A \times \{v\}$ is a δ-class. But this forces $v \in U$ which is contrary to our assumption. This proves the Claim.

The Claim just proven implies that U is a class of a congruence on $\mathbf{A}$. Since we have shown in the first paragraph of the argument for Case 1 that $U \neq A^{m-1} = A$, we cannot have U a class of the universal congruence, 1. Since $\mathbf{A}$ is simple, U is a 0-class. Thus $U = \{0\}$ for some element $0 \in A$. By the Claim, if $t(x, \bar{y})$ depends on x in $\mathbf{A}$ and $\bar{s} \in A^n$, then $t^{\mathbf{A}}(0, \bar{s}) = 0$. This is precisely what it means for 0 to be an absorbing element for $\mathbf{A}$. By Lemma 2.1 and our assumption that $|A| > 2$, 0 is the unique absorbing element for $\mathbf{A}$. Hence, any algebra in Case 1 has a unique absorbing element and therefore belongs to class of algebras of type (a) of the theorem.

Proof for Case 2: Now assume that $\mathbf{A}^m$ has no congruence δ such that $0 < \delta < \eta_0'$. Since $\theta \not\geq \eta_0'$, we have $\theta \wedge \eta_0' < \eta_0'$. Because we are in Case 2, this means that $\theta \wedge \eta_0' = 0$. Hence $C(\theta, \eta_0')$ and so $\theta \leq (0 : \eta_0')$. As η_0 and η_0' are complements, we also have $\eta_0 \wedge \eta_0' = 0$ and so $C(\eta_0, \eta_0')$ also holds. Thus $\eta_0 \leq (0 : \eta_0')$. Altogether this implies that $\theta \vee \eta_0 \leq (0 : \eta_0')$. But $\eta_0 \not\geq \theta$, by choice, and $\eta_0 \prec 1$ since $\mathbf{A}$ is simple. Hence $\theta \vee \eta_0 = 1$ and so $(0 : \eta_0') = 1$ and it follows that $C(1, \eta_0')$ holds. If $\mathbf{B} = \mathbf{A} \times \{u\}$, $u \in A^{m-1}$, then $\mathbf{B}$ is a subalgebra of $\mathbf{A}^m$ isomorphic to $\mathbf{A}$ as we explained in the argument for Case 1. Now, restricting congruences to $\mathbf{B}$ we get that $C(1_{\mathbf{A}}|_{\mathbf{B}}, \eta_0'|_{\mathbf{B}})$ holds. But

$$(1_{\mathbf{A}})|_{\mathbf{B}} = 1_{\mathbf{B}} = \eta_0'|_{\mathbf{B}},$$

so $C(1_{\mathbf{B}}, 1_{\mathbf{B}})$ holds and $\mathbf{B}$ is abelian. Since $\mathbf{A}$ is isomorphic to $\mathbf{B}$, $\mathbf{A}$ is abelian, too.

Our arguments have shown that a skew congruence below some η_0' implies the existence of an absorbing element for $\mathbf{A}$. If a finite power of $\mathbf{A}$ has a skew congruence, but has no skew congruence below any η_0', then $\mathbf{A}$ is abelian. Hence any idempotent simple algebra $\mathbf{A}$ must belong to at least one of the classes described in the theorem. Now we show that no idempotent simple algebra belongs to two classes.

If $\mathbf{A}$ has an absorbing element 0, then $\mathbf{A}^2$ has an **ideal congruence** which is skew. That is, for $I = A \times \{0\}$ it is the case that $\theta = \mathrm{Cg}(I \times I)$ has I as its only nontrivial congruence class. This congruence satisfies $0 < \theta < \eta_1$ and is therefore skew. If $\mathbf{A}$ is abelian, then $\mathbf{A}^2$ has a congruence which has the diagonal $D = \{(x, x) | x \in A\}$ as a congruence class. This congruence is clearly distinct from the product congruences $0, \eta_0, \eta_1$ and 1, so it is skew. Hence any algebra in classes (a) or (b) cannot belong to class (c). To show that classes (a) and (b) are disjoint, assume that $\mathbf{A}$ is abelian and has an absorbing element $0 \in A$. Since we have assumed that $|A| > 2$ we can also assume, by Lemma 2.1, that $\mathbf{A}$ is not essentially unary. Let $t(x, y, \bar{z})$ be a term depending on x and y in $\mathbf{A}$ and choose $a \in A - \{0\}$. 0 is an absorbing element, so

$$t^{\mathbf{A}}(\underline{0}, 0, a, \ldots, a) = 0 = t^{\mathbf{A}}(\underline{0}, a, a, \ldots, a).$$

Changing the underlined occurrences of 0 to a we obtain

$$t^{\mathbf{A}}(a, 0, a, \ldots, a) = 0 \neq a = t^{\mathbf{A}}(a, a, a, \ldots, a).$$

This is a failure of the term condition. It contradicts our assumption that $\mathbf{A}$ is abelian. We conclude that classes (a) and (b) of the theorem are disjoint. This completes the proof of the theorem. □

COROLLARY 2.4 *If $\mathbf{A}$ is an idempotent simple algebra and $\mathbf{A}^2$ is skew-free, then every finite power of $\mathbf{A}$ is skew-free.*

Proof: Theorem 2.3 shows that if some finite power of $\mathbf{A}$ has a skew congruence, then $\mathbf{A}$ has an absorbing element or is abelian. In the former case, $\mathbf{A}^2$ has skew congruences associated with ideal congruences while in the latter case $\mathbf{A}^2$ has a skew congruence associated with the "diagonal congruence". Hence if some finite power of $\mathbf{A}$ has a skew congruence, then $\mathbf{A}^2$ has a skew congruence. □

3 An Affine Module Embedding

In [3], C. Herrmann proves that every abelian algebra in a congruence modular variety is affine. This implies that an idempotent abelian algebra which generates a congruence modular variety is an affine module. Starting with an abelian algebra, Herrmann constructs a 1-1 function into an affine algebra in the first half of his paper. In the second half, he proves that this function is an isotopy. For idempotent algebras, his function is an isomorphism. Herrmann's proof depends quite heavily on congruence modularity, but we shall see in this section that it is possible to filter out much of this dependency in the first half of his proof. In the few remaining places where it seems impossible to avoid at least some modularity requirement, we shall see that simplicity and idempotence can substitute for modularity. In this way we will have modified the first half of his proof to produce an embedding into an affine module. Hence we prove that any idempotent simple algebra which is abelian is a subreduct of an affine module.

For this section, $\mathbf{A}$ denotes a simple, idempotent, abelian algebra of more than two elements. We single out a part of the proof of Theorem 2.3 that we shall need later.

LEMMA 3.1 *If η_i, $i = 0, 1$ are the kernels of the coordinate projections of $\mathbf{A}^2$, then $0 \prec \eta_i$.*

Proof: In the proof of Case 1 of Theorem 2.3 we show that when $|A| > 2$ and there is a congruence $0 < \delta < \eta_i$, then $\mathbf{A}$ has a unique absorbing element. We show later in the proof of Theorem 2.3 that such an algebra cannot be abelian. □

LEMMA 3.2 *There is a largest proper congruence Δ on $\mathbf{A} \times \mathbf{A}$ which contains $D = \{(x, x) \mid x \in A\}$ in a Δ-class. This Δ has D as a congruence class and satisfies $\Delta\eta_0 = \Delta\eta_1 = 0$. Furthermore, $\mathbf{B} = (\mathbf{A} \times \mathbf{A})/\Delta$ is simple and an extension of $\mathbf{A}$.*

Proof: Saying that D is equal to a class of some congruence is just another way of saying that $\mathbf{A}$ is abelian. Hence, there *is* a proper congruence which contains D in a congruence class. Now suppose that there is some congruence θ which properly contains D in a θ-class. Since the containment is proper, we can find $a \neq b$ in A such that $(a,b)\ \theta\ (a,a)$ and so $\theta\eta_0 > 0$. By Lemma 3.1 and the fact that $\mathbf{A}$ is simple, each η_i is a coatom and also an atom in $\mathbf{Con}\mathbf{A}^2$. It follows $\theta = \eta_0$ or else $\theta = 1$. Since $D \times D \subseteq \theta$ and $D \times D \nsubseteq \eta_0$, we have $\theta = 1$. Hence for any proper congruence γ on $\mathbf{A} \times \mathbf{A}$ we have that γ contains D in a γ-class iff D is a γ-class.

Let Σ be the set of all congruences on $\mathbf{A} \times \mathbf{A}$ which have D as a congruence class. Define Δ to equal the join of all members of Σ. Clearly Δ has D as a congruence class, so Δ is the largest element in Σ. Since Δ is the largest congruence which has D as a congruence class and no proper congruence contains D in a congruence class we have $\Delta \prec 1$ in $\mathbf{Con}\ \mathbf{A} \times \mathbf{A}$. Hence $\mathbf{B} = (\mathbf{A} \times \mathbf{A})/\Delta$ is simple. Since $\eta_i \nleq \Delta$, clearly, and each η_i is an atom, we must have

$$\Delta\eta_0 = \Delta\eta_1 = 0.$$

In this paragraph we have justified all the claims of the lemma except that we haven't yet shown that $\mathbf{B}$ is an extension of $\mathbf{A}$.

To show that $\mathbf{A}$ embeds in $\mathbf{B}$, choose an element $0 \in A$ and consider the function $\mathbf{A} \to \mathbf{B}$ defined by $x \mapsto (x,0)/\Delta$. This function is a homomorphism since $\mathbf{A}$ is idempotent and it is an embedding since $\Delta\eta_0 = 0$. $\square$

We shall think of the fourth power of $\mathbf{A}$ as 2×2 matrices of elements of $\mathbf{A}$ and write $\mathbf{A}^{2\times 2}$ to denote this. We order the entries as follows:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

According to our conventions this means that

$$\pi_{01}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a,b) \text{ while } \pi_{10}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (b,a).$$

Thus Δ_{01}, for example, is the congruence on $\mathbf{A}^{2\times 2}$ which contains all pairs of 2×2 matrices of the form

$$\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right\rangle, \quad \langle (a,b),(e,f)\rangle \in \Delta.$$

LEMMA 3.3 *For Δ as described in Lemma 3.2, the following properties hold.*

(*i*) $(x,y)\ \Delta\ (u,v) \Leftrightarrow (y,x)\ \Delta\ (v,u)$.

(*ii*) $(x,y)\ \Delta\ (u,v) \Leftrightarrow (x,u)\ \Delta\ (y,v)$.

Proof: Let $\dagger$ denote the canonical involutory automorphism of $\mathbf{A}^2$. If one applies $\dagger$ coordinatewise to $\mathbf{A}^{2\times 2} = (\mathbf{A}^2)^2$, then one obtains an automorphism of $\mathbf{A}^{2\times 2}$ which permutes the subalgebras of $\mathbf{A}^{2\times 2}$. This automorphism permutes the congruences of $\mathbf{A}^2$ considered as subalgebras. We shall use † to denote both the described automorphism of $\mathbf{A}^2$ and the induced permutation of congruences of $\mathbf{A}^2$.

Clearly $\Delta^\dagger$ is a congruence on $\mathbf{A}^2$ which contains the diagonal as a congruence class. It follows that $\Delta^\dagger \subseteq \Delta$ by Lemma 3.2. But now applying † to both sides of this inclusion we get $\Delta^{\dagger\dagger} \subseteq \Delta^\dagger$ and therefore $\Delta = \Delta^{\dagger\dagger} \subseteq \Delta^\dagger \subseteq \Delta$. Hence $\Delta = \Delta^\dagger$ which means precisely that (i) holds.

Now let ‡ denote the automorphism of $\mathbf{A}^{2\times 2}$ which may be described as "transposing matrices" or equivalently as "switching coordinates 1 and 2". Viewing Δ as a subset of $\mathbf{A}^{2\times 2}$, part (ii) of the lemma is the statement that $\Delta^\ddagger = \Delta$. By the same argument we used in the last paragraph, to prove that $\Delta^\ddagger = \Delta$ it will suffice for us to prove that $\Delta^\ddagger \subseteq \Delta$.

$\Delta^\ddagger$ is a subalgebra of $\mathbf{A}^{2\times 2}$ since Δ is. Since Δ satisfies part (i) of the lemma, $\Delta^\ddagger$ is a symmetric relation. Since $D \times D \subseteq \Delta$, we get that $\Delta^\ddagger$ is reflexive and since Δ is reflexive, we get $D \times D \subseteq \Delta^\ddagger$. Hence $\Delta^\ddagger$ is a reflexive, symmetric, compatible relation on $\mathbf{A}^2$ which contains $D \times D$. Let θ denote the transitive closure of $\Delta^\ddagger$. θ is a congruence on $\mathbf{A}^2$ which contains $D \times D$ and also contains $\Delta^\ddagger$. Since $\Delta\eta_0 = 0$, we get

$$(a, b)\ \Delta\ (a, c) \Rightarrow b = c.$$

Hence

$$(a, a)\ \Delta^\ddagger\ (b, c) \Rightarrow b = c.$$

From this it follows that

$$(a, a)\ \theta\ (b, c) \Rightarrow b = c.$$

Thus, θ is a proper congruence on $\mathbf{A}^2$ and D is a θ-class. By the maximality of Δ, we get that $\Delta^\ddagger \subseteq \theta \subseteq \Delta$ which implies that $\Delta^\ddagger = \Delta$. Thus (ii) holds. □

LEMMA 3.4 *The following hold.*

I. $\mathbf{A}^2/\Delta$ *is abelian.*

II. *If* χ *is a coatom of* **Con** $\mathbf{A}^{2\times 2}$, *then the following conditions are equivalent.*

(i) $\Delta_{01}\Delta_{23} \leq \chi$ *and* χ *contains some pair of the form*

$$\langle M, N\rangle = \left\langle \begin{bmatrix} x & y \\ x & y \end{bmatrix}, \begin{bmatrix} u & z \\ u & z \end{bmatrix} \right\rangle$$

where $\langle (x, y), (u, z)\rangle \notin \Delta$.

(ii) $\Delta_{02}\Delta_{13} \leq \chi$ *and* χ *contains some pair of the form*

$$\langle P, Q\rangle = \left\langle \begin{bmatrix} x & x \\ y & y \end{bmatrix}, \begin{bmatrix} u & u \\ z & z \end{bmatrix} \right\rangle$$

where $\langle (x, y), (u, z)\rangle \notin \Delta$.

(iii) $\Delta_{01}\Delta_{23} + \Delta_{02}\Delta_{13} \leq \chi$.

III. There is a unique congruence χ on $\mathbf{A}^{2\times 2}$ which is a coatom of $\mathbf{Con}\ \mathbf{A}^{2\times 2}$ and satisfies the equivalent conditions in part II.

Proof: If $\mathbf{B} = \mathbf{A}^2/\Delta$, then $\mathbf{B}$ is simple by Lemma 3.2 and $\mathbf{B}\times\mathbf{B}$ is naturally isomorphic to $\mathbf{A}^{2\times 2}/(\Delta_{01}\Delta_{23})$. Throughout this proof we shall identify $\mathbf{B}\times\mathbf{B}$ and $\mathbf{A}^{2\times 2}/(\Delta_{01}\Delta_{23})$. Choose any pair $\langle M, N\rangle$ as described in items $II(ii)$ above. Define

$$\psi = [(\Delta_{01}\Delta_{23}) + \mathrm{Cg}(M, N)]/(\Delta_{01}\Delta_{23})$$

and let $\theta = (\Delta_{01}\Delta_{23} + \Delta_{02}\Delta_{13})/(\Delta_{01}\Delta_{23})$. Since $\langle M, N\rangle \in \Delta_{02}\Delta_{13}$ we have $\psi \leq \theta$. We shall prove that $\psi \leq \theta < 1$ in $\mathbf{Con}\ \mathbf{B}^2$ and that there is a largest proper congruence in the interval $[\psi, 1]$ of $\mathbf{Con}\ \mathbf{B}^2$. Together, these two facts imply that the largest proper congruence above ψ is also the largest proper congruence above θ. This will prove that in $\mathbf{Con}\ \mathbf{A}^{2\times 2}$, a coatom satisfies $II(i)$ if and only if it satisfies $II(iii)$. By symmetry, we will get that a coatom satisfies $II(ii)$ if and only if it satisfies $II(iii)$. This will prove II. Along the way we will establish parts I and III of the lemma.

If $E = \{(x, x) | x \in B\}$ is the diagonal of $\mathbf{B}^2$, then the congruence ψ on $\mathbf{B}^2$ contains $E \times E$. The reason for this is that ψ contains the pair

$$\langle M/\Delta_{01}\Delta_{23}, N/\Delta_{01}\Delta_{23}\rangle = \left\langle \begin{pmatrix} (x\ y)/\Delta \\ (x\ y)/\Delta \end{pmatrix}, \begin{pmatrix} (u\ v)/\Delta \\ (u\ v)/\Delta \end{pmatrix} \right\rangle$$

which is a pair of elements of E. Our assumption that $\langle (x, y), (u, z)\rangle \notin \Delta$ means that this pair is a pair of distinct elements of E. The fact that the diagonal subalgebra $\mathbf{E} \leq \mathbf{B}^2$ is simple (since it is isomorphic to $\mathbf{B}$) together with $\psi|_E \neq 0$ implies that $E \times E \subseteq \psi \leq \theta$.

Next we prove that in $\mathbf{Con}\ \mathbf{B}^2$ we have $\theta < 1$. This is equivalent to the claim that

$$\Delta_{01}\Delta_{23} + \Delta_{02}\Delta_{13} < 1$$

in $\mathbf{Con}\ \mathbf{A}^{2\times 2}$. We prove this by exhibiting a proper subset of $A^{2\times 2}$ which is a union of $(\Delta_{01}\Delta_{23} + \Delta_{02}\Delta_{13})$-classes. That subset is $\Delta \subseteq A^{2\times 2}$. Δ is a proper subset of $A^{2\times 2}$ since $\Delta < 1$ in $\mathbf{Con}\ \mathbf{A}^2$. We first argue that Δ, as a subset of $A^{2\times 2}$, is a union of $\Delta_{01}\Delta_{23}$-classes. Applying the automorphism $\ddagger$ to this statement then yields that $\Delta^\ddagger$ is a union of $\Delta_{02}\Delta_{13}$-classes. But $\Delta = \Delta^\ddagger$ by Lemma 3.3 (ii). Hence it will follow that Δ is a union of $(\Delta_{01}\Delta_{23} + \Delta_{02}\Delta_{13})$-classes.

To see that Δ is a union of $\Delta_{01}\Delta_{23}$-classes, take matrices

$$R = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ S = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

such that $R \in \Delta$ and $\langle R, S\rangle \in \Delta_{01}\Delta_{23}$. The fact that $R \in \Delta$ means precisely that $(a, b) \equiv_\Delta (c, d)$. The fact that $\langle R, S\rangle \in \Delta_{01}\Delta_{23}$ means precisely that $(a, b) \equiv_\Delta (e, f)$ and $(c, d) \equiv_\Delta (g, h)$. By transitivity, we get $(e, f) \equiv_\Delta (g, h)$, so $S \in \Delta$. As explained in the last paragraph, this proves that $\Delta_{01}\Delta_{23} + \Delta_{02}\Delta_{13} < 1$ in $\mathbf{Con}\ \mathbf{A}^{2\times 2}$ and so $\theta < 1$ in $\mathbf{Con}\ \mathbf{B}^2$.

We now prove that in $\mathbf{Con}\ \mathbf{B}^2$ we have $\theta\eta_0 = \theta\eta_1 = 0$. Suppose instead that for some $b \neq c$ we have $\langle (a, b), (a, c)\rangle \in \theta\eta_0$. Then θ restricts nontrivially to the simple subalgebra $\mathbf{F} = \{a\}\times\mathbf{B}$. Hence $F \times F \subseteq \theta$. But now for every $x, y \in B - \{a\}$ we have $(a, x) \equiv_\theta (a, a) \equiv_\theta (x, x)$

and $(a, y) \equiv_\theta (a, a) \equiv_\theta (y, y)$, so the same type of argument shows that $G \times G, H \times H \subseteq \theta$ when $\mathbf{G} = \mathbf{B} \times \{x\}$ and $\mathbf{H} = \mathbf{B} \times \{y\}$ if $x \neq a \neq y$. Using the fact that $|B| \geq |A| > 2$, we choose $x, y \in B$ so that $|\{a, x, y\}| = 3$ and we get that θ restricts nontrivially to every subalgebra of the form $\{u\} \times \mathbf{B}$ since θ contains $\langle (u, x), (u, y) \rangle$. This implies that $\theta = 1$ which we know to be false. Our conclusion is that $\theta\eta_0 = 0$. Similarly $\theta\eta_1 = 0$.

We have shown that $\mathbf{B}^2$ has congruences ψ and θ where $E \times E \subseteq \psi \leq \theta < 1$ for E equal to the diagonal of $\mathbf{B}^2$. We have also shown that $\theta\eta_0 = \theta\eta_1 = 0$. These conditions imply that E is a θ-class (and also a ψ-class), so $\mathbf{B}$ is abelian. This establishes part I of the lemma. Applying Lemma 3.2 to $\mathbf{B}$ we see that there is a largest proper congruence $\Delta^{\mathbf{B}}$ on $\mathbf{B}^2$ which contains E in a congruence class. Lemma 3.2 proves that $\Delta^{\mathbf{B}} \prec 1$. We have $\psi \leq \theta \leq \Delta^{\mathbf{B}} \prec 1$ since θ is a proper congruence which contains E in a congruence class. $\Delta^{\mathbf{B}}$ is the largest proper congruence in $\mathbf{Con}\ \mathbf{B}^2$ which has E as a congruence class, so $\Delta^{\mathbf{B}}$ is the largest proper congruence above ψ and is also the largest proper congruence above θ. Hence a coatom in $\mathbf{Con}\ \mathbf{B}^2$ is above ψ if and only if it is above θ. This holds for the coimages of ψ and θ in $\mathbf{Con}\ \mathbf{A}^{2\times 2}$, so a coatom satisfies condition $II(i)$ if and only if it satisfies condition $II(iii)$. By symmetry we get that II holds.

For III, our only choice is to take χ to be the coimage of $\Delta^{\mathbf{B}}$. From the equivalence of the conditions in II, it doesn't matter if we take this coimage with respect to the natural map

$$\mathbf{A}^{2\times 2} \to \mathbf{A}^{2\times 2}/(\Delta_{01}\Delta_{23}) \cong \mathbf{B}^2$$

or with respect to the natural map

$$\mathbf{A}^{2\times 2} \to \mathbf{A}^{2\times 2}/(\Delta_{02}\Delta_{13}) \cong \mathbf{B}^2.$$

In either case we get the largest proper congruence of $\mathbf{A}^{2\times 2}$ above $\Delta_{01}\Delta_{23} + \Delta_{02}\Delta_{13}$. □

Now we define a sequence of algebras and embeddings between them.

(*i*) Set $\mathbf{A}_0 = \mathbf{A}$ and choose $0_0 \in A_0$.

(*ii*) Let $\mathbf{A}_{n+1} = (\mathbf{A}_n \times \mathbf{A}_n)/\Delta^{\mathbf{A}_n}$, $0_{n+1} = (0_n, 0_n)/\Delta^{\mathbf{A}_n}$.

(*iii*) Let $\delta_n : \mathbf{A}_n \to \mathbf{A}_n^2 : x \mapsto (x, 0_n)$.

(*iv*) Let $t_n : \mathbf{A}_n^2 \to \mathbf{A}_{n+1} : (x, y) \mapsto (x, y)/\Delta^{\mathbf{A}_n}$. (The canonical map.)

(*v*) Let $\epsilon_n = t_n \circ \delta_n : \mathbf{A}_n \to \mathbf{A}_{n+1}$. Let $\epsilon_{ij} = \epsilon_{j-1} \circ \cdots \circ \epsilon_{i+1} \circ \epsilon_i$ if $j > i$ and let $\epsilon_{ii} = \mathrm{id}_{A_i}$.

LEMMA 3.5 *If $\mathcal{V} = \mathcal{V}(\mathbf{A})$, then the following hold.*

(*i*) *$\mathbf{A}_n \in \mathcal{V}$ for all n.*

(*ii*) *$\epsilon_n : \mathbf{A}_n \to \mathbf{A}_{n+1}$ is an embedding for all n.*

(*iii*) *Each $\mathbf{A}_n$ is simple and abelian.*

(*iv*) *For all* $a, b, c, d \in A_n$,

$$t_{n+1}(t_n(a,b), t_n(c,d)) = t_{n+1}(t_n(a,c), t_n(b,d)).$$

Proof: Item (*i*) is immediate from the fact that $\mathcal{V} = \mathbf{HSP}(\mathcal{V})$. Item (*ii*) follows from induction based on our proof in Lemma 3.2 that the function

$$\mathbf{A} \to (\mathbf{A} \times \mathbf{A})/\Delta : x \mapsto (x, 0)/\Delta$$

is an embedding. Item (*iii*) follows by induction from our proofs that $(\mathbf{A} \times \mathbf{A})/\Delta$ is simple (Lemma 3.2) and abelian (Lemma 3.4).

Item (*iv*) is the statement that the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{A}_n^{2\times 2} & \xrightarrow{t_n} & \mathbf{A}_{n+1}^2 \\
t_n \downarrow & & t_{n+1} \downarrow \\
\mathbf{A}_{n+1}^2 & \xrightarrow{t_{n+1}} & \mathbf{A}_{n+2}
\end{array}$$

The map across the top is t_n applied to the rows of a matrix in $\mathbf{A}_n^{2\times 2}$. The map on the left is t_n applied to the columns of a matrix in $\mathbf{A}_n^{2\times 2}$. Since the composite maps in both directions are the canonical maps, it suffices to prove that they have the same kernel. But the kernels of these two composite maps are each coatoms of $\mathbf{Con}\ \mathbf{A}^{2\times 2}$ since $\mathbf{A}_{n+2}$ is simple. The map across the top and down the right side has a kernel which satisfies condition $II(i)$ of Lemma 3.4 while the map down the left side and across the bottom has a kernel which satisfies $II(ii)$ of Lemma 3.4. The result of Lemma 3.4 proves that these kernels are the same. □

We now define the direct limit $\hat{\mathbf{A}}$ in $\mathcal{V}$ of the system $\{\epsilon_n : \mathbf{A}_n \to \mathbf{A}_{n+1}\}$ in a concrete way. Let X denote the set of partial functions from the natural numbers into $\bigcup A_n$ which satisfy the following conditions.

1. If $a \in X$ and $n \in \operatorname{dom} a$, then $a(n) \in A_n$.

2. If $a \in X$ and $n \in \operatorname{dom} a$, then $n+1 \in \operatorname{dom} a$ and $a(n+1) = \epsilon_n(a(n))$.

Let Θ be the equivalence relation on X defined by $a \equiv_\Theta b$ if $a(n) = b(n)$ for all $n \in (\operatorname{dom} a \cap \operatorname{dom} b)$ (equivalently, for some $n \in (\operatorname{dom} a \cap \operatorname{dom} b)$). The underlying set of $\hat{\mathbf{A}}$ will be taken to be $\hat{A} = X/\Theta$.

If f is an m-ary $\mathcal{V}$-operation, then there is an induced operation $f^{\hat{\mathbf{A}}}$ on $\hat{A}$ which we define as follows. For $a_0, \ldots, a_{m-1} \in X$, let $f^{\hat{\mathbf{A}}}(a_0/\Theta, \ldots, a_{m-1}/\Theta)$ be the Θ-class of the partial function whose domain is $\bigcap \operatorname{dom} a_i$ and which is defined by

$$f^{\hat{\mathbf{A}}}(a_0, \ldots, a_{m-1})(n) \stackrel{\text{def}}{=} f^{\mathbf{A}_n}(a_0(n), \ldots, a_{m-1}(n))$$

for $n \in \bigcap \operatorname{dom} a_i$. Now we make some further definitions.

(*i*) Let $i_m : \mathbf{A}_m \to \hat{\mathbf{A}}$ be the function which assigns to each $a \in A_m$ the Θ-class of the partial function $i_m(a)$ defined by $i_m(a)(n) = \epsilon_{mn}(a)$ whenever $n \geq m$.

(*ii*) Let $0 \in \hat{A}$ to be the Θ-class of the everywhere-defined function $0(n) = 0_n$.

(*iii*) Define $t : \hat{\mathbf{A}}^2 \to \hat{\mathbf{A}}$ as follows. For $R, S \in \hat{A}$ choose $r \in R$ and $s \in S$. Let t be the function that assigns to the pair (R, S) the Θ-class of the partial function $t(R, S)$ defined by $t(R, S)(n+1) = t_n(r(n), s(n))$ whenever $n \in (\mathrm{dom}\, r \cap \mathrm{dom}\, s)$.

LEMMA 3.6 *If $\mathcal{V} = \mathcal{V}(\mathbf{A})$, then the following hold.*

(*i*) *If f is a $\mathcal{V}$-operation, then $f^{\hat{\mathbf{A}}}$ is a well-defined operation on $\hat{A}$.*

(*ii*) *$i_n : \mathbf{A}_n \to \hat{\mathbf{A}}$ is an embedding for all n and $\hat{A} = \bigcup i_n(A_n)$.*

(*iii*) *$\hat{\mathbf{A}} \in \mathcal{V}$.*

(*iv*) *$\hat{\mathbf{A}}$ is simple and abelian.*

(*v*) *0 is a well-defined element of $\hat{A}$ and t is a well-defined binary operation on $\hat{\mathbf{A}}$ and the following equations hold:*

  (*a*) $t(x, 0) = x$.

  (*b*) $t(x, x) = 0$.

  (*c*) $t(t(x, y), t(z, u)) = t(t(x, z), t(y, u))$.

(*vi*) *$t : \hat{\mathbf{A}}^2 \to \hat{\mathbf{A}}$ is a homomorphism.*

Proof: To show that $f^{\hat{\mathbf{A}}}$ is a well-defined function, we need to show that if $a_i \in X$, $i < m$, then the partial function defined by

$$f^{\hat{\mathbf{A}}}(a_0, \ldots, a_{m-1})(n) = f^{\mathbf{A}_n}(a_0(n), \ldots, a_{m-1}(n))$$

is also in X and that if $a_i \equiv_\Theta a_i'$, $i < m$, then $f^{\hat{\mathbf{A}}}(\bar{a}) \equiv_\Theta f^{\hat{\mathbf{A}}}(\bar{a}')$. The first part requires showing that

$$\begin{aligned} \epsilon_n(f^{\mathbf{A}_n}(a_0(n), \ldots, a_{m-1}(n))) &= f^{\mathbf{A}_{n+1}}(a_0(n+1), \ldots, a_{m-1}(n+1)) \\ &= f^{\mathbf{A}_{n+1}}(\epsilon_n(a_0(n)), \ldots, \epsilon_n(a_{m-1}(n))). \end{aligned}$$

This is just the statement that ϵ_n is a homomorphism which is immediate from its definition. The second part follows immediately, since $a_i \equiv_\Theta a_i'$ means $a_i(m) = a_i'(m)$ wherever they are both defined, so $f^{\hat{\mathbf{A}}}(\bar{a})(m) = f^{\hat{\mathbf{A}}}(\bar{a}')(m)$ wherever they are both defined.

To show (*ii*) we first need to show that for $a \in A_m$ the partial function $i_m(a)$ is in X. This means that we must show that $\epsilon_n(i_m(a)(n)) = i_m(a)(n+1)$ for $n \geq m$. This just requires observing that $\epsilon_n \circ \epsilon_{mn}(a) = \epsilon_{mn+1}(a)$. Now we must show that each i_m is a 1-1 homomorphism. To see that i_m is 1-1, it suffices to note that if $a \neq b$ in $\mathbf{A}_m$, then $i_m(a)(m) = a \neq b = i_m(b)(m)$, so $i_m(a)$ and $i_m(b)$ differ on a natural number that belongs to their common domain. Thus, $i_m(a) \not\equiv_\Theta i_m(b)$. Checking that i_m is a homomorphism reduces to checking that

$$i_m(f^{\mathbf{A}_m}(a_0, \ldots, a_k))(m) = f^{\mathbf{A}_m}(i_m(a_0)(m), \ldots, i_m(a_k)(m)).$$

Clearly both sides equal $f^{\mathbf{A}_m}(a_0, \ldots, a_k)$. To finish (*ii*) we must show that $\hat{A} = \bigcup i_n(A_n)$. For this it is enough to show that for each $a \in X$ there is an $n < \omega$ and an element $b \in A_n$ such that $i_n(b)(n) = a(n)$. (Then we will have $i_n(b) = a$.) We may simply choose $b = a(n)$.

To show that $\hat{\mathbf{A}} \in \mathcal{V}$ it suffices to observe that a variety fails to contain $\hat{\mathbf{A}}$ if and only if it fails to contain some finitely generated subalgebra of $\hat{\mathbf{A}}$. By (*ii*), $\hat{A} = \bigcup A_n$, so any such variety will also fail to contain some $\mathbf{A}_n$. $\mathcal{V}$ contains all $\mathbf{A}_n$, so $\mathcal{V}$ contains $\hat{\mathbf{A}}$.

Statement (*iv*) follows from (*ii*), since any algebra which is the nested union of simple or abelian subalgebras has the same properties. (The union $\hat{A} = \bigcup A_n$ is nested since $i_n = i_{n+1} \circ \epsilon_n$. Hence the image of i_n is contained in the image of i_{n+1}.)

0 is a well-defined element of $\hat{A}$, since $\epsilon_n(0_n) = t_n(0_n, 0_n) = 0_{n+1}$. To see that t is well-defined, we must first show that for $R, S \in \hat{A}$ and $r \in R$ and $s \in S$ we have $t(R, S) \in X$ where $t(R,S)(n+1) = t_n(r(n), s(n))$ whenever $n \in (\text{dom } r \cap \text{dom } s)$. Then we need to verify that the Θ-class of $t(R, S)$ doesn't depend on our choice of r and s. The first part follows from Lemma 3.5 (*iv*) as we show here.

$$\begin{aligned}
\epsilon_{n+1}(t(R,S)(n+1)) &= \epsilon_{n+1}(t_n(r(n), s(n))) \\
&= t_{n+1} \circ \delta_{n+1} \circ t_n(r(n), s(n)) \\
&= t_{n+1}(t_n(r(n), s(n)), 0_{n+1}) \\
&= t_{n+1}(t_n(r(n), s(n)), t_n(0_n, 0_n)) \\
&= t_{n+1}(t_n(r(n), 0_n), t_n(s(n), 0_n)) \\
&= t_{n+1}(t_n(\delta_n(r(n))), t_n(\delta_n(s(n))) \\
&= t_{n+1}(\epsilon_n(r(n)), \epsilon_n(s(n))) \\
&= t_{n+1}(r(n+1), s(n+1)) \\
&= t(R,S)(n+2).
\end{aligned}$$

For the second part of this verification, it is clear that if r and r' agree on their common domain and s and s' agree on their common domain, then $t_n(r(n), s(n)) = t_n(r'(n), s'(n))$ whenever n belongs to the domain of r, r', s and s'.

To show that $t(x, 0) = x$, choose $R \in X$ and $r \in R$. We must show that $t_n(r(n), 0_n) = r(n+1)$ for some (equivalently *all*) $n \in \text{dom } r$. This is accomplished with

$$\begin{aligned}
t_n(r(n), 0_n) &= t_n(\delta_n(r(n))) \\
&= \epsilon_n(r(n)) \\
&= r(n+1).
\end{aligned}$$

To show that $t(x, x) = 0$, choose $R \in X$ and $r \in R$. We must show that $t_n(r(n), r(n)) = 0_{n+1}$ for some $n \in \text{dom } r$. We show this with: $t_n(r(n), r(n)) = t_n(0_n, 0_n) = 0_{n+1}$. Here we used that fact that $(r(n), r(n)) \equiv_{\Delta \mathbf{A}_n} (0_n, 0_n)$.

To establish the equation $t(t(x, y), t(z, u)) = t(t(x, z), t(y, u))$ choose $P, Q, R, S \in X$ and $p \in P$, $q \in Q$, $r \in R$ and $s \in S$. Then we must show that

$$t_{n+1}(t_n(p(n), q(n)), t_n(r(n), s(n))) = t_{n+1}(t_n(p(n), r(n)), t_n(q(n), s(n)))$$

for some n where p, q, r and s are all defined. This equation follows from Lemma 3.5 (*iv*).

To show that $t : \hat{\mathbf{A}}^2 \to \hat{\mathbf{A}}$ is a homomorphism, we must show that if f is an m-ary fundamental operation, then t and f commute on any $2 \times m$ matrix of elements of $\hat{\mathbf{A}}$. This verification reduces to showing that each $t_n : \mathbf{A}_n^2 \to \mathbf{A}_{n+1}$ is a homomorphism. But by definition t_n is the canonical homomorphism between these two algebras. $\Box$

LEMMA 3.7 *If $p(x, y, z) = t(x, t(y, z))$, then*

(i) $p(x, y, y) = x$ *and* $p(x, x, y) = y$.

(ii) *p commutes with itself and with all the operations of $\hat{\mathbf{A}}$.*

(iii) *$\langle \hat{\mathbf{A}}; p(x, y, z) \rangle$ is an idempotent, abelian, simple algebra.*

Proof: For (i), note that $p(x, y, y) = t(x, t(y, y)) = t(x, 0) = x$ by Lemma 3.6 $(v)(a)$ and $(v)(b)$. Using Lemma 3.6 $(v)(c)$ as well, we get

$$\begin{aligned} p(x, x, y) &= t(x, t(x, y)) \\ &= t(t(x, 0), t(x, y)) \\ &= t(t(x, x), t(0, y)) \\ &= t(0, t(0, y)) \\ &= t(t(y, y), t(0, y)) \\ &= t(t(y, 0), t(y, y)) \\ &= t(y, 0) = y. \end{aligned}$$

Property (ii) follows from the fact that p is in the clone on $\hat{A}$ generated by t and t has the properties mentioned in (ii) by Lemma 3.6.

Since $\hat{\mathbf{A}}$ is idempotent and simple and $p(x, y, z)$ is idempotent, the expansion $\langle \hat{\mathbf{A}}; p(x, y, z) \rangle$ is an idempotent simple algebra. This expansion generates a congruence permutable variety by part (i) of the lemma. Part (ii) proves that $p(x, y, z)$ commutes with all the operations of $\langle \hat{\mathbf{A}}; p(x, y, z) \rangle$, so by Proposition 5.7 of [2] we get that $\langle \hat{\mathbf{A}}; p(x, y, z) \rangle$ is abelian. $\Box$

THEOREM 3.8 *If $\mathbf{A}$ is a simple, idempotent, abelian algebra, then there is an embedding $i : \mathbf{A} \to \hat{\mathbf{A}}$ where $\hat{\mathbf{A}} \in \mathcal{V}(\mathbf{A})$ is a simple reduct of an affine module. Furthermore, the subgroup of this affine module generated by $i(A)$ is $\hat{A}$.*

Proof: Let $\langle \hat{\mathbf{A}}; p(x, y, z) \rangle$ be the expansion described in Lemma 3.7 of the simple algebra $\hat{\mathbf{A}}$ of Lemma 3.6. $\langle \hat{\mathbf{A}}; p(x, y, z) \rangle$ is an idempotent, simple, abelian algebra by Lemma 3.7. By Theorem 9.16 of [2] and the fact that $\langle \hat{\mathbf{A}}; p(x, y, z) \rangle$ is idempotent and simple, we get that $\langle \hat{\mathbf{A}}; p(x, y, z) \rangle$ is a simple affine module. It follows that $\hat{\mathbf{A}}$ is a reduct of a simple affine module. Since $\mathbf{A} = \mathbf{A}_0$ and $i_0 : \mathbf{A}_0 \to \hat{\mathbf{A}}$ is an embedding (by Lemma 3.6 (ii)) we have proven the first claim of this theorem (taking $i = i_0$).

We may assume that the ring associated with the module structure on $\langle \hat{\mathbf{A}}; p(x, y, z) \rangle$ acts faithfully. If we take 0 to be the zero element of this module, then the equations $t(x, x) = 0$ and $t(x, 0) = x$ force $t(x, y) = x - y$ in this module. If one now looks back to the definition of t and i_n, one will find that $i_{n+1}(A_{n+1}) = t(i_n(A_n), i_n(A_n))$. Since $\hat{A} = \bigcup i_n(A_n)$, we get that $\hat{A}$ is generated under $t(x, y) = x - y$ by the set $i_0(A_0) = i(A)$. This proves the second claim of the theorem. $\Box$

Theorem 1.1 is a consequence of Theorems 2.3 and 3.8.

4 Restrictions on Subalgebras

In this section we refine Theorem 1.1 in the case that $\mathbf{A}$ has no nontrivial proper subalgebras.

THEOREM 4.1 *If $\mathbf{A}$ is an idempotent algebra with no nontrivial proper subalgebras, then exactly one of the following conditions is true.*

(i) $\mathbf{A}$ *is term equivalent to the 2-element set,*

(ii) $\mathbf{A}$ *is term equivalent to the 2-element semilattice,*

(iii) $\mathbf{A}$ *is a reduct of an affine module and* $\mathbf{A}$ *is not essentially unary or*

(iv) *Every member of* $\mathsf{HSP}_{fin}(\mathbf{A})$ *is congruence distributive.*

Proof: Our proof shall be quite similar to the proof of Theorem 2.3 except that we shall apply our assumption that $\mathbf{A}$ has no nontrivial proper subalgebras in appropriate spots in order to strengthen our conclusion. Note that since $\mathbf{A}$ is idempotent and has no proper nontrivial subalgebras, $\mathbf{A}$ is simple; for any nontrivial proper congruence class is a subuniverse.

We omit the argument for the case $|A| = 2$; this case can be handled as we described at the beginning of Section 2. Also, we will only explain why $\mathbf{A}$ must satisfy "at least one" of the conditions $(i) - (iv)$. The argument that shows that $\mathbf{A}$ satisfies no more than one of these conditions can be found in the proof of Theorem 2.3.

We will see that the condition in (iv) can be strengthened to $(iv)'$: If $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{A}^m$ but not isomorphic to a subalgebra of $\mathbf{A}^{m-1}$, then $\mathbf{ConB} \cong \mathbf{2}^m$. From this it follows that every member of $\mathsf{SP}_{fin}(\mathbf{A})$ (and thus every member of $\mathsf{HSP}_{fin}(\mathbf{A})$) is congruence distributive. To prove the theorem it will suffice to prove that if $\mathbf{A}$ does not satisfy $(iv)'$ and $|A| > 2$, then $\mathbf{A}$ is a reduct of an affine module. Assume that $\mathbf{A}$ does not satisfy condition $(iv)'$. Then for some $m > 1$ there is some subalgebra $\mathbf{B} \leq \mathbf{A}^m$ where $\mathbf{B}$ is not isomorphic to a subalgebra of $\mathbf{A}^{m-1}$, but where $\mathbf{ConB}$ is not isomorphic to $\mathbf{2}^m$. We assume that m is chosen minimally for this. In particular, the representation $\mathbf{B} \leq \mathbf{A}^m$ is irredundant and subdirect. If η_i is the kernel of the coordinate i^{th} projection of $\mathbf{A}^m$, then we use the same symbol, η_i, to denote the restriction of this congruence to $\mathbf{B}$. The irredundance of the representation $\mathbf{B} \leq \mathbf{A}^m$ implies that $0 < \eta_i'$ for each i and this implies that all product congruences are distinct. Furthermore, the (meet semilattice) of product congruences on $\mathbf{B}$ is order-isomorphic to $\mathbf{2}^m$. It follows that $\mathbf{B}$ has a congruence θ which is not a product congruence. We may choose i so that $\eta_i \not\geq \theta$ since $\theta \neq 0$. Arguing as in the proof of Theorem 2.3, we may assume that $i = 0$ and $\theta \not\geq \eta_0'$. Again we split our argument into two cases. Case 1: $\mathbf{B}$ has a congruence δ such that $0 < \delta < \eta_0'$. Case 2: $\mathbf{B}$ has no congruence δ such that $0 < \delta < \eta_0'$.

Proof for Case 1: Write $\mathbf{A}^m$ as $\mathbf{A} \times \mathbf{A}^{m-1}$. Each η_0'-class of $\mathbf{B}$ is a subuniverse of the form $(A \times \{c\}) \cap B$ where $c \in A^{m-1}$. Since $A \times \{c\}$ is a subuniverse of $\mathbf{A} \times \mathbf{C}$ which generates an algebra isomorphic to $\mathbf{A}$ and $\mathbf{A}$ has no proper nontrivial subalgebras, it follows that either $A \times \{c\} \subseteq B$ or else $|(A \times \{c\}) \cap B| = 1$. Define

$$C = \{c \in A^{m-1} \mid A \times \{c\} \subseteq B\}.$$

We claim that C is a subuniverse of $\mathbf{A}^{m-1}$. To see this, choose a fundamental operation f, say of arity ℓ, and elements $c_0, \ldots, c_{\ell-1} \in C$. Choose distinct $a, b \in A$. Then from the definition of C we have $(a, c_i), (b, c_i) \in B$ for all i, so

$$p := f((a, c_0), \ldots, (a, c_{\ell-1})) = (a, f(c_0, \ldots, c_{\ell-1})) \in B$$

and similarly $q := (b, f(c_0, \ldots, c_{\ell-1})) \in B$. But p and q are distinct elements of B contained in the subuniverse $A \times \{f(c_0, \ldots, c_{\ell-1})\}$. Since the subalgebra generated by this subuniverse is isomorphic to $\mathbf{A}$ and $\mathbf{A}$ has no proper nontrivial subalgebras, it must be that

$$A \times \{f(c_0, \ldots, c_{\ell-1})\} \subseteq B$$

which means that $f(c_0, \ldots, c_{\ell-1}) \in C$. This proves that C is a subuniverse. Define $U \subseteq C$ by

$$U = \{u \in C \mid \delta|_{A\times\{u\}} = 1_{A\times\{u\}}\}.$$

Since $0 < \delta < \eta_0'$ it follows that $\mathbf{B}$ has a nontrivial δ-class (which equals an η_0'-class) and also $\mathbf{B}$ has a nontrivial η_0'-class which is not a δ-class. The nontrivial η_0'-classes of $\mathbf{B}$ are precisely the sets of the form $A \times \{c\}$ where $c \in C$. Hence $0 < \delta < \eta_0'$ implies that U is a nonempty, proper subset of C.

Claim. If $t(x, \bar{y})$ is an $(n+1)$-ary term which depends on x in $\mathbf{A}$ and $\bar{s} \in C^n$, then $t^{\mathbf{C}}(U, \bar{s}) \subseteq U$.

Proof of Claim: Assume otherwise that $t(x, \bar{y})$ depends on x in $\mathbf{A}$ and for some $u \in U$ we have $t^{\mathbf{C}}(u, \bar{s}) = v \notin U$. Since $t^{\mathbf{A}}(x, \bar{y})$ depends on x, there is some $\bar{r} \in A^n$ such that $t^{\mathbf{A}}(x, \bar{r})$ is non-constant. Recall that $u \in U$ means precisely that $A \times \{u\}$ is a nontrivial δ-class. Applying the polynomial $t^{\mathbf{A}\times\mathbf{C}}(x, \overline{(r_i, s_i)})$ to this class, we obtain that the set

$$t^{\mathbf{A}\times\mathbf{C}}(A \times \{u\}, \overline{(r_i, s_i)}) = t^{\mathbf{A}}(A, \bar{r}) \times \{v\}$$

is a nontrivial subset contained in $A \times \{v\}$. Since this subset is a polynomial image of a δ-class, δ restricts nontrivially to $A \times \{v\}$. This implies that $A \times \{v\}$ is a δ-class. But this forces $v \in U$ which is contrary to our assumption. This proves the Claim.

The Claim just proven implies that U is a class of a congruence on $\mathbf{C}$. Let ψ be a congruence on $\mathbf{C}$ which is maximal for the property that U is a ψ-class. Since $U \neq C$ we get that $\psi < 1$. Now $\mathbf{C}$ is a subalgebra of $\mathbf{A}^{m-1}$, so there exists a $k < m$ such that $\mathbf{C}$ is isomorphic to a subalgebra of $\mathbf{A}^k$ but not isomorphic to a subalgebra of $\mathbf{A}^{k-1}$. By the minimality of m we have that $\mathbf{Con C} \cong \mathbf{2}^k$ for this value of k. In $\mathbf{2}^k$ there is a unique way to represent the bottom element as a meet of meet-irreducible elements. Since $\mathbf{C}$ is a subdirect power of $\mathbf{A}$, it follows that the maximal congruences of $\mathbf{C}$ are exactly the kernels of homomorphisms of $\mathbf{C}$ onto $\mathbf{A}$. Choose a maximal proper congruence $\beta \in \mathbf{Con C}$ such that $\psi \leq \beta$. Let α be a complement of β in $\mathbf{Con C}$ ($\cong \mathbf{2}^k$). We have $\alpha \wedge \psi = 0$, so $\psi < \alpha \vee \psi$. If U was a union of α-classes, then U would be a union of $\alpha \vee \psi$-classes. But this would violate the maximality of ψ. We conclude that U is not a union of α-classes. There must be some α-class V such that $V \nsubseteq U$, but $V \cap U \neq \emptyset$. V is a subuniverse since it is a congruence class. The facts that α complements β and β is the kernel of a homomorphism onto $\mathbf{A}$ implies that the subalgebra $\mathbf{V} \leq \mathbf{C}$ which is generated by V is isomorphic to a nontrivial subalgebra of

A: i.e., $\mathbf{V} \cong \mathbf{A}$. Now $U \cap V$ is a proper subuniverse of V, so this set must contain only one element. Say $U \cap V = \{0\}$. Let's interpret what the previous Claim means when $\bar{s}$ is chosen from V^n (we use the fact that $\mathbf{V} \cong \mathbf{A}$): If $t(x, \bar{y})$ is an $(n+1)$-ary term which depends on x in $\mathbf{V}$ and $\bar{s} \in V^n$, then $t^{\mathbf{V}}(0, \bar{s}) = 0$. This means precisely that 0 is an absorbing element for $\mathbf{V}$. Choose $v \in V - \{0\}$. From the definition of an absorbing element, it follows that $\{0, v\}$ is a subuniverse of V. This is impossible, since $\mathbf{V} \cong \mathbf{A}$ and $\mathbf{A}$ is an algebra with more than two elements which has no proper nontrivial subalgebras. This shows that Case 1 can never occur when $|A| > 2$.

Proof for Case 2: Now assume that $\mathbf{B}$ has no congruence δ such that $0 < \delta < \eta_0'$. Since $\theta \not\geq \eta_0'$, we have $\theta \wedge \eta_0' < \eta_0'$. Because we are in Case 2, this means that $\theta \wedge \eta_0' = 0$. Hence $C(\theta, \eta_0')$ and so $\theta \leq (0 : \eta_0')$. As η_0 and η_0' are complements, we also have $\eta_0 \wedge \eta_0' = 0$ and so $C(\eta_0, \eta_0')$ also holds. Thus $\eta_0 \leq (0 : \eta_0')$. Altogether this implies that $\theta \vee \eta_0 \leq (0 : \eta_0')$. But $\eta_0 \not\geq \theta$, by choice, and $\eta_0 \prec 1$ since $\mathbf{A}$ is simple. Hence $\theta \vee \eta_0 = 1$ and so $(0 : \eta_0') = 1$ and it follows that $C(1, \eta_0')$ holds. If V is a nontrivial η_0'-class, then the subalgebra of $\mathbf{B}$ generated by V is isomorphic to a subalgebra of $\mathbf{B}/\eta_0 \cong \mathbf{A}$, so $\mathbf{V} \cong \mathbf{A}$. But V is a class of an abelian congruence. Hence $\mathbf{V}$, and therefore $\mathbf{A}$, is an abelian algebra. Theorem 1.1 proves that $\mathbf{A}$ is a subreduct of an affine module. We shall prove that $\mathbf{A}$ is a reduct of an affine module.

As proved in the last section, there is a largest proper congruence Δ on $\mathbf{A}^2$ which has the diagonal of $\mathbf{A}^2$ contained in a single Δ-class. Choose $0 \in A$ arbitrarily and let $\mathbf{A}' = \mathbf{A}^2/\Delta$ and define

(*i*) $\delta : \mathbf{A} \to \mathbf{A}^2 : a \mapsto (a, 0)$,

(*ii*) $t : \mathbf{A}^2 \to \mathbf{A}' : (a, b) \mapsto (a, b)/\Delta$ and

(*iii*) $\epsilon = t \circ \delta$.

We proved (in Lemma 3.2 and we reiterated in Lemma 3.5) that ϵ is a 1-1 homomorphism. We claim that ϵ is onto as well. This amounts to proving that $t(A, A) = t(A, 0)$. To prove this, choose any $b \in A$ with $b \neq 0$. Let $U = A \times \{0\}$ and let $V = \{b\} \times A$. U and V are subuniverses of $\mathbf{A}^2$ which generate subalgebras isomorphic to $\mathbf{A}$. Since $\delta(A) = U$ we get that $\epsilon(A) = t \circ \delta(A) = t(U)$. The homomorphism ϵ is non-constant, so it must be that t is non-constant on U. $\mathbf{U} \cong \mathbf{A}$ is simple, so we get that t is 1-1 on U. Now $(b, 0) \in U \cap V$ and

$$t(b, 0) \neq t(0, 0) = t(b, b)$$

where the non-equality follows from the fact that $b \neq 0$ and t is 1-1 on U. This proves that t is non-constant on V since $t(b, 0) \neq t(b, b)$. Since $\mathbf{V} \cong \mathbf{A}$ we conclude that t is 1-1 on V.

$t(\mathbf{U})$ and $t(\mathbf{V})$ are subalgebras of $\mathbf{A}'$ which are isomorphic to $\mathbf{A}$. Furthermore, $t(U) \cap t(V)$ contains the distinct elements $t(b, 0)$ and $t(b, b)$. In this case we must have $t(U) = t(V)$ since otherwise one of the algebras $t(\mathbf{U})$ or $t(\mathbf{V})$, both of which are isomorphic to $\mathbf{A}$, has a nontrivial proper subuniverse: $t(U) \cap t(V)$. We have proven the following statement: For any distinct elements $0, b \in A$ it is the case that $t(A, 0) = t(b, A)$. This together with our assumption that $|A| > 2$ implies that $t(A, 0) = t(A, A)$.

Now that we know ϵ is onto, we get that each ϵ_n in Lemma 3.5 is an isomorphism. It follows that $\mathbf{A} \cong \hat{\mathbf{A}}$ in Lemma 3.6. This proves that $\mathbf{A}$ is a reduct of a module in Case 2 and concludes the proof of the theorem. □

One may wonder if in Theorem 4.1 (*iii*) it must be that $\mathbf{A}$ is itself an affine module. The answer is "yes" if $\mathbf{A}$ is finite. This can be deduced from theorems in the next section. But in general the answer is "no". Another question that arises is whether $\mathbf{A}$ must generate a congruence distributive variety in Theorem 4.1 (*iv*). The answer is again "yes" when $\mathbf{A}$ is finite, but "no" in general. Here are examples which support these claims.

Example. Let $\mathbf{F}_2$ denote the 2-element field and let V be the $\mathbf{F}_2$-space with basis $\mathcal{B} = \{e_n \mid n < \omega\}$. Let $\mathbf{R} = \mathbf{F}_2\langle\alpha, \beta\rangle$ be the free $\mathbf{F}_2$-algebra in the noncommuting variables α and β. We define an $\mathbf{R}$-module structure on V by defining actions of α and β on $\mathcal{B}$ and extending these definitions to V by linearity. We define $\alpha(e_n) = e_{n-1}$ if $n > 0$ and $\alpha(e_0) = 0$; $\beta(e_n) = e_{n^2+1}$. V is a faithful simple $\mathbf{R}$-module as is proved on pages 196–197 of [7]. We define $\mathbf{A}$ to be the algebra which is the reduct of ${}_{\mathbf{R}}V$ to the operations $rx + (1-r)y$, $r \in R$. Clearly, $\mathbf{A}$ is idempotent.

The translations $x \mapsto x + a$ form a transitive group of automorphisms of $\mathbf{A}$, so to see that $\mathbf{A}$ has no proper nontrivial subalgebras it suffices to prove that if $b \neq 0$, then the subuniverse generated by $\{0, b\} \subseteq V$ is all of V. Certainly the subuniverse generated by $\{0, b\}$ contains all elements of the form $rb + (1-r)0 = rb$. Hence this subuniverse contains all elements of V which belong to the submodule of ${}_{\mathbf{R}}V$ generated by b. But $Rb = V$ since ${}_{\mathbf{R}}V$ is simple, so we are done.

We have shown that $\mathbf{A}$ is an algebra of the sort described in Theorem 4.1 (*iii*). We now show that $\mathbf{A}$ is not affine. Let I be the ideal of $\mathbf{R}$ generated by $\{\alpha, \beta\}$. Since $\mathbf{R}/I \cong \mathbf{F}_2$ it follows that for all $r \in R$ we have $r \in I$ or $(1-r) \in I$, but not both. Hence the set of all idempotent operations of the form

$$r_1x_1 + \cdots + r_nx_n \quad \text{where } r_i \in I \text{ for all but one } i$$

is a clone on V which contains each $rx + (1-r)y$, $r \in R$, but does not contain $x - y + z$. Since ${}_{\mathbf{R}}V$ has $x - y + z$ as its unique Mal'cev operation and this operation is not in the clone of $\mathbf{A}$, it follows that $\mathbf{A}$ is not affine. □

The next example was supplied by Ágnes Szendrei.

Example. Let A be an infinite set and consider all operations f on A which have the following properties:

(*i*) $f(x, \ldots, x) = x$ for all $x \in A$.

(*ii*) Assume that the variables on which f depends are $x_{i_0}, \ldots, x_{i_{n-1}}$. There is a finite set $S \subseteq A$ such that $f(\bar{a}) \in S$ whenever $a_{i_j} \neq a_{i_k}$ for some $j, k < n$.

When f depends on all of its variables, these properties say that f is idempotent, but has finite range "off the diagonal". We let $\mathbf{A}$ be the algebra with universe A and basic operations chosen so that each operation on A satisfying (*i*) and (*ii*) is the interpretation of a basic operation. The following facts are easily verified.

- All operations in the clone of $\mathbf{A}$ are the interpretation of basic operations. (That is, the operations on A satisfying (i) and (ii) are closed under composition and contain the projections.)

- $\mathbf{A}$ is idempotent.

- $\mathbf{A}$ has no proper nontrivial subuniverse.

- $\mathbf{A}$ is not abelian.

- $\mathbf{A}$ has no ternary operation $m(x, y, z)$ which depends on its middle variable and satisfies $m(x, y, x) = x$ on A.

These items imply that $\mathbf{A}$ is an algebra which falls under case (iv) of Theorem 4.1. If $\mathcal{V}(\mathbf{A})$ is a congruence distributive variety, then for some q this variety would have terms $m_i(x, y, z)$, $i < q$, such that the following list of equations hold in $\mathcal{V}(\mathbf{A})$:

(i) $m_0(x, y, z) = x$, $m_{q-1}(x, y, z) = z$,

(ii) $m_i(x, y, x) = x$ for all i,

(iii) $m_i(x, x, z) = m_{i+1}(x, x, z)$ for even i, and

(iv) $m_i(x, z, z) = m_{i+1}(x, z, z)$ for odd i.

The second equation implies that each m_i is independent of its middle variable. This can be coupled with equations (iii) and (iv) to deduce that $m_i(x, y, z) = m_j(x, y, z)$ for all i and j. Finally, applying (i) we get that

$$x = m_0(x, y, z) = m_{q-1}(x, y, z) = z$$

is an equation of $\mathcal{V}(\mathbf{A})$. But this is false, since $\mathbf{A}$ is nontrivial. We conclude that $\mathcal{V}(\mathbf{A})$ is not congruence distributive. □

Of course, if $\mathbf{A}$ is finite and falls under case (iv) of Theorem 4.1, then the fact that $\mathcal{V}(\mathbf{A})$ is locally finite and $\mathsf{HSP}_{fin}(\mathbf{A})$ is congruence distributive is enough to imply that $\mathcal{V}(\mathbf{A})$ is congruence distributive.

The last two pathological examples indicate the limits of Theorem 4.1. However, we can give a more complete description of idempotent algebras without proper subalgebras satisfying additional hypotheses. The next theorem is an example of such a result.

THEOREM 4.2 *Let $\mathbf{A}$ be a non-unary algebra with a minimal clone which has no nontrivial proper subalgebras. Then exactly one of the following conditions is true.*

(i) $\mathbf{A}$ *is term equivalent to the 2-element semilattice,*

(ii) $\mathbf{A}$ *is term equivalent to an affine vector space over a prime field, or*

(iii) $\mathbf{A}$ *is term equivalent to a 2-element algebra with a majority operation.*

Proof: Since **A** is non-unary and has a minimal clone, it is idempotent. According to Lemma 2.1 and the proof of Lemma 3.3 in [14], if **A** is idempotent, has no nontrivial proper subalgebras and $|A| > 2$, then **A**

(a) has a local discriminator operation,

(b) is locally affine, or

(c) has an element 0 and a local operation $x * y$ such that $0 * x = x * 0 = 0$ and $x * y = x$ when $x \neq 0 \neq y$.

We assume that $|A| > 2$ and analyze each of these three cases.

Assume that **A** has a local discriminator operation. Choose distinct $a, b \in A$ and an operation $d(x, y, z)$ in $\mathrm{Clo}_3(\mathbf{A})$ whose restriction to $\{a, b\}$ is the discriminator on that set. It cannot be that $d(x, y, z)$ is a projection, so d generates $\mathrm{Clo}(\mathbf{A})$. The fact that $\{a, b\}$ is closed under d implies that $\{a, b\}$ is closed under all members of $\mathrm{Clo}(\mathbf{A})$. Hence $\{a, b\}$ is a subuniverse. **A** has no nontrivial proper subuniverses, so $|A| = 2$ which contradicts the assumption in the last paragraph.

The contradiction in Case (c) is obtained in exactly the same way: choose $a \in A - \{0\}$. Then choose $b(x, y) \in \mathrm{Clo}_2(\mathbf{A})$ so that b interpolates the local term operation $x * y$ on $\{0, a\}$. The operation $b(x, y)$ is not a projection, so it generates the clone of **A**. But $\{0, a\}$ is closed under b, hence it is a subuniverse. This implies that $|A| = 2$.

It must be that **A** is locally affine. Fix an affine representation of **A** over the appropriate ring, denoted **R**, and choose an operation $r_1x_1 + \cdots + r_nx_n \in \mathrm{Clo}(\mathbf{A})$ which is not a projection. We may assume that $r_1 \notin \{0, 1\}$, so by setting x_1 equal to x and all other variables equal to y we get that $r_1x + (1 - r_1)y$ is an operation in the clone which is not a projection. This operation generates the clone. It is a consequence of this that **R** is generated as a ring by the element r_1. The expansion of **A** to an **R**–module is still simple. There are no infinite simple modules over 1–generated rings. (1–generated rings are commutative, simple modules over commutative rings are 1–dimensional vector spaces, 1–dimensional vector spaces over 1–generated rings are finite.) But a finite, locally affine algebra is affine. Hence, **A** is a 1-dimensional affine vector space over a finite field. The minimality of the clone implies that field of scalars is a prime field.

What remains to check is the case where $|A| = 2$. In this case, the classification of clones on a 2-element set given in [10] proves that **A** must be affine over the 2-element field, equivalent to the 2-element semilattice or to the 2-element majority algebra. □

5 Minimal Varieties

As we described in the Introduction, the study of simple algebras is closely related to the study of minimal varieties. A full classification of the simple algebras in a given variety leads immediately to a full classification of the minimal subvarieties, but in fact the determination of minimal subvarieties can usually be accomplished with only a partial knowledge of the simple algebras. For example, one does not need a description of all simple groups to see that the minimal varieties of groups are precisely the varieties of elementary abelian p-groups. It suffices to know that a 1-generated simple group has prime cardinality. It turns out that the partial description we have of idempotent simple algebras is detailed enough to serve as a basis for a classification of all *locally finite*, idempotent, minimal varieties. However, our only achievement would be to reproduce the main result of [15]: Call a minimal idempotent variety **exceptional** if it is not

(i) term equivalent to the variety of sets,

(ii) term equivalent to the variety of semilattices,

(iii) a variety of affine modules, or

(iv) a congruence distributive variety.

THEOREM 5.1 (Á. Szendrei) *There is no exceptional, locally finite, idempotent, minimal variety.*

This result can be obtained as a corollary to Theorem 4.1. Any locally finite minimal variety is generated by a finite simple algebra which has no proper nontrivial subalgebras. Such an algebra must be term equivalent to a 2-element set, a 2-element semilattice, a reduct of an affine module or it must be that $\mathsf{HSP}_{fin}(\mathbf{A}) = \mathcal{V}(\mathbf{A})$ is congruence distributive. The only detail we must attend to is a proof that when $\mathbf{A}$ is a finite simple reduct of an affine module and $\mathbf{A}$ has no proper nontrivial subalgebras, then $\mathbf{A}$ is an affine module or is term equivalent to the 2-element set. This fact follows immediately from the results of this section, but it is only a byproduct of our efforts. Our real goal is not to duplicate the characterization of locally finite, idempotent, minimal varieties described above. We are interested only in whether there exist exceptional, idempotent, minimal varieties and this goal necessarily leads us into the realm of non-locally finite varieties. The principal result of this section is that no exceptional variety contains a subreduct of an affine module. This result can be used to complete the proof of Theorem 5.1 that we have just sketched.

Theorem 5.1 indicates that an exceptional, idempotent, minimal variety is not locally finite. Referring to Theorem 1.1, it is clear that an exceptional variety will not contain a simple algebra of type (a). For if $\mathbf{A}$ is such a simple algebra and $0 \in A$ is absorbing, then for any $a \in A - \{0\}$, $\{0, a\}$ is a subuniverse. Hence, there is no non-locally finite minimal variety which contains a simple algebra of type (a). This means that an exceptional variety must be generated by an infinite simple algebra of type (b) or (c). This gives us the following result about non-locally finite minimal varieties. (In this proposition we say that $\mathbf{A}$ is an $(x - y + z)$**-reduct** of the affine module $\mathbf{M}$ if $\mathbf{A}$ is a reduct of $\mathbf{M}$, but the subclone of $\mathbf{M}$ generated by $x - y + z$ and the operations of $\mathbf{A}$ is the full clone of $\mathbf{M}$. $\mathbf{A}$ is a **proper** $(x - y + z)$-reduct of $\mathbf{M}$ if $x - y + z$ is not in the clone of $\mathbf{A}$.)

PROPOSITION 5.2 *Let $\mathcal{V}$ be an exceptional, idempotent, minimal variety. Then $\mathcal{V}$ is generated by an infinite simple algebra $\mathbf{A}$ where* (*i*) **A** *is a simple, proper* $(x - y + z)$*-reduct of an affine module or*

(*ii*) *Every finite power of* **A** *is skew-free.* □

The argument for this has essentially already been given. If **A** is of type (c) from Theorem 1.1, then we are in case (ii) of this proposition. If **A** is of type (b), then we can replace **A** with the simple algebra $\hat{\mathbf{A}} \in \mathcal{V}$ which we described in Theorem 3.8. Since $\mathcal{V}$ is exceptional, $\hat{\mathbf{A}}$ is a proper $(x-y+z)$-reduct of an affine module. Now we are in case (i) of this proposition.

In this section we show that there is no exceptional, idempotent, minimal variety generated by an algebra of the type described in Proposition 5.2 (i).

LEMMA 5.3 *If a variety contains a nontrivial subreduct of an affine algebra, then it contains one which is a reduct of a simple module over a simple ring.*

Proof: Assume that the variety $\mathcal{V}$ contains **A**, which is a subreduct of the affine algebra **M**. Fix an affine representation for the operations of **A**. We may assume that the clone of **M** is generated by the operations of **A** and the operation $p(x, y, z) = x - y + z$. Furthermore, we may assume that **M** is generated under these operation by the set A. If we take the reduct of **M** to the operations of **A**, and call this reduct $\mathbf{M}'$, then we may assume that A is maximal under inclusion as a subuniverse of $\mathbf{M}'$ which generates an algebra belonging to $\mathcal{V}$. Under all these (permissible) assumptions, **A** is a reduct of **M** (not just a subreduct). Here is why. The operation $p(x, y, z)$ is a homomorphism $p : \mathbf{M}^3 \to \mathbf{M}$ which is a left inverse to the diagonal homomorphism $\delta : \mathbf{M} \to \mathbf{M}^3 : x \mapsto (x, x, x)$. Both p and δ remain homomorphisms when we replace **M** with its reduct $\mathbf{M}'$. Hence we have a homomorphism $p \circ \delta|_{\mathbf{A}} : \mathbf{A} \to \mathbf{M}'$ which is the identity on **A**, since $p \circ \delta = \mathrm{id}_{\mathbf{M}}$. It follows that

$$A = p \circ \delta|_{\mathbf{A}}(A) \subseteq p(A, A, A).$$

Since $\mathbf{A}^3 \in \mathcal{V}$ and $(\ker p)|_{\mathbf{A}^3}$ is a congruence on $\mathbf{A}^3$, we get that

$$p(\mathbf{A}, \mathbf{A}, \mathbf{A}) \cong \mathbf{A}^3/(\ker p)|_{\mathbf{A}^3} \in \mathcal{V}.$$

But $p(\mathbf{A}, \mathbf{A}, \mathbf{A})$ is the subalgebra of $\mathbf{M}'$ with universe $p(A, A, A) \supseteq A$. The maximality condition on A implies that $A = p(A, A, A)$ and therefore that A is closed under $x - y + z$. Since this operation and the operations of **A** generate the clone of **M**, A is a subuniverse of **M**. We chose **M** so that A generates **M**, so $A = M$.

Let **N** be a simple algebra generating a minimal subvariety of $\mathsf{HSP}(\mathbf{M})$. Replacing **N** by its linearization if necessary, we may assume that **N** is a reduct of a module. There exists a cardinal κ, a subalgebra $\mathbf{P} \leq \mathbf{M}^\kappa$ and a congruence θ such that $\mathbf{P}/\theta \cong \mathbf{N}$. Let $\mathbf{P}'$ be the reduct of **P** to the operations of **A**. Then $\mathbf{P}'$ is a subalgebra of $\mathbf{A}^\kappa$ and the equivalence relation θ is a congruence on $\mathbf{P}'$. $\mathbf{N}' := \mathbf{P}'/\theta$ is the reduct of **N** to the operations of **A**. This makes $\mathbf{N}'$ a reduct of the simple affine algebra **N** which is itself a reduct of a module. Since **N** generates a minimal variety, the corresponding ring is simple. Furthermore, $\mathbf{N}' \in \mathsf{HSP}(\mathbf{A}) \subseteq \mathcal{V}$. Hence, $\mathbf{N}'$ fulfills the claims of the lemma. □

In Lemma 5.3, the algebra $\mathbf{N}'$ constructed is an $(x - y + z)$-reduct of a simple, linear, affine algebra over a simple ring. Expanding $\mathbf{N}'$ by $x - y + z$ and a constant 0 interpreted as a trivial subalgebra, one obtains a simple module over a simple ring.

Observe that if we attempt to apply Lemma 5.3 to an abelian idempotent simple algebra, then we gain some information and we lose some information. Starting with a simple subreduct of an affine module, we end up with a reduct of a simple affine module over a simple ring. What we gain is the simplicity of the coefficient ring. What we lose is the simplicity of our abelian algebra. We shall find that the simplicity of the coefficient ring is more important for us than the simplicity of the abelian algebra.

For the rest of this section, we assume that $\mathbf{A}$ is a reduct of a simple affine $\mathbf{R}$-module $\mathbf{M}$ where $\mathbf{R}$ is a simple ring. We assume also that $\mathbf{A}$ generates an exceptional minimal variety, so $\mathbf{A}$ is a proper $(x - y + z)$-reduct of $\mathbf{M}$. We do not assume that $\mathbf{A}$ is simple. Each term operation of $\mathbf{A}$ may be expressed uniquely as

$$t^{\mathbf{A}}(x_0, \dots, x_{n-1}) = r_0x_0 + \dots + r_{n-1}x_{n-1}$$

where each $r_i \in R$ and $\Sigma_{i<n} r_i = 1$ (since $t(x, \dots, x) = x$). If some $r \in R$ appears as the i^{th} coefficient of some term t, then it occurs as the coefficient of x in the term $t(y, \dots, y, x, y, \dots, y)$ where x is in the i^{th} position. Hence $r \in R$ is a coefficient in some term iff $rx + (1 - r)y$ represents a term operation of $\mathbf{A}$. The set of distinct binary term operations of $\mathbf{A}$ may be identified with a subset $U \subseteq R$ under the correspondence $rx + (1 - r)y \leftrightarrow r$. This set U of coefficients of terms must satisfy some very special properties.

Definition 5.4 If $\mathbf{R}$ is a ring, then a set $U \subseteq R$ will be called a **unit interval** if the following conditions are met.

(*i*) U is a proper subset of R.

(*ii*) $0, 1 \in U$.

(*iii*) $U \neq \{0, 1\}$.

(*iv*) If $a, b, c \in U$, then $ab + (1 - a)c \in U$.

(*v*) $\mathbf{R}$ is generated by U as an abelian group.

Note that condition (*iv*) implies that U is closed under multiplication (take $c = 0$) and also under the function $x \mapsto (1 - x)$ (take $b = 0$, $c = 1$).

THEOREM 5.5 *Assume that* $\mathbf{A}$ *is not essentially unary. If* $\mathbf{A}$ *is a proper* $(x-y+z)$*-reduct of an affine module over a simple ring, then the set of coefficients of term operations of* $\mathbf{A}$ *is a unit interval.*

Proof: The set U of coefficients of binary term operations $ax + (1 - a)y$ satisfies (*ii*) since the projection operations $p_0(x, y) = x = 1x + 0y$ and $p_1(x, y) = y = 0x + 1y$ are term operations. (iv) is satisfied, since if $a, b, c \in U$, then $ax + (1-a)y$, $bx + (1-b)y$ and $cx + (1-c)y$ represent binary terms, so

$$a(bx + (1-b)y) + (1-a)(cx + (1-c)y) = (ab + (1-a)c)x + (1 - (ab + (1-a)c))y$$

also represents a binary term. This proves that $ab + (1-a)c \in U$. Using the fact that $x - y + z$ commutes with all term operations of $\mathbf{A}$, we see that the collection of binary terms of $\mathbf{A}$ generate all binary terms of $\langle \mathbf{A}; x - y + z \rangle$ under the operation $x - y + z$. Thus the additive subgroup of $\mathbf{R}$ generated by U contains all elements of R which occur as the coefficient x in some binary term of $\langle \mathbf{A}; x - y + z \rangle$. But the binary terms of $\langle \mathbf{A}; x - y + z \rangle$ up to equivalence are the set of all operations $rx + (1-r)y$, $r \in R$. Hence U generates $\mathbf{R}$ under $x - y + z$ and (v) holds.

Assume that $U = \{0, 1\}$. Since $\mathbf{A}$ is not essentially unary, there is a term which depends on all $n > 1$ of its variables: $r_0 x_0 + \cdots + r_{n-1} x_{n-1}$. Since all coefficients come from $U - \{0\}$ $= \{1\}$, this term is simply $x_0 + \cdots + x_{n-1}$. But now if we set the first two variables equal to x_1, we get a new term $2x_1 + \cdots + x_{n-1}$. Since all coefficients belong to $U = \{0, 1\}$ we must have $2 = 0$ or $2 = 1$ in $\mathbf{R}$. But $2 = 1$ forces $\mathbf{R}$, and therefore $\mathbf{A}$ to be trivial. We must have $2 = 0$. Now property (v) implies that $\mathbf{R}$ is the 2-element field. $\mathbf{A}$ can only be a reduct of a 1-dimensional affine vector space over $\mathbf{R}$. But the clone of a 1-dimensional affine vector space over the 2-element field is a minimal clone. Hence, the only reducts of a 1-dimensional affine vector space over the 2-element field are the full clone of the affine vector space and the clone of projection operations. Our assumptions tell us that $\mathbf{A}$ is not essentially unary and $\mathbf{A}$ is not affine. This is a contradiction to $U = \{0, 1\}$.

To finish we must prove (i). Suppose that $U = R$. Then the clone of $\mathbf{A}$ contains every operation of $\mathbf{M}$ of the form $rx + (1-r)y$, $r \in U = R$, but it does not contain $x - y + z$. Corollary 2 of [13] proves that $x - y + z$ is in the clone generated by $\{rx + (1-r)y \mid r \in R\}$ if and only if $\mathbf{R}$ has no homomorphism onto a 2-element field. Hence $U = R$ implies that our simple ring $\mathbf{R}$ has a homomorphism onto a 2-element field. This means that $\mathbf{R}$ is a 2-element field. Now we have $U = R = \{0, 1\}$ even though we have already shown that $U = \{0, 1\}$ is false. This contradiction proves that $U \subset R$. $\square$

A simple ring may have more than one unit interval. For example, if U is a unit interval of $\mathbf{R}$ and α is an automorphism of $\mathbf{R}$, then $\alpha(U)$ is another unit interval. If $\mathbf{R}$ is the field obtained by adjoining to the field of rationals the number $\sqrt{2}$ and α is the nontrivial automorphism of this field, then $U := \mathbf{Q}[\sqrt{2}] \cap [0, 1]$, $U' := \alpha(U)$ and in this case $U'' :=$ $U \cap U'$ are all unit intervals of $\mathbf{Q}[\sqrt{2}]$. So, there may be many unit intervals, some properly containing others. (The observations of this paragraph are due to G.M. Bergman.)

Which simple rings have unit intervals? We focus on this question now. First, we explain how to construct some examples.

Definition 5.6 Let $\mathbf{R}$ be a ring. A **positive cone** of an ordering of $\mathbf{R}$ is a subset $P \subseteq R$ satisfying

(i) $P + P \subseteq P$,

(ii) $P \cdot P \subseteq P$ and

(iii) $P \cap -P \subseteq \{0\}$.

That is, P is closed under addition and multiplication and does not contain both r and $-r$ if $r \neq 0$. The **ordering of R** associated with P is

$$a \leq b \Leftrightarrow b - a \in P.$$

We call $\langle \mathbf{R}; \leq \rangle$ a **partially ordered ring**.

The words "positive", "negative" and the symbol $<$ have their usual meanings, as does the interval notation $[x, y] := \{z \in R \mid x \leq z \leq y\}$. "Ordering" will always mean "partial ordering". If n is an integer, then the element $n \cdot 1 \in R$, defined to be a sum of n copies of 1, will be denoted simply by n.

Definition 5.7 Let $\langle \mathbf{R}; \leq \rangle$ be a partially ordered ring. $\mathbf{R}$ is **archimedean** if $na \leq b$ for all positive integers n implies $a \leq 0$ whenever $a, b \in R$. An element $e \in R$ is a **strong order unit** if whenever $a \in R$ there is a positive integer n such that $a \leq ne$. We say that $\langle \mathbf{R}; \leq \rangle$ is **strongly archimedean** if it is archimedean and 1 is a positive strong order unit.

If $\mathbf{R}$ is a subfield of the real numbers and $\leq$ is the restriction to $\mathbf{R}$ of the usual ordering, then $\langle \mathbf{R}; \leq \rangle$ is an example of a strongly archimedean partially ordered, simple ring. The next theorem describes one half of the connection between unit intervals and strongly archimedean partial orderings.

THEOREM 5.8 *A simple ring with a strongly archimedean partial ordering has a unit interval.*

Proof: Let $\langle \mathbf{R}; \leq \rangle$ be a simple ring with a strongly archimedean partial ordering. We denote the center of $\mathbf{R}$, which is a field, by $Z(\mathbf{R})$. Define

$$P' = \left\{ \frac{p}{f} \in R \mid p \in P,\ f \in (P \cap Z(\mathbf{R})) - \{0\} \right\}.$$

P' is the closure of P under division by positive elements of $\mathbf{F}$. An easy exercise shows that P' is the positive cone of a partial ordering which extends the one given by P. Any extension of a strongly archimedean partial ordering is another strongly archimedean partial ordering, so P' determines such an ordering. P' is closed under division by positive central elements (even by division by the central elements that are P'-positive but not P-positive). Replacing P with P' we may assume that our ordering has the property that the positive cone is closed under division by positive central elements.

Let $U = \{r \in R \mid 0 \leq r \leq 1\}$. We claim that U is a unit interval in $\mathbf{R}$. We must show that

(i) U is a proper subset of R.

(*ii*) $0, 1 \in U$.

(*iii*) $U \neq \{0, 1\}$.

(*iv*) If $a, b, c \in U$, then $ab + (1 - a)c \in U$.

(*v*) $\mathbf{R}$ is generated by U as an abelian group.

Since $U \subseteq P \subset R$, (*i*) is satisfied. (*ii*) clearly holds. (*iii*) holds since all rationals between 0 and 1 belong to U. For (*iv*), assume that $a, b, c \in U$. Then $0 \leq a, b, c \leq 1$. We get that $0 \leq 1 - a \leq 1$, so each of $a, b, c, (1 - a)$ is between 0 and 1. From this we deduce that

$$\begin{aligned} 0 &\leq ab + (1 - a)c \\ &\leq a \cdot 1 + (1 - a) \cdot 1 = 1 \end{aligned}$$

Hence $ab + (1 - a)c \in U$. Finally, assume that $r \in R$. Since our ordering is a strongly archimedean partial ordering, we can find positive integers $m, n \in F$ such that $-r < m$ and $m + r < n$. These inequalities imply that $0 < m + r < n$. Hence $(m + r)/n \in U$. The abelian group generated by U contains all multiples of $(m + r)/n$, so it contains $m + r$. But this group also contains all multiples of $1 \in U$, so this group contains m. Since it contains m and $m + r$ it also contains r. Our choice of $r \in R$ was arbitrary, so the abelian group generated by U is R. This proves that U is a unit interval in $\mathbf{R}$. □

Strongly archimedean simple rings give us some examples of simple rings with unit intervals. Our next goal is to prove the converse of Theorem 5.8, which is that all simple rings with unit intervals are strongly archimedean partially ordered rings. The converse implication is really the only one that is useful to us; we will find the partial ordering easier to work with than unit intervals. (Theorem 5.8 is included only to assure us that we lose no generality in considering orderings rather than unit intervals.)

Let $\mathbf{R}$ be a simple ring with a unit interval U. To continue the analysis, it will be convenient to associate to $\mathbf{R}$ an auxilary algebraic structure $\widehat{\mathbf{R}}$. The universe of $\widehat{\mathbf{R}}$ is R and the basic operations of $\widehat{\mathbf{R}}$ are just the binary operations $[u](x, y) := ux + (1 - u)y$, $u \in U$. We will refer to $\widehat{\mathbf{R}} = \langle R; [u], u \in U \rangle$ as simply 'the auxilary structure'. Note that $\widehat{\mathbf{R}}$ is a reduct of the affine $\mathbf{R}$-module structure of ${}_{\mathbf{R}}\mathbf{R}$. It is worth pointing out that property (*iv*) of a unit interval implies that the binary term operations of $\widehat{\mathbf{R}}$ are exactly those of the form $[u](x, y)$, $u \in U$. This implies, that no term operation of $\widehat{\mathbf{R}}$ agrees with $x - y + z$, for then

$$[u](x, y) - [v](x, y) + [w](x, y) = [u - v + w](x, y)$$

would force U to be closed under $x - y + z$. Since $0 \in U \neq R$ and U generates R as an abelian group, U is not closed under $x - y + z$. Thus, the auxilary structure $\widehat{\mathbf{R}}$ is a proper $(x - y + z)$-reduct of ${}_{\mathbf{R}}\mathbf{R}$.

LEMMA 5.9 *If $a_0x_0 + \cdots + a_{n-1}x_{n-1}$ represents a term operation of $\widehat{\mathbf{R}}$ and $a_i = 1$ for some i, then $a_j = 0$ for all $j \neq i$.*

Proof: This proof is based on an idea from [12]. Assume that, say, $a_0 = 1$. Then

$$a_0x + a_1z + a_2y + \cdots + a_{n-1}y = x - a_1y + a_1z$$

represents a term operation of $\widehat{\mathbf{R}}$. But, we shall argue, the set $I = \{r \in R|\ x - ry + rz \in \mathrm{Clo}_3\widehat{\mathbf{R}}\}$ is a proper ideal of $\mathbf{R}$. To see that I is an ideal, note that $a, b \in I$ implies that

$$(x - bz + by) - ay + az = x - (a - b)y + (a - b)z,$$

so $a - b \in I$. Since I is closed under subtraction it is an abelian group. I is an additive subgroup of $\mathbf{R}$ and U generates $\mathbf{R}$ as an additive subgroup, so to show that I is an ideal it suffices to show that if $a \in I$ and $u \in U$ we have $au, ua \in I$. This is proved by the lines

$$x - a(uy + (1 - u)z) + az = x - auy + auz$$

and

$$u(x - ay + az) + (1 - u)x = x - uay + uaz.$$

The ideal I must be proper, for if $1 \in I$ then the definition of I would force $x - y + z$ into the clone of $\widehat{\mathbf{R}}$ and we know this to be false. By the simplicity of $\mathbf{R}$, $I = \{0\}$. Hence, $a_1 = 0$. The same type of argument proves that $a_i = 0$ for all $i > 0$. □

THEOREM 5.10 *A simple ring with a unit interval has a strongly archimedean partial ordering.*

Proof: Let U be a unit interval of the simple ring $\mathbf{R}$. Let P be the set of elements of R that can be expressed as sums of elements of U. Our goal will be to show that P is a positive cone for a partial ordering which is extendible to a strongly archimedean partial ordering.

Clearly P is closed under sums. Furthermore, the fact that U is closed under products and P is the set of sums of elements of U implies that P is closed under products. The bulk of our argument will be to prove that $P \cap -P \subseteq \{0\}$. If this is not the case, then there is an $r \in P - \{0\}$ such that $r \in -P$. From the definition of P this means that there are elements $u_1, \ldots, u_m \in U$ and $v_1, \ldots, v_n \in U$ such that

$$u_1 + \cdots + u_m = -(v_1 + \cdots + v_n) = r \neq 0.$$

It follows that the u_i, v_i are not all zero even though $(\Sigma u_i) + (\Sigma v_i) = 0$. We set

$$V = \{u \in U \mid \exists w_1, \ldots, w_k \in U(u + w_1 + \cdots + w_k = 0)\}.$$

V contains $\{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ and so V properly contains $\{0\}$. Furthermore, since U is closed under multiplication, it follows that $U \cdot V \subseteq V$ and $V \cdot U \subseteq V$. If I is the abelian group generated by V, then our conclusions mean that I is a nonzero ideal of $\mathbf{R}$. To repeat and amplify, $P \cap -P \not\subseteq \{0\}$ implies that $I = R$.

Claim. $I \neq R$.

Proof of Claim: We begin with a little detour to uncover a strange property of the elements of V. We will show that if $u \in V$, then there exists a positive integer n such that

$r(1-r)^n u = 0$ for any $r \in U$. (This fact was discovered by Ágnes Szendrei.) To show this, fix some $u \in V$. Since $u \in V$ there exist $w_1, \ldots, w_k \in U$ such that $u + w_1 + \cdots + w_k = 0$. We claim that $r(1-r)^n u = 0$ for $n = k$ and for any $r \in U$.

Fix specific choices of r, u and w_i as described in the previous paragraph. Since $r, u, w_i \in U$, we get that $rx + (1-r)y$, $ux + (1-u)y$ and $w_i x + (1-w_i)y$ are operations of the auxilary structure, $\widehat{\mathbf{R}}$. We can compose the operation $rx + (1-r)y$ with itself repeatedly in order to produce operations of larger arity. We will change variables to those of the form x_σ where σ is a sequence of 0's and 1's in order to make it clearer how the composition should be done. Begin with $rx_0 + (1-r)x_1$. If we substitute $rx_{\epsilon 0} + (1-r)x_{\epsilon 1}$ in for the occurrence of x_ϵ in $rx_0 + (1-r)x_1$, we obtain a 4-ary operation:

$$r(rx_{00} + (1-r)x_{01}) + (1-r)(rx_{10} + (1-r)x_{11}) = r^2 x_{00} + r(1-r)x_{01} + r(1-r)x_{10} + (1-r)^2 x_{11}.$$

If $t(x_{00}, x_{01}, x_{10}, x_{11})$ denotes this 4-ary operation, we can build an 8-ary operation by substituting $t(x_{\epsilon 00}, x_{\epsilon 01}, x_{\epsilon 10}, x_{\epsilon 11})$ in for x_ϵ in $rx_0 + (1-r)x_1$. Continuing, for any n we can build a 2^n-ary operation of the variables x_σ, $\sigma \in \{0,1\}^n$, where the coefficient of x_σ is $r^m(1-r)^{n-m}$ precisely when m is the number of 0's in the string σ. We let $s(\bar{x})$ be such an operation for $n = k+1$, Define τ_i, $i = 0, 1, \ldots, k$, to be the element of $\{0,1\}^{k+1}$ which has a 0 in the i^{th} position and 1's elsewhere. For each i, the coefficient of x_{τ_i} is $r(1-r)^k$. Substitute into the variable x_{τ_i} of $s(\bar{x})$ the operation $w_i z_i + (1-w_i)x$ if $i > 0$ and substitute $uz_0 + (1-u)x$ into x_{τ_0}. If σ is a sequence which does not have exactly one occurrence of 0, then set x_σ equal to x. With these substitutions into $s(\bar{x})$ we end up with a new $(k+2)$-ary operation in the clone generated by the operations $rx + (1-r)y$, $ux + (1-u)y$ and $w_i x + (1-w_i)y$:

$$r(1-r)^k uz_0 + r(1-r)^k w_1 z_1 + \cdots + r(1-r)^k w_k z_k + Cx$$

for some $C \in R$. This operation is in the clone of $\widehat{\mathbf{R}}$. If you set all variables equal to x and use the fact that $u + w_1 + \cdots + w_k = 0$, you get

$$r(1-r)^k ux + r(1-r)^k w_1 x + \cdots + r(1-r)^k w_k x + Cx = Cx.$$

But this operation is idempotent, so $Cx = x$. Thus,

$$r(1-r)^k uz_0 + r(1-r)^k w_1 z_1 + \cdots + r(1-r)^k w_k z_k + x$$

is in the clone of $\widehat{\mathbf{R}}$. Now we apply Lemma 5.9 to this operation: the coefficient of x equals 1, so $r(1-r)^k u = r(1-r)^k w_i = 0$ for all i. (All that we care about is that $r(1-r)^k u = 0$, though.)

Now we return to our proof that $I \neq R$. Assume instead that $I = R$. Then

$$1 = \pm u_1 \pm \cdots \pm u_\ell$$

for some choice of elements $u_i \in V$. By the results of the last paragraph, there exist m_i such that $r(1-r)^{m_i} u_i = 0$ for any $r \in U$ and for each i. Since U is closed under the operation $x \mapsto (1-x)$, what we have said about r also holds for $(1-r)$: $(1-r)r^{m_i} u_i = 0$ for all i. Choose any N greater than all m_i. Then for any $r \in U$ we have

$$r(1-r)^N u_i = 0 = (1-r)r^N u_i$$

for all i. Hence we have

$$r(1-r)^N = r(1-r)^N(u_1+\cdots+u_\ell) = 0 = (1-r)r^N$$

for all $r \in U$.

We claim that there are polynomials with integer coefficients, $p(x), q(x) \in \mathbf{Z}[x]$, such that

$$x^{N-1}\cdot p(x) + (1-x)^{N-1}\cdot q(x) = 1.$$

Otherwise, the ideal $J = (x^{N-1}, (1-x)^{N-1})$ is proper in $\mathbf{Z}[x]$, so $\mathbf{Z}[x]$ has a maximal ideal M containing J. $\mathbf{Z}[x]/M$ is a field which has an element $\bar{x} := x/M$ such that $\bar{x}^{N-1} = 0 = (1-\bar{x})^{N-1}$. This is impossible because it implies that $\bar{x} = 0 = (1-\bar{x})$ or $1 = 0$. Now, using p and q we see that

$$\begin{aligned} r(1-r) &= r(1-r)[r^{N-1}\cdot p(r) + (1-r)^{N-1}\cdot q(r)] \\ &= r(1-r)^N\cdot p(r) + (1-r)r^N\cdot q(r) \\ &= 0 \end{aligned}$$

whenever $r \in U$. This proves that for any $r \in U$ we have $r^2 = r$. We write $U \models x^2 = x$ for this.

Choose $a, b \in U$. Recall that U is closed under multiplication and the function $x \mapsto (1-x)$. Hence

$$1 - ((1-a)(1-b)) = a + b - ab \in U.$$

Furthermore, taking $c = 1-b$ in property (iv) of a unit interval we get $ab+(1-a)(1-b) \in U$, so

$$1 - (ab + (1-a)(1-b)) = a + b - 2ab \in U.$$

For $x = a+b-ab$ and for $x = a+b-2ab$ we must have $x^2 = x$. We calculate (using $a^2 = a$, $b^2 = b$, $(ab)^2 = ab$) that

$$\begin{aligned} a+b-ab &= (a+b-ab)^2 \\ &= a^2 + ab - a^2b + ba + b^2 - bab - aba - ab^2 + abab \\ &= a + ab - ab + ba + b - bab - aba - ab + ab \\ &= a + ba + b - bab - aba. \end{aligned}$$

Hence $aba + bab = ab + ba$. A further calculation shows that

$$\begin{aligned} a+b-2ab &= (a+b-2ab)^2 \\ &= a^2 + ab - 2a^2b + ba + b^2 - 2bab - 2aba - 2ab^2 + 4abab \\ &= a + ab - 2ab + ba + b - 2bab - 2aba - 2ab + 4ab \\ &= a + ab + ba + b - 2bab - 2aba. \end{aligned}$$

This yields $2aba + 2bab = 3ab + ba$. Our two conclusions together give us that

$$3ab + ba = 2(aba + bab) = 2(ab + ba) = 2ab + 2ba.$$

This simplifies to $ab = ba$. Since $a, b \in U$ were chosen arbitrarily, $U \models xy = yx$. As U generates $\mathbf{R}$ as an abelian group we must have $\mathbf{R} \models xy = yx$. $\mathbf{R}$ is a commutative simple ring, so $\mathbf{R}$ is a field. Since $U \models x^2 = x$, U is contained in the set of idempotents of the field $\mathbf{R}$. Thus $U \subseteq \{0,1\}$. This contradicts the fact that U is a unit interval: U must properly contain $\{0,1\}$. This contradiction finishes the proof of the Claim.

The impact of the Claim is that P is the positive cone of a partial ordering of $\mathbf{R}$. In this ordering, if $u \in U$, then $u, 1-u \in U \subseteq P$ so $0 \leq u \leq 1$ as one might hope. We argue that 1 is a positive strong order unit. Positivity of 1 follows from $1 \in U \subseteq P$. Now we must show that every element of R is majorized by a positive integer. Fix $r \in R$. Since $\mathbf{R}$ is generated as an abelian group by U, we can find $u_1, \ldots, u_i \in U$ and $v_1, \ldots, v_j \in U$ such that

$$-r = u_1 + \cdots + u_i - v_1 - \cdots - v_j.$$

For each $v_k \in U$ we have $(1 - v_k) \in U$, so we can re-express this as

$$-r = u_1 + \cdots + u_i + (1 - v_1) + \cdots + (1 - v_j) - j.$$

Since $u_1 + \cdots + u_i + (1 - v_1) + \cdots + (1 - v_j) \in P$, this expresses $-r$ as $p - j$ where $p \in P$ and j is a nonnegative integer. We conclude that $r \leq j$. Since r is majorized by a nonnegative integer and 1 is positive, r is majorized by a positive integer. This proves that 1 is a positive strong order unit under the ordering determined by P.

It can be shown by example that the P-ordering does not have to be archimedean. So replace P with a maximal extension. This does not affect the fact the 1 is a positive strong order unit, but it implies extra properties of P. For example, by maximality, P must be closed under division by positive central elements. Define P' to be the set of all $x \in R$ such that $-\frac{1}{n} \leq x$ holds for all positive integers n. Then $P \subseteq P'$, $P' + P' \subseteq P'$ and (since 1 is a positive strong order unit) $P' \cdot P' \subseteq P'$. Furthermore, using the fact that 1 is a positive strong order unit, it is easy to see that

$$K := P' \cap -P' = \bigcap_{n<\omega} \left[-\frac{1}{n}, \frac{1}{n}\right]$$

is an ideal of $\mathbf{R}$. Since $1 \notin K$, we have $K = \{0\}$. P' is the positive cone of an ordering which extends the P-ordering, so $P = P'$.

Since 1 is a strong order unit and P is closed under division by positive integers, $na \leq b$ holds for all positive n iff $a \leq \frac{1}{n}$ holds for all positive n. The latter condition is equivalent to $a \in -P' = -P$ which may be expressed as $a \leq 0$. This proves that P is the positive cone of an archimedean partial ordering, so $\mathbf{R}$ is a strongly archimedean partially ordered ring. □

For us, the most important fact about strongly archimedean partial orderings of simple rings is contained in the next theorem. The proof evolved out of email correspondence with K.H. Leung.

THEOREM 5.11 *A simple ring with a strongly archimedean partial ordering has no non-trivial zero divisors.*

Proof: Replacing P by an extension if necessary, we may assume that P is closed under division by positive central elements. Assume that $r^2 = 0$ in $\mathbf{R}$. Using the fact that 1 is a strong order unit, select and fix a positive integer q such that $-r \leq q$. Since $0 \leq r+q$, we get that, for any positive integer m,

$$0 \leq (r+q)^m = mrq^{m-1} + q^m = mq^{m-1}(r + \frac{q}{m}).$$

Since P is closed under division by positive central elements, the integer mq^{m-1} is positive and central, and $0 \leq mq^{m-1}(r+\frac{q}{m})$, we get that $0 \leq r + \frac{q}{m}$. Thus, $-\frac{q}{m} \leq r$ for fixed q and increasing m. It follows that $-\frac{1}{n} \leq r$ for all n, so $r \in P$ by the archimedean property. But if $r^2 = 0$, then we also have $(-r)^2 = 0$. We get $-r \in P$ as well. Hence, $r \in P \cap -P \subseteq \{0\}$ which means that $r = 0$. This proves that $\mathbf{R}$ has no nonzero nilpotent elements.

Now assume that for certain $a, b \in R$ we have $ab = 0$. Then each element of bRa is nilpotent. From our earlier conclusion, $bRa = \{0\}$. This implies that the product of the ideals generated by b and a is

$$(b)(a) = (RbR)(RaR) = R(bRa)R = \{0\}.$$

But the only ideals of $\mathbf{R}$ are $\{0\}$ and R. Since $RR = R \neq \{0\}$ we conclude that $(a) = \{0\}$ or $(b) = \{0\}$. Hence $a = 0$ or $b = 0$. □

We are prepared to prove the main result of this section.

THEOREM 5.12 *Let $\mathcal{V}$ be a minimal idempotent variety and assume that $\mathcal{V}$ contains a nontrivial subreduct of an affine module. Then $\mathcal{V}$ is equivalent to the variety of sets or else is affine. In particular, $\mathcal{V}$ is not exceptional.*

Proof: Assume that $\mathcal{V} = \mathcal{V}(\mathbf{A})$ where $\mathbf{A}$ is a nontrivial subreduct of an affine module. By Lemma 5.3, we may assume that $\mathbf{A}$ is a reduct of an affine module $\mathbf{N}$ over a simple ring, $\mathbf{R}$. Fix such a representation for $\mathbf{A}$. If $\mathbf{A}$ is affine or essentially unary, then we are done, so assume that $\mathbf{A}$ is neither. Theorem 5.5 proves that $\mathbf{R}$ has a unit interval U, and that the binary term operations of $\mathbf{A}$ are precisely the operations $ux + (1-u)y$, $u \in U$. Using the same argument as in the proof of Lemma 5.3 we may adjust our choice of $\mathbf{A}$ so that it is a reduct of any affine $\mathbf{R}$-module in $\mathsf{HSP}(\mathbf{N})$. Let $\mathbf{B}$ be the reduct of ${}_{\mathbf{R}}\mathbf{R}$ to the operations of $\mathbf{A}$. Since the binary term operations of $\mathbf{B}$ are the operations $ux+(1-u)y$, $u \in U$, it is clear that the universe of the subalgebra $\mathbf{C} \leq \mathbf{B}$ generated by $0, 1 \in B = R$ is

$$C = \{u \cdot 1 + (1-u) \cdot 0 = u \mid u \in U\} = U \subseteq R.$$

We define $I \subseteq C$ to be an **ideal** in $\mathbf{C}$ if whenever $t(x, \bar{y})$ is an $(n+1)$-ary term that depends on x in $\mathbf{C}$ and $u \in I$ we have $t^{\mathbf{A}}(u, \bar{c}) \in I$ for any $\bar{c} \in C^n$. (So, an absorbing element is nothing more than a 1-element ideal.)

Claim. If $I = C - \{0\}$, then I is an ideal of $\mathbf{C}$.

Proof of Claim: We need to show that if $p(\bar{x}) = c_0x_0 + \cdots + c_{m-1}x_{m-1}$ is the interpretation of a term operation of $\mathbf{C}$ which depends on all of its variables and $\bar{d} \in C^m$ with at least one $d_i \in I = C - \{0\} = U - \{0\}$, then $p(\bar{d}) \in I$. This is trivial if $m = 1$, so we assume that $m > 1$. Since p depends on all variables, $c_i \in U - \{0\}$ for each i. Now $c_i, d_i \in U = C$

for all i, so $c_i d_i \in U$ for all i. This implies that $0 \leq c_i d_i$ for all i. Furthermore, $c_i \neq 0$ for all i and $d_j \neq 0$ for at least one j. Since $\mathbf{R}$ has no nontrivial zero divisors, $0 < c_j d_j$ for some j. We conclude that $0 < \Sigma_{i<m} c_i d_i = p(\bar{d})$, so $p(\bar{d}) \in C - \{0\} = I$. Since $p(\bar{x})$ and $\bar{d} \in C^m$ were chosen arbitrarily, the claim is proved.

The relation $\theta = (I \times I) \cup \{(0,0)\}$ is a congruence of $\mathbf{C}$, since $I = C - \{0\}$ is an ideal, and the quotient $\mathbf{C}/\theta$ has two elements, $0/\theta$ and I/θ. The latter element is a 1-element ideal (otherwise known as an absorbing element). Thus $\mathbf{C}/\theta$ is a 2-element idempotent simple algebra in $\mathcal{V}$ with at least one absorbing element. It must be equivalent to either a 2-element semilattice or a 2-element set. We have assumed that $\mathcal{V}$ is minimal, so $\mathcal{V}$ must be equivalent to the variety of sets or the variety of semilattices. Since $\mathbf{A}$ is not essentially unary it must be that $\mathcal{V}$ is not equivalent to the variety of sets. But $\mathbf{A} \in \mathcal{V}$ is a nontrivial abelian algebra; $\mathcal{V}$ cannot be equivalent to the variety of semilattices either. This contradiction finishes the proof. $\square$

Let $\mathcal{V}$ be an exceptional idempotent minimal variety. If $\mathcal{V}$ does not satisfy the commutator equation $[\alpha, \beta] = \alpha \wedge \beta$, equivalent to the implication $[\alpha, \alpha] = 0 \Rightarrow \alpha = 0$, then some member of $\mathcal{V}$ has a nontrivial abelian congruence. A nontrivial congruence class of an abelian congruence generates an abelian subalgebra, by idempotence, so a failure of $[\alpha, \beta] = \alpha \wedge \beta$ implies the existence of a nontrivial abelian algebra in the variety. It is our conjecture that any idempotent abelian algebra is a subreduct of a module. If this conjecture is true, then Theorem 5.12 can be applied to obtain a contradiction to $\mathcal{V}$ being exceptional. To summarize: if every idempotent abelian algebra is a subreduct of a module, then any exceptional idempotent minimal variety satisfies the commutator equation $[\alpha, \beta] = \alpha \wedge \beta$. This equation implies congruence meet semidistributivity. A proof of the conjecture stated in this paragraph would be a good first step toward proving that there is no exceptional idempotent minimal variety.

6 Simple Modes

As mentioned in the Introduction, a mode is an idempotent, entropic algebra. Of the numerous articles have been written about modes, most develop specific examples and restrict attention to only those examples. Hard evidence that the current list of examples is comprehensive is lacking. In response to this we initiated in [5] a classification all locally finite varieties of modes. This project was not completed in [5]. Here is what we accomplished. We showed that when $\mathcal{V}$ is a locally finite variety of modes, then

$$\mathcal{V} = (\mathcal{V}_1 \times \mathcal{V}_2) \circ \mathcal{V}_5$$

where $\mathcal{V}_1$ is locally strongly solvable, $\mathcal{V}_2$ is affine and $\mathcal{V}_5$ has a semilattice term. $(\mathcal{V}_1 \times \mathcal{V}_2)$ denotes the varietal product of $\mathcal{V}_1$ and $\mathcal{V}_2$ while the previous displayed line means that $\mathcal{V}$ is a Mal'cev product of $(\mathcal{V}_1 \times \mathcal{V}_2)$ and $\mathcal{V}_5$. The structure of $\mathcal{V}_2$ may be said to be well-understood. $\mathcal{V}_2$ is term equivalent to a variety of affine modules over a finite commutative ring. We investigated $\mathcal{V}_5$ and its non-locally finite analogues in [6] and now these varieties are fairly well-understood as well. (We know all such varieties up to term equivalence. A mode with a semilattice term turns out to be a subreduct of a semimodule over a commutative semiring satisfying $1+r=1$.) Varieties of the form $\mathcal{V}_1$ are not yet well-understood. The classification of locally finite modes may be completed by (i) classifying mode varieties of the form $\mathcal{V} = \mathcal{V}_1$ and (ii) fully describing the nature of the product of the subvarieties $\mathcal{V}_1 \times \mathcal{V}_2$ and $\mathcal{V}_5$. This product appears to be a generalization of the regularization of a variety – indeed it is exactly this in the case of groupoids – and so the algebras in $(\mathcal{V}_1 \times \mathcal{V}_2) \circ \mathcal{V}_5$ may turn out to be 'generalized Płonka sums of locally solvable algebras'.

Let us give a more explicit description of what the decomposition theorem of [5] means. Let $\mathbf{A}$ be an idempotent algebra and let $\alpha \prec \beta$ be a covering pair of congruences on $\mathbf{A}$. If B is a β-class, then B is a subuniverse. Let $\mathbf{B}$ be the subalgebra generated by B. We define a **relative quotient of A** to be any algebra of the form $\mathbf{B}/(\alpha|_B)$. The algebras in an idempotent variety $\mathcal{V}$ which are relative quotients are precisely those algebras whose universe is a minimal congruence class of an algebra in $\mathcal{V}$. In particular, any simple algebra in $\mathcal{V}$ is a relative quotient. It happens that the relative quotients of a finite mode come in three types. A relative quotient of a finite mode is term equivalent to

(**1**) a set;

(**2**) a 1-dimensional affine vector space over a finite field; or

(**5**) a 2-element semilattice.

The three-fold classification of relative quotients of finite modes are numbered according to the tame congruence theory labeling scheme delineated in [4]. Types **3** and **4** do not appear in modes. Now we are in a position to give a more informative description of the decomposition $\mathcal{V} = (\mathcal{V}_1 \times \mathcal{V}_2) \circ \mathcal{V}_5$. Say that a finite algebra **has type i**, for $i = \mathbf{1}, \mathbf{2}$ or **5** if all its relative quotients are of type i. Say that a locally finite algebra **has type i** if all its finitely generated subalgebras have type i. The main result of [5] is that if $\mathcal{V}$ is a locally finite variety of modes, then the collection of all algebras in $\mathcal{V}$ of type i comprises a subvariety, $\mathcal{V}_i$. Lastly, these subvarieties combine so that $\mathcal{V} = (\mathcal{V}_1 \times \mathcal{V}_2) \circ \mathcal{V}_5$.

A classification of arbitrary mode varieties, including those that are not locally finite, might begin with a full description of those modes which arise as relative quotients. A smaller project would be to classify the simple modes. It was this project that led to the results in this paper. Beginning with the complete characterization of the finite simple modes (which is an immediate consequence of the results in [5]) it is natural to call a simple mode **standard** if it is term equivalent to a 2-element set, a 2-element semilattice or a 1-dimensional affine vector space. Otherwise, we shall call it **non-standard**. We shall show that any non-standard simple mode is a subreduct of a 1-dimensional vector space. This result, together with Theorem 5.12 implies that the minimal varieties of modes are precisely those varieties generated by a standard simple mode.

According to remarks made at the beginning of this paper, we only need to consider simple modes with more than two elements. Clearly every non-standard simple mode has more than two elements (in fact, has infinitely many). Hence, we must show that every simple mode of more than two elements is a subreduct of a 1-dimensional vector space.

THEOREM 6.1 *If* $\mathbf{A}$ *is a simple mode with more than two elements, then* $\mathbf{A}$ *is a subalgebra of a simple reduct of a 1-dimensional affine vector space.*

Proof: A simple mode is an idempotent simple algebra and therefore must belong to one of the three classes described in Theorem 1.1. We shall prove that there is no simple mode of more than two elements in classes (a) and no simple mode in class (c). This will prove at least that a simple mode with more than two elements is a subalgebra of a simple reduct of a module. A little more work will establish that this module is a vector space.

It follows from Lemma 2.1 and our assumption that $|A| > 2$ that $\mathbf{A}$ is not essentially unary. Let $t(\bar{x})$ be a term which depends on all $n > 1$ of its variables. Since $\mathbf{A}$ is entropic, $t^{\mathbf{A}} : \mathbf{A}^n \to \mathbf{A}$ is a homomorphism. If we restrict $t^{\mathbf{A}}$ to the diagonal of $\mathbf{A}^n$, then $t^{\mathbf{A}}$ is a bijection since $t^{\mathbf{A}}$ is idempotent. Hence $t^{\mathbf{A}} : \mathbf{A}^n \to \mathbf{A}$ is onto, but not 1-1. It follows that $\ker t^{\mathbf{A}}$ is a proper, nonzero congruence of $\mathbf{A}^n$. If $\ker t^{\mathbf{A}} = \eta_\sigma$ where σ is a proper, non-empty subsequence of n, then $t^{\mathbf{A}}$ depends only the variables whose subscripts appear in σ. This is impossible, since t was chosen so that $t^{\mathbf{A}}$ depends on all variables. Hence $\ker t^{\mathbf{A}} \neq \eta_\sigma$ for any σ which means that $\ker t^{\mathbf{A}}$ is a skew congruence on $\mathbf{A}^n$. This proves that there is no simple mode in class (c) of Theorem 1.1.

Now we must show that there is no simple mode with more than two elements in class (a) of Theorem 1.1. Assume that $\mathbf{B}$ is such a mode and that 0 is the unique absorbing element of $\mathbf{B}$. Choose $a \neq b$ such that $a, b \in B - \{0\}$. As $\mathbf{B}$ is simple we have $(0, a) \in \mathrm{Cg}(a, b)$, so by Mal'cev's congruence generation lemma there exists $p(x) \in \mathrm{Pol}_1\mathbf{B}$ such that $p(a) = 0 \neq p(b)$ or $p(b) = 0 \neq p(a)$. Without loss of generality we may assume that $p(a) = 0 \neq p(b)$. $\mathbf{B}$ is a mode, so $p : \mathbf{B} \to \mathbf{B}$ is a homomorphism. Since $p(a) \neq p(b)$, p is not a constant homomorphism; the simplicity of $\mathbf{B}$ forces p to be a 1-1 homomorphism. Thus, p is an isomorphism from $\mathbf{B}$ to $p(\mathbf{B})$. But $0 \in p(B)$ is clearly an absorbing element for $p(\mathbf{B})$ since it is an absorbing element for $\mathbf{B}$. Since $a \mapsto 0$ under the isomorphism $p : \mathbf{B} \to p(\mathbf{B})$, it follows that a is an absorbing element for $\mathbf{B}$. Hence 0 and a are distinct absorbing elements for $\mathbf{B}$. This is impossible for algebras in class (a) of Theorem 1.1.

To finish the proof of the theorem, we must show that if $\mathbf{C}$ is a simple mode which is a subalgebra of a simple reduct of a module, then $\mathbf{C}$ is in fact a subalgebra of a simple reduct of a vector space. From Theorem 3.8, $\mathbf{C}$ is a subalgebra of a simple algebra $\widehat{\mathbf{C}} \in \mathcal{V}(\mathbf{C})$ and $\langle \widehat{\mathbf{C}} \ ; \ p(x, y, z) \rangle$ is a simple affine module. Thus $\widehat{\mathbf{C}}$ is a simple $(x - y + z)$-reduct of a 1-dimensional vector space. The fundamental operations of $\widehat{\mathbf{C}}$ are idempotent and commute with themselves, since $\mathcal{V}(\mathbf{C})$ is a variety of modes. The operation $p(x, y, z)$ is idempotent, commutes with itself and all the fundamental operations of $\mathbf{C}$ by Lemma 3.7. Hence $\langle \widehat{\mathbf{C}} \ ; \ p(x, y, z) \rangle$ is an affine module which is at the same time a mode. If we assume that $\langle \widehat{\mathbf{C}} \ ; \ p(x, y, z) \rangle$ is a faithful $\mathbf{R}$-module (which we may assume), then $\mathbf{R}$ is a commutative ring. (For when $\mathbf{R}$ acts faithfully on $\langle \widehat{\mathbf{C}} \ ; \ p(x, y, z) \rangle$ and $r, s \in R$, then the affine module operations $r(x) + (1 - r)(y)$ and $s(x) + (1 - s)(y)$ commute iff $rs = sr$.) It is well-known that any simple (affine) module over a commutative ring is a 1-dimensional vector space, so this completes our proof. □

The simple modes of the form $\hat{\mathbf{A}}$ are either 1-dimensional affine vector spaces or they are simple, proper $(x - y + z)$-reducts of 1-dimensional affine vector spaces. By Theorem 5.5, the corresponding field must have a unit interval. By Theorem 5.10, the field must have a strongly archimedean partial order. We obtain the following corollary to these theorems.

COROLLARY 6.2 *If* $\mathbf{A}$ *is a non-standard simple mode, then* $\mathbf{A}$ *is a subalgebra of a simple* *mode $\hat{\mathbf{A}} \in \mathcal{V}(\mathbf{A})$ where $\hat{\mathbf{A}}$ is a proper $(x-y+z)$-reduct of a 1-dimensional affine vector space over a strongly archimedean ordered field $\langle \mathbf{F}; \leq \rangle$. If $t(\bar{x}) = \Sigma a_i x_i$ is the affine representation of a term of $\hat{\mathbf{A}}$, then $0 \leq a_i \leq 1$ for each i and $\Sigma a_i = 1$.* $\square$

The important open question that concerns us most in this section is:

Question: Which fields have a strongly archimedean partial ordering?

This question must be answered in order to classify the non-standard simple modes. It may be that any strongly archimedean partial ordering of a field can be extended to a total ordering. If this is so, then the answer to the previous question is "the subfields of the real numbers" since, as was shown by Hilbert, any totally ordered archimedean ring is isomorphic to a subring of the real numbers. One can show fairly easily that to prove the statement "If a field $\mathbf{F}$ has a strongly archimedean partial ordering, then $\mathbf{F}$ is isomorphic to a subfield of the real numbers", it suffices to prove the statement when $\mathbf{F}$ is a subfield of the complex numbers. If every field with a strongly archimedean partial ordering is isomorphic to a subfield of the real numbers, then every nonstandard simple mode is a subreduct of a 1-dimensional affine vector space over the real numbers.

Theorem 5.12 implies that a non-standard simple mode does not generate a minimal variety. This has the following consequence.

COROLLARY 6.3 *If $\mathcal{V}$ is a minimal variety of modes, then $\mathcal{V}$ is term equivalent to the variety of sets, the variety of semilattices or a variety of affine vector spaces.* $\square$

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