HAUSDORFF PROPERTIES OF TOPOLOGICAL ALGEBRAS

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Abstract. Let P be a property of topological spaces. Let $[P]$ be the class of all varieties $\mathcal{V}$ having the property that any topological algebra in $\mathcal{V}$ has underlying space satisfying property P. We show that if P is preserved by finite products, and if $\neg P$ is preserved by ultraproducts, then $[P]$ is a class of varieties that is definable by a Maltsev condition.

The property that all T_0 topological algebras in $\mathcal{V}$ are j-step Hausdorff (H_j) is preserved by finite products, and its negation is preserved by ultraproducts. We partially characterize the Maltsev condition associated to $T_0 \Rightarrow \mathrm{H}_j$ by showing that this topological implication holds in every $(2j+1)$-permutable variety, but not in every $(2j+2)$-permutable variety.

Finally, we show that the topological implication $T_0 \Rightarrow T_2$ holds in every k-permutable, congruence modular variety.

1. Introduction

A topological space X is T_0 if whenever a and b are distinct points of X there is a closed subset of X containing one of the points that does not contain the other. X is T_1 if for each $a \in X$ the singleton set $\{a\}$ is closed. X is T_2, or *Hausdorff*, if for each $a \in X$ the intersection of the closures of the neighborhoods of a is $\bigcap \mathrm{cl}(N) = \{a\}$. The implications $T_2 \Longrightarrow T_1$ and $T_1 \Longrightarrow T_0$ follow immediately from these (nonstandard) definitions: if X is T_2 then each singleton set $\{a\}$ is the intersection of closed sets, hence is closed (so $T_2 \Longrightarrow T_1$); if X is T_1 and $a, b \in X$ are distinct then $\{a\}$ is a closed set containing one of the points and not containing the other (so $T_1 \Longrightarrow T_0$). This paper is one of a series of papers concerned with determining when the converse implications (and related implications) hold for topological algebras.

The significance of investigations of this type resides in the following observation: the class of all topological algebras in any variety is determined by its T_0 members. That is, if $\mathcal{A}$ is a topological algebra in $\mathcal{V}$, and

$$\theta = \{(a,b) \in A \times A \mid a \in \mathrm{cl}(b) \ \ \& \ \ b \in \mathrm{cl}(a)\},$$

then θ is a congruence on $\mathcal{A}$, $\mathcal{A}/\theta$ endowed with the quotient topology is a T_0 topological algebra in $\mathcal{V}$, and the topology on $\mathcal{A}$ consists of the sets of the form $\nu^{-1}(U)$ where U is open in $\mathcal{A}/\theta$ and ν is the natural homomorphism from $\mathcal{A}$ to $\mathcal{A}/\theta$. (See

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The results contained in this paper also appear in the second author's Ph. D. thesis [16].

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[5] for details.) Thus, the T_0 topological algebras in a variety are of fundamental interest, and it seems important to understand when they satisfy stronger topological properties.

It is a classical result that any T_0 topological group is T_2. W. Taylor extended this result with the following theorem.

Theorem 1. [19] *If $\mathcal{V}$ is congruence permutable, then any topological algebra in $\mathcal{V}$ satisfies*

$$T_0 \Longrightarrow T_2.$$

H. P. Gumm then generalized Taylor's result with:

Theorem 2. [10]

(1) *If $\mathcal{V}$ is k-permutable, then any topological algebra in $\mathcal{V}$ satisfies*

$$T_0 \Longrightarrow T_1.$$

(2) *If $\mathcal{V}$ is 3-permutable, then any topological algebra in $\mathcal{V}$ satisfies*

$$T_0 \Longrightarrow T_2.$$

Gumm's results were sharpened in J. P. Coleman's papers [4, 5]. Among other things, Coleman proved the converse of the first claim in Theorem 2 thereby showing that the implication $T_0 \Longrightarrow T_1$ for all topological algebras in a variety is equivalent to k-permutable for some k.

In this paper we show that if P is a topological property that is preserved by finite products and whose negation is preserved by ultraproducts, then the satisfaction of P by all topological algebras in a variety is characterizable by a Maltsev condition. This result applies to either of the properties $T_0 \Longrightarrow T_1$ or $T_0 \Longrightarrow T_2$.

In order to understand the topological consequences of k-permutability for a fixed k, Coleman defined new separation conditions called j-step Hausdorffness for each $j \geq 1$ (H_j for short). The relative strengths of the T_i conditions and the H_j conditions are indicated by

$$T_0 \Longleftarrow T_1 \Longleftarrow \cdots \Longleftarrow \mathrm{H}_4 \Longleftarrow \mathrm{H}_3 \Longleftarrow \mathrm{H}_2 \Longleftarrow \mathrm{H}_1 \Longleftrightarrow T_2,$$

where none of the unidirectional arrows are reversible. Coleman showed

Theorem 3. [4, Theorem 3.2] *If $k \geq 3$, then for every topological algebra in a k-permutable variety,*

$$T_0 \implies \mathrm{H}_{k-2}$$

Coleman also showed that, in a sense, Theorem 3 is sharp for $k = 4$. Specifically, he showed that T_0 topological algebras in 4-permutable varieties must be H_2 but there exist T_0 topological algebras in 4-permutable varieties that are not H_1. While this does not characterize the Maltsev condition for $T_0 \Longrightarrow H_j$ for any j, it does completely determine the relationship between these Maltsev conditions and the Maltsev condition for 4-permutability. The question of whether Theorem 3 is sharp in this sense for larger values of k was left open.

We introduce symmetrized versions of Coleman's H_j conditions, which we label sH_j. Although each sH_j, $j > 1$, is strictly weaker than the corresponding H_j for topological spaces, we show that for topological algebras in k-permutable varieties $\mathrm{H}_j \Longleftrightarrow \mathrm{sH}_j$ for each j and k (Theorem 19). We use the symmetrized conditions to prove that $T_0 \Longrightarrow \mathrm{H}_{\lfloor \frac{k}{2} \rfloor}$ for topological algebras in k-permutable varieties (Theorem 20). This result improves Coleman's theorem and is the best possible result of this type, for we also construct, for each $k > 1$, a topological algebra in a k-permutable variety that satisfies $\mathrm{H}_{\lfloor \frac{k}{2} \rfloor}$ but not $\mathrm{H}_{\lfloor \frac{k}{2} \rfloor - 1}$ (Theorem 21).

Coleman made an interesting suggestion regarding the implication $T_0 \implies T_2$. Certainly $T_0 \implies T_2$ is stronger than $T_0 \implies T_1$, and the latter implication is equivalent to k-permutability for some k, so it is natural to wonder what condition together with k-permutability for some k characterizes $T_0 \implies T_2$. Coleman suggested that "a reasonable conjecture is that congruence modularity together with n-permutability is necessary and/or sufficient for $T_0 \implies T_2$ to hold." Two partial results regarding this suggestion appear in the paper [2] by W. Bentz. Bentz proved that the implication $T_0 \implies T_2$ holds in any k-permutable variety that has a majority term. Then Bentz introduced, for each $k \geq 2$, a k-permutable variety $\mathcal{W}_k$ for which he could prove $T_0 \implies T_2$ but could not prove modularity. Bentz raised the question of whether his $\mathcal{W}_k$'s were counterexamples to the necessity part of Coleman's conjecture.

In this paper we prove that $T_0 \implies T_2$ holds in any modular, k-permutable variety. Then we prove that Bentz's $\mathcal{W}_k$'s are indeed modular. We leave open the question of whether modularity is necessary for the implication $T_0 \implies T_2$, although we do point out that Polin's variety fails to satisfy $T_0 \implies T_2$ and this variety is considered by some to be "barely nonmodular".

2. Preliminaries

We assume the reader is familiar with the basics of universal algebra and general topology. A *topological algebra* is a structure $\mathcal{A} = \langle A; \tau; O \rangle$, where $\langle A; O \rangle$ is an algebra and τ is a topology on A, such that each fundamental operation $F_i \in O$ is continuous with regard to the product topology on each power of A.

By definition, a variety $\mathcal{V}$ is *congruence k-permutable* if whenever $\mathcal{A} \in \mathcal{V}$ and α and β are congruences on $\mathcal{A}$, then the k-fold alternating compositions $\alpha \circ \beta \circ \alpha \circ \cdots$ and $\beta \circ \alpha \circ \beta \circ \cdots$ are equal. This definition will play absolutely no role in this paper. Rather, we will work with the Hagemann-Mitschke terms characterizing this property:

Theorem 4. [11] *A variety $\mathcal{V}$ is k-permutable if and only if there exist ternary $\mathcal{V}$-terms $p_0, \ldots, p_k$ such that the following are identities of $\mathcal{V}$:*

$$\begin{aligned} &p_0(x,y,z) \approx x \\ &p_i(x,x,z) \approx p_{i+1}(x,z,z) \ \textit{for}\ 0 \leq i \leq k-1 \\ &p_k(x,y,z) \approx z \end{aligned}$$

By definition, a variety $\mathcal{V}$ is *congruence modular* if all algebras in $\mathcal{V}$ have modular congruence lattices. This definition also plays no role in this paper. We will work only with the Day terms and the Gumm terms, which each characterize modularity:

Theorem 5. [6] *A variety $\mathcal{V}$ is modular if and only if there exist quaternary $\mathcal{V}$-terms $m_0, \ldots, m_n$ such that the following are identities of $\mathcal{V}$:*

$$\begin{aligned} &m_0(x,y,z,w) \approx x \\ &m_i(x,x,w,w) \approx m_{i+1}(x,x,w,w) \quad 0 \leq i < n,\ i \textit{ even} \\ &m_i(x,y,y,w) \approx m_{i+1}(x,y,y,w) \quad 0 \leq i < n,\ i \textit{ odd} \\ &m_n(x,y,z,w) \approx w \end{aligned}$$

Theorem 6. [9] *A variety $\mathcal{V}$ is modular if and only if there exist ternary $\mathcal{V}$-terms $q_0, \ldots, q_{n-1}, p$ such that the following are identities of $\mathcal{V}$:*

$$\begin{aligned} q_0(x,y,z) &\approx x \\ q_i(x,y,x) &\approx x && \textit{for all } i \\ q_i(x,x,y) &\approx q_{i+1}(x,x,y) && 0 \le i < n-1,\ i \textit{ even} \\ q_i(x,y,y) &\approx q_{i+1}(x,y,y) && 0 \le i < n-1,\ i \textit{ odd} \\ q_{n-1}(x,y,y) &\approx p(x,y,y) \\ p(x,x,y) &\approx y \end{aligned}$$

Our interest in the properties of k-permutability and modularity is limited to the fact that the existence of continuous operations satisfying the identities of any of the last three theorems restricts the topology of a topological algebra.

Now we turn from algebraic preliminaries to topological preliminaries. Recall from the Introduction that a space X is Hausdorff, or T_2, if for each $a \in X$ the intersection of the closures of the neighborhoods of a is $\{a\}$. This definition of T_2 suggests the following generalization.

Definition 7. Suppose that A is a topological space. For each $a \in A$ and $n < \omega$ define $\Delta_n(a)$ recursively by

$$\Delta_0(a) = A$$

$$\Delta_{n+1}(a) = \{\, b \mid \forall \text{ open } U, V \text{ with } a \in U, b \in V, \quad U \cap V \cap \Delta_n(a) \neq \emptyset \,\}$$

This definition implies that $\Delta_1(a)$ is the intersection of the closures of the neighborhoods of a. Thus $\Delta_1(a)$ is a closed subspace of A containing a. Each $\Delta_{n+1}(a)$ is the intersection of the closures of neighborhoods of a in the subspace $\Delta_n(a)$ under the relative topology. In particular, $\Delta_n(a)$ is closed in A for all a and n. We say that a point $a \in A$ is *j-step Hausdorff* if $\Delta_j(a) = \{a\}$. We say that a space is *j-step Hausdorff*, or H_j, if each of its points is j-step Hausdorff. Clearly, a space is H_1 if and only if it is Hausdorff since both properties say exactly that $\Delta_1(a) = \{a\}$ for all $a \in A$. Since each $\Delta_n(a)$ is closed, and since H_j asserts that $\Delta_j(a) = \{a\}$ for all $a \in A$, it follows that $\mathrm{H}_j \Longrightarrow T_1$. Coleman proved in [4] that all the conditions H_j are distinct and strictly stronger than T_1.

Coleman defined the concept of j-step Hausdorffness in terms of the complement $\Gamma_n(a) = A \setminus \Delta_n(a)$. We prefer to work with $\Delta_n(a)$ instead of $\Gamma_n(a)$ because of the usefulness of the following extension of the notation.

Definition 8. For each $n \geq 0$, let the symbol Δ_n denote the binary relation defined by

$$a \,\Delta_n\, b :\Longleftrightarrow a \in \Delta_n(b)$$

The usefulness of switching from Γ_n to Δ_n is clear from the next result.

Lemma 9. *Let A be a topological space. For each $k \geq 0$, Δ_k is a reflexive binary relation on A that is compatible with every continuous map $f : A^n \to A$, for $n \geq 0$.*

Proof. The reflexivity is clear. We will prove compatibility by induction on k, the result being clear for $k = 0$ (Δ_0 is the universal relation on A). So, suppose the result true of k, let $f : A^n \to A$, and, for $1 \leq i \leq n$, let $a_i, b_i \in A$ with $a_i \,\Delta_{k+1}\, b_i$. We have to show that

$$f(a_1, \ldots, a_n) \,\Delta_{k+1}\, f(b_1, \ldots, b_n)$$

Thus, let U, V be open sets such that

$$f(a_1,\dots,a_n)\in U, f(b_1,\dots,b_n)\in V$$

By continuity, we can find open sets A_i, B_i such that $a_i\in A_i$, $b_i\in B_i$ for each i, $1\le i\le n$, and

$$f(A_1,\dots,A_n)\subseteq U, f(B_1,\dots,B_n)\subseteq V$$

Since $a_i\,\Delta_{k+1}\,b_i$, we can pick

$$c_i\in A_i\cap B_i\cap\Delta_k(b_i)$$

so we have

$$f(c_1,\dots,c_n)\in U\cap V$$

and, by the induction hypothesis,

$$f(c_1,\dots,c_n)\,\Delta_k\,f(b_1,\dots,b_n)$$

Thus

$$U\cap V\cap\Delta_k(f(b_1,\dots,b_n))\neq\emptyset$$

and, since U and V were arbitrary,

$$f(a_1,\dots,a_n)\,\Delta_{k+1}\,f(b_1,\dots,b_n)$$

as desired. □

Corollary 10. *If $\mathcal{A}$ is a topological algebra, Δ_k is a reflexive and compatible binary relation on $\mathcal{A}$, for every $k\ge 0$.*

The relation Δ_k need not be symmetric, except of course when $k=0$ (since $\Delta_0=A\times A$) and when $k=1$ (since Δ_1 is the closure of the diagonal of $A\times A$).

We will henceforth adopt the following equivalent definition of j-step Hausdorffness.

Definition 11. Let A be a topological space. For each $j\ge 0$, we will say that A is *j-step Hausdorff*, or H_j, if the following condition holds for all $a,b\in A$:

$$a\,\Delta_j\,b\implies a=b \qquad (\mathrm{H}_j)$$

In other words, H_j is the assertion that Δ_j is the equality relation.

We introduce a new family of separation conditions related to the H_j's.

Definition 12. Let A be a topological space. For every $j\ge 0$, we will say that A is *j-step Hausdorff up to symmetry*, or sH_j, if the following condition holds for all $a,b\in A$:

$$a\,\Delta_j\,b\ \wedge\ b\,\Delta_j\,a\Longrightarrow a=b \qquad (\mathrm{sH}_j)$$

Thus sH_j asserts that Δ_j is antisymmetric.

Remark. All sH_j conditions are distinct: Coleman's examples of spaces which are H_j but not H_{j-1} ([4, Theorem 2.4]) also satisfy sH_j but not sH_{j-1}. We will presently give a construction, generalizing Coleman's, which yields many more examples.

3. Topological Properties Definable by Maltsev Conditions

This section, which proves the result mentioned in the first paragraph of the abstract, can be read independently of the rest of the paper.

For the purposes of this section only, we consider topological algebras to be 2-sorted first-order structures $\mathcal{A} = \langle A; B; O, R, \epsilon \rangle$ where A and B are the sorts, O is a set of operations on A, R is a set $\{\Delta_n\}_{n=0}^{\infty}$ of binary relations on A, and ϵ is a binary relation from A to B. The sort A is intended to represent the set of elements of the topological algebra, sort B is intended to represent a basis for the topology, the operations in O are intended to be the operations of the topological algebra, the relations $\Delta_n \in R$ are intended to be the ones defined in the preceding section, and ϵ is intended to denote the relation of membership of elements of A in elements of B. (N.B.: actual set membership in A or B will be indicated with $\in$.)

The class of all 2-sorted structures of the type that we are considering whose symbols have their intended meanings is a first-order axiomatizable class. Indeed, the following statements are first-order, and they define the class of all topological algebras of a given signature, each with a specified basis B, with the correct interpretation of the relation symbols.

(1) For all $U, V \in B$, if $x \,\epsilon\, U \iff x \,\epsilon\, V$ for all $x \in A$, then $U = V$.
(2) There exists a $U \in B$ such that $x \,\epsilon\, U$ holds for no $x \in A$.
(3) For all $U, V \in B$ there exists a $W \in B$ such that $x \,\epsilon\, W$ if and only if $x \,\epsilon\, U$ and $x \,\epsilon\, V$.
(4) For all $x \in A$ there exists a $U \in B$ such that $x \,\epsilon\, U$.
(5) For all $x_1, \dots, x_n \in A$ and all $U \in B$, if $f(x_1, \dots, x_n) \,\epsilon\, U$, then there exist $V_1, \dots, V_n \in B$ such that $f(y_1, \dots, y_n) \,\epsilon\, U$ whenever $y_i \,\epsilon\, V_i$.
(6) Each symbol Δ_n interprets as the relation from Definition 8.

Statement (1) asserts that basis elements U and V may be distinguished by their "elements"; that is, by their ϵ-related elements of A. Thus any $U \in B$ may be identified with its subset of ϵ-related elements. Statements (2)–(4) assert that the collection of subsets of A that correspond to elements of B under this identification contains $\emptyset$, is closed under finite intersection, and has union equal to A. Thus (1)–(4) assert that B is a basis for a topology on A. Statement (5) asserts that the operations of $\mathcal{A}$ are continuous in this topology. Statement (6) asserts that the relation symbols Δ_n are names for the relations defined in the preceding section.

Lemma 13. *Each of of the statements* (1)–(6) *is equivalent to a first-order sentence. Statements* (2)–(5) *are equivalent to Horn sentences. Statement* (6) *is equivalent to a sentence* $(\forall x, y \in A)[y \;\Delta_n\; x \iff \Phi_n(x, y)]$ *where* $\Phi_n(x, y)$ *is a factorable formula.*

Proof. Recall that a Horn formula is a formula that in prenex form looks like $Q_1 x_1, \dots, Q_k x_k \left(\bigwedge \Psi_i\right)$ where each Q_i is a quantifier and each Ψ_i is a formula. Moreover, each Ψ_i has the form $\psi_1 \vee \cdots \vee \psi_k$ with each ψ_j atomic or negated atomic and at most one ψ_j atomic in any Ψ_i. A Horn sentence is a Horn formula that is a sentence.

A formula $\Phi(x_1, \dots, x_m)$ is *factorable* if whenever $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$ is a product of structures, and $\mathbf{a}_1, \dots, \mathbf{a}_m \in A$, then

$$\mathcal{A} \models \Phi(\mathbf{a}_1, \dots, \mathbf{a}_m) \iff \mathcal{A}_i \models \Phi(a_{i1}, \dots, a_{im}) \text{ for all } i \in I.$$

(In [13] such formulas are called *multiplicative*, in [14] they are called *filtering*, and in [20] they are called *formulas evaluated coordinatewise*.) The class of factorable formulas has the following closure properties (see [20]):

(F1) Any atomic formula is factorable.
(F2) The class of factorable formulas is closed under $\forall, \exists$ and $\wedge$.
(F3) If $\alpha(\bar{x}, \bar{y})$ and $\beta(\bar{x}, \bar{y})$ are factorable, then so is

$$(\exists \bar{x})[\alpha(\bar{x}, \bar{y})] \wedge (\forall \bar{x})[\alpha(\bar{x}, \bar{y}) \implies \beta(\bar{x}, \bar{y})].$$

Now we write (1)–(6) in first-order/Horn/factorable form.

[Sentence (1)] (First-order)

$$(\forall U, V \in B)\,[[(\forall x \in A)(x \,\epsilon\, U \Leftrightarrow x \,\epsilon\, V)] \Rightarrow U = V]\,.$$

[Sentence (2)] (Horn)

$$(\exists U \in B)(\forall x \in A)\,[x \not\epsilon\, U]\,.$$

[Sentence (3)] (Horn)

$$\begin{aligned}(\forall U, V \in B)(\exists W \in B)(\forall x \in A) \quad & [((x \not\epsilon\, W) \vee (x \,\epsilon\, U)) \\ & \wedge((x \not\epsilon\, W) \vee (x \,\epsilon\, V)) \\ & \wedge((x \not\epsilon\, U) \vee (x \not\epsilon\, V) \vee (x \,\epsilon\, W))].\end{aligned}$$

[Sentence (4)] (Horn)

$$(\forall x \in A)(\exists U \in B)\,[(x \,\epsilon\, U)]\,.$$

[Sentence (5)] (Horn)

$$(\forall \bar{x} \in A^n)(\forall U \in B)(\exists \bar{V} \in B^n)(\forall \bar{y} \in A^n)\left[\left(\bigwedge_{i=1}^{n} \Psi_n\right) \wedge \Theta\right]$$

where Ψ_i is the Horn clause $[(f(\bar{x}) \not\epsilon\, U) \vee (x_i \,\epsilon\, V_i)]$ and Θ is the Horn clause $[(f(\bar{x}) \not\epsilon\, U) \vee \bigvee_{i=1}^{n} (y_i \not\epsilon\, V_i) \vee (f(\bar{y}) \,\epsilon\, U)]$.

[Sentence (6)] ($(\forall x, y \in A)[y \,\Delta_n\, x \Leftrightarrow \Phi_n(x, y)]$ with $\Phi_n(x, y)$ factorable)

We define Φ_n inductively. Let $\Phi_0(x, y_0)$ be $[x = x]$. This is factorable by (F1) above. Now assume that $\Phi_k(x, y_k)$ is factorable, and let $\Phi_{k+1}(x, y_{k+1})$ be

$$(\exists U, V \in B)[\alpha(U, V, x, y_{k+1})] \wedge (\forall U, V \in B)[\alpha(U, V, x, y_{k+1}) \implies \beta(U, V, x, y_{k+1})]$$

where

$$\alpha(U, V, x, y_{k+1}) = [(x \,\epsilon\, U) \wedge (y_{k+1} \,\epsilon\, V)]$$

and

$$\beta(U, V, x, y_{k+1}) = (\exists y_k)[(y_k \epsilon U) \wedge (y_k \epsilon V) \wedge (\Phi_k(x, y_k))].$$

It follows from (F1), (F2) and induction that both α and β are factorable. Since $\Phi_{k+1}(x, y_{k+1})$ has the form described in (F3) it is factorable.

To see that $\Phi_n(x, y)$ defines the relation Δ_n from Definition 8 in any topological algebra, note first that the initial clause $(\exists U, V \in B)[\alpha(U, V, x, y_n)]$ expands to

$$(\exists U, V \in B)[(x \,\epsilon\, U) \wedge (y_{k+1} \,\epsilon\, V)],$$

which holds in every topological algebra. Thus, what needs to be verified inductively is that $(\forall U, V \in B)[\alpha(U, V, x, y_{k+1}) \implies \beta(U, V, x, y_{k+1})]$ is equivalent to $y_{k+1} \,\Delta_{k+1}\, x$. Expanding the formula we are considering,

$$(\forall U, V \in B)[((x \,\epsilon\, U) \wedge (y_{k+1} \,\epsilon\, V)) \implies (\exists y_k)[(y_k \epsilon U) \wedge (y_k \epsilon V) \wedge (\Phi_k(x, y_k))],$$

we see by inspection that this is a direct translation of Definitions 7 and 8. □

Sentence (1), which expresses extentionality, is not equivalent to a Horn sentence since it is not preserved by products. Specifically, if $\mathcal{A}$ satisfies Sentence (1), U, V are distinct sets in B, and U is "empty" in the sense that no $x \in A$ satisfies $x \epsilon U$, then in $\mathcal{A} \times \mathcal{A}$ the sets $U \times V, V \times U \in B \times B$ are different because they are different coordinatewise. But they are both "empty". Hence extentionality is not preserved under products. Throughout this section we shall be concerned with topological properties that are preserved by products and ultraproducts. Thus, we now restrict our attention to 2-sorted structures $\mathcal{A} = \langle A; B; O, R, \epsilon\rangle$ axiomatized by Sentences (2)–(6). Such structures are precisely those derived from topological algebras $\langle A; \tau; O\rangle$ with all symbols having their intended meanings, except that the sort B denotes only a set of names for the basis elements of the topology, and we allow multiple names for any basis element.

Let $\mathcal{A}_i = \langle A_i; B_i; O, R, \epsilon\rangle, i \in I$, be a family of nonempty structures satisfying Sentences (2)–(6). The product $\prod_{i\in I} \mathcal{A}_i = \left\langle \prod_{i\in I} A_i; \prod_{i\in I} B_i; O, R, \epsilon\right\rangle$ satisfies (2)–(5), since these sentences are Horn, hence the product corresponds to a topological algebra. It is immediate from the well known definition that the topology on $\prod_{i\in I} A_i$ generated by $\prod_{i\in I} B_i$ is what is usually called the *box topology.* For finite products the box topology is identical with the *product topology.* Since the topological relation Δ_n is definable by a factorable formula, the symbol Δ_n interprets as the relation introduced in Definition 8 in a product if and only if it interprets as that relation in each factor.

Now let $\mathcal{U}$ be an ultrafilter on I. The *ultraproduct* (*over* $\mathcal{U}$) of the sets $A_i, i \in I$, is the set $\prod_{\mathcal{U}} A_i$, defined to be the quotient of the product set $A = \prod_{i\in I} A_i$ by the equivalence relation $\theta_{\mathcal{U}} = \{(\mathbf{a}, \mathbf{b}) \in A^2 \mid [\![\mathbf{a} = \mathbf{b}]\!] \in \mathcal{U}\}$ where $[\![\mathbf{a} = \mathbf{b}]\!] = \{i \in I \mid \mathbf{a}_i = \mathbf{b}_i\}$ denotes the set of coordinates where $\mathbf{a}$ and $\mathbf{b}$ are equal. The ultraproduct $\prod_{\mathcal{U}} \mathcal{A}_i = \langle \prod_{\mathcal{U}} A_i; \prod_{\mathcal{U}} B_i; O, R, \epsilon\rangle$ satisfies (2)–(5), since these sentences are first-order and assumed to hold in each coordinate, hence the ultraproduct corresponds to a topological algebra. The topology on $\prod_{\mathcal{U}} A_i$ generated by $\prod_{\mathcal{U}} B_i$ is called the *ultraproduct topology.* The ultraproduct topology can be constructed in a different way, as follows: First give the product $\prod_{i\in I} A_i$ the box topology (whose basis is $\prod_{i\in I} B_i$). Then give the quotient set $\prod_{\mathcal{U}} A_i$ the quotient topology induced by the natural map $\nu : \prod_{i\in I} A_i \to \left(\prod_{i\in I} A_i\right)/\theta_{\mathcal{U}} : \mathbf{a} \mapsto \mathbf{a}/\theta_{\mathcal{U}}$.

Lemma 14. *Each of the following topological properties is expressible by a first-order sentence that is preserved by products.*

(1) T_0, T_1, T_2.
(2) H_j, sH_j, $j = 1, 2, \ldots$.
(3) $T_0 \Longrightarrow T_i$, $i = 1, 2$.
(4) $T_0 \Longrightarrow \mathrm{H}_j$, $T_0 \Longrightarrow \mathrm{sH}_j$, $j = 1, 2, \ldots$.

Proof. We first show that each property T_i, H_j, or sH_j is expressible by a factorable sentence. First, notice that the relation "$x \in \mathrm{cl}(y)$" is expressible by the factorable formula $(\exists U \in B)[\alpha(U, x, y)] \wedge (\forall U \in B)[\alpha(U, x, y) \Rightarrow \beta(U, x, y)]$ (an instance of (F3)) where $\alpha(U, x, y)$ is $(x \epsilon U)$ and $\beta(U, x, y)$ is $(y \epsilon U)$ (instances of (F1)).

T_0 is expressed by the factorable sentence

$$(\exists x, y \in A)[\alpha(x, y)] \wedge (\forall x, y \in A)[\alpha(x, y) \Longrightarrow \beta(x, y)]$$

where $\alpha(x, y)$ is the factorable formula $(x \in \mathrm{cl}(y)) \wedge (y \in \mathrm{cl}(x))$, and $\beta(x, y)$ is $(x = y)$. T_1 is expressed by the factorable sentence that is the same as the one for T_0 except $\alpha(x, y)$ is changed to $(x \in \mathrm{cl}(y))$.

Since $T_2 = H_1$, we may now turn to H_j and sH_j. H_j is expressible as

$$(\exists x, y \in A)[\alpha(x,y)] \wedge (\forall x, y \in A)[\alpha(x,y) \Longrightarrow \beta(x,y)]$$

where $\alpha(x,y)$ is $\Phi_j(x,y)$ and $\beta(x,y)$ is $(x = y)$. The property sH_j is expressible as

$$(\exists x, y \in A)[\alpha(x,y)] \wedge (\forall x, y \in A)[\alpha(x,y) \Longrightarrow \beta(x,y)]$$

where $\alpha(x,y)$ is $\Phi_j(x,y) \wedge \Phi_j(y,x)$ and $\beta(x,y)$ is $(x = y)$. This completes the proof of parts (1) and (2) of the lemma since factorable sentences are preserved by products.

If Q and R are factorable sentences, then $Q \Longrightarrow R$ is a sentence that is preserved by products. To see this, suppose that each $\mathcal{A}_i$ satisfies $Q \Longrightarrow R$, but $\prod_{i \in I} \mathcal{A}_i$ does not satisfy it. Then the product satisfies Q and does not satisfy R. Since these sentences are factorable, every factor satisfies Q and some factor fails to satisfy R. But then the factor that fails R also fails $Q \Longrightarrow R$, contrary to assumption. Thus, it follows from what we proved above that the properties in parts (3) and (4) are expressible by sentences that are preserved by products. □

The previous lemma shows that if P denotes one of the implications $T_0 \Longrightarrow T_i$ or $T_0 \Longrightarrow \mathrm{H}_j$, then P is preserved by products and (since P is first-order expressible) $\neg P$ is preserved by ultraproducts. This leads us to the main theorem of this section.

Theorem 15. *Let P be a property of topological spaces, and let $[P]$ be the class of all varieties $\mathcal{V}$ having the property that any topological algebra in $\mathcal{V}$ has underlying space satisfying property P. If P is preserved by finite products, and if $\neg P$ is preserved by ultraproducts, then $[P]$ is a class of varieties that is definable by a Maltsev condition.*

Remark. Our assumption that P is preserved by finite products includes the preservation of P by the empty product. Hence our assumption implies that P is true for a 1-element space. Moreover, since products are unique only up to isomorphism, the assumption that P is preserved by products of one factor is equivalent to the assumption that P is an isomorphism invariant.

Proof. W. Taylor showed in [18] that a nonempty class $\mathcal{K}$ of varieties is definable by a Maltsev condition if and only if

(1) $\mathcal{K}$ is closed under the formation of equivalent varieties;
(2) $\mathcal{K}$ is closed under the formation of subvarieties;
(3) $\mathcal{K}$ is closed under the formation of finite products of varieties;
(4) if $\mathcal{V} \in \mathcal{K}$ and $\mathcal{V}$ is generated by all reducts of members of $\mathcal{W}$ to the type of $\mathcal{V}$, then $\mathcal{W} \in \mathcal{K}$; and
(5) if the equations Σ define a variety in $\mathcal{K}$ of signature σ, then there exist finite subsets $\Sigma_0 \subseteq \Sigma$ and $\sigma_0 \subseteq \sigma$ with Σ_0 defining a variety in $\mathcal{K}$ of signature σ_0.

We apply this result to $\mathcal{K} = [P]$. This class is nonempty, since our assumption that P is preserved by finite products implies that any variety of trivial algebras is in $[P]$.

The underlying space of a topological algebra in a variety that is:

(1) equivalent to some $\mathcal{V} \in [P]$,
(2) is a subvariety of some $\mathcal{V} \in [P]$, or
(4) is some $\mathcal{W}$ whose reducts to the type of some $\mathcal{V} \in [P]$ are $\mathcal{V}$-algebras

is also the underlying space of a $\mathcal{V}$-algebra. Thus Taylor's conditions (1), (2) and (4) hold for $[P]$.

If $\mathcal{V}, \mathcal{V}' \in [P]$ and $\mathcal{A}$ is a topological algebra in $\mathcal{V} \times \mathcal{V}'$, then according to Proposition 5 of Chapter 1 of [8] there are topological algebras $\mathcal{B} \in \mathcal{V}$ and $\mathcal{B}' \in \mathcal{V}'$ such that $\mathcal{A} \cong \mathcal{B} \times \mathcal{B}'$ *as topological algebras*. In particular, the underlying space of $\mathcal{A}$ is the product of the underlying spaces of $\mathcal{B}$ and $\mathcal{B}'$. Hence if P is a property that is preserved by finite products, then (1)–(4) of Taylor's characterization hold for $[P]$.

We now prove that if $\neg P$ is preserved by ultraproducts, then property (5) of Taylor's characterization holds for $[P]$. Let $\mathcal{V} \in [P]$ be a variety of signature σ that is axiomatized by the set of equations Σ. Define I to be the set of all pairs (σ_0, Σ_0) of finite subsets $\sigma_0 \subseteq \sigma, \Sigma_0 \subseteq \Sigma$. For each $i = (\sigma_0, \Sigma_0) \in I$ define

$$U_i = \{(\sigma_0', \Sigma_0') \in I \mid \sigma_0 \subseteq \sigma_0', \Sigma_0 \subseteq \Sigma_0'\}.$$

No U_i is empty, since $i \in U_i$. Since

$$U_{(\sigma_0,\Sigma_0)} \cap U_{(\sigma_0',\Sigma_0')} = U_{(\sigma_0 \cup \sigma_0', \Sigma_0 \cup \Sigma_0')},$$

the collection of all $U_i, i \in I$, is a filter on I. Let $\mathcal{U}$ be an ultrafilter extending this filter. For each $i = (\sigma_0, \Sigma_0) \in I$ let $\mathcal{V}_i$ be the variety of signature σ_0 that is axiomatized by Σ_0.

We are done if some $\mathcal{V}_i \in [P]$, so suppose that no $\mathcal{V}_i \in [P]$. Then for each i we can find a topological algebra $\mathcal{A}_i \in \mathcal{V}_i$ whose underlying space fails to satisfy P. If $i = (\sigma_0, \Sigma_0)$, then we can expand $\mathcal{A}_i$ to a topological algebra $\hat{\mathcal{A}}_i$ of signature σ by defining each operation in $\sigma - \sigma_0$ to be an arbitrary constant operation on A_i of the right arity. Since constant operations are continuous, the family $\hat{\mathcal{A}}_i$, $i \in I$, consists of topological algebras of signature σ whose underlying spaces fail to have property P. Since the property of being a topological algebra is first-order (Sentences (2)–(6) from the beginning of the section), and since we have assumed that $\neg P$ is preserved by ultraproducts, we get that $\prod_{\mathcal{U}} \hat{\mathcal{A}}_i$ is a topological algebra of signature σ that fails to have property P. Moreover, $\prod_{\mathcal{U}} \hat{\mathcal{A}}_i \in \mathcal{V}$, as we now argue. Choose any equation $\varepsilon \in \Sigma$. Let σ_0 be the set of operation symbols that occur in ε, and let $\Sigma_0 = \{\varepsilon\}$. Then $(\sigma_0, \Sigma_0) = i$ for some $i \in I$. According to our definitions, $[\![\varepsilon]\!]$ contains $U_i \in \mathcal{U}$, so $[\![\varepsilon]\!] \in \mathcal{U}$. By Łos's Theorem $\prod_{\mathcal{U}} \hat{\mathcal{A}}_i$ satisfies ε. Since ε was arbitrary, $\prod_{\mathcal{U}} \hat{\mathcal{A}}_i$ satisfies Σ, and therefore $\prod_{\mathcal{U}} \hat{\mathcal{A}}_i \in \mathcal{V}$. As the underlying space of $\prod_{\mathcal{U}} \hat{\mathcal{A}}_i$ fails to satisfy P, we conclude that $\mathcal{V} \notin [P]$, a contradiction. The assumption that led to this contradiction is that no $\mathcal{V}_i \in [P]$. Hence (5) is established. □

Corollary 16. *Let P be one of the implications $T_0 \Longrightarrow T_i$ or $T_0 \Longrightarrow \mathrm{H}_j$. The class $[P]$ is definable by a Maltsev condition.*

Remarks. Although we have not defined $T_{2\frac{1}{2}}, T_3, T_{3\frac{1}{2}}$ and T_4 in this paper, the reader can easily locate their definitions. We leave it as an exercise for the interested reader to show that $T_0 \Longrightarrow T_{2\frac{1}{2}}$ and $T_0 \Longrightarrow T_3$ are properties that are preserved by products and whose negations are preserved by ultraproducts, hence these topological implications correspond to Maltsev conditions.

The implication $T_0 \Longrightarrow T_4$ is not preserved by finite products, nor is its negation preserved by ultraproducts. Yet for this property P the class $[P]$ is definable by a Maltsev condition. This is because any nontrivial variety contains topological algebras that are T_0 but not T_4! (Hence $[P]$ is the Maltsev-definable class of trivial varieties.) The reason that this is true is that if $\mathcal{A} \in \mathcal{V}$ is any nontrivial algebra equipped with the discrete topology and $\mathcal{A}^{\omega_2}$ is given the product topology, then the ultrapower $\prod_{\mathcal{U}} \mathcal{A}^{\omega_2}$ where $\mathcal{U}$ is a nonprincipal ultrafilter on a countable set is a T_0 topological algebra in $\mathcal{V}$ that is not T_4. (See the corollary to Theorem 8.2 of [1] for details.)

The property $T_0 \Longrightarrow T_{3\frac{1}{2}}$ is preserved by finite products, but its negation is not preserved by ultraproducts (see [1]). Therefore Theorem 15 does not apply to show that the class $[P]$ for this P is definable by a Maltsev condition. However, it is a classical result due to Pontryagin that $T_0 \Longrightarrow T_{3\frac{1}{2}}$ holds for the variety of groups (see [12]). Thus, it may be interesting to determine whether or not this implication corresponds to a Maltsev condition. We conjecture that it does.

4. k-Permutable Varieties Satisfy $T_0 \Longrightarrow \mathrm{H}_{\lfloor \frac{k}{2} \rfloor}$

Lemma 17. *Let $\mathcal{A}$ be a T_0 topological algebra in a k-permutable variety. Let $a, b \in A$ and suppose $a \in \Delta_j(b)$, for some $j \geq 0$. Then*

$$p_{1+j}(b, a, a) = b = p_{k-1-j}(a, a, b)$$

Proof. First note that $\mathcal{A}$ is T_1, by Theorem 2. We prove the first equality. A symmetric argument yields the second one. The proof follows by induction on j. The equality clearly holds for $j = 0$. Let $j > 0$, suppose the result holds for $j - 1$ and let $a \in \Delta_j(b)$. Suppose, by way of contradiction, that $p_{1+j}(b, a, a) \neq b$; thus

$$p_j(b, b, a) = p_{1+j}(b, a, a) \in A \setminus \{ b \}$$

By T_1, $A \setminus \{ b \}$ is open, so by continuity there exist open sets $V \ni b$, $U \ni a$ such that

$$p_j(b, V, U) \subseteq A \setminus \{ b \}$$

Since $a \in \Delta_j(b)$, we can take an element $c \in U \cap V \cap \Delta_{j-1}(b)$; thus

$$p_j(b, c, c) \in A \setminus \{ b \}$$

contradicting the induction hypothesis. □

Remark. Since $\Delta_j(b) \subseteq \Delta_l(b)$ for each $l \leq j$, we have, for $a \in \Delta_j(b)$,

$$p_1(b, a, a) = p_2(b, a, a) = \cdots = p_{1+j}(b, a, a) = b$$

From this lemma, Coleman's Theorem 3 follows easily:

New proof of Theorem 3. Suppose $k \geq 3$ and $\mathcal{A}$ is a T_0 topological algebra in a k-permutable variety $\mathcal{V}$. Choose $a, b \in A$ with $a \in \Delta_{k-2}(b)$. Since $k - 2 \geq 1$, $\Delta_{k-2} \subseteq \Delta_1$, so $a \,\Delta_1\, b$ and thus also $b \,\Delta_1\, a$ (since Δ_1 is symmetric, as noted after Corollary 10). Using the first equality in Lemma 17 we get

$$p_{k-1}(b, a, a) = p_{1+k-2}(b, a, a) = b$$

and using the second equality together with $b \in \Delta_1(a)$ we get

$$p_{k-2}(b, b, a) = p_{k-1-1}(b, b, a) = a.$$

Thus

$$a = p_{k-2}(b, b, a) = p_{k-1}(b, a, a) = b.$$

This shows that Δ_{k-2} is the equality relation, so $\mathcal{A}$ is H_{k-2}. □

Lemma 18. *Let $\mathcal{A}$ be an algebra in a k-permutable variety. If θ is a reflexive, antisymmetric and compatible relation on $\mathcal{A}$, then θ is the identity relation.*

Proof. Choose $a, b \in A$ with $a \,\theta\, b$. Clearly, $p_1(b, a, a) = b$. If $p_i(b, a, a) = b$, then

$$b = p_i(b, a, a) \;\theta\; p_i(b, b, a) \;\theta\; p_i(b, b, b) = b$$

so $\langle b, p_i(b, b, a)\rangle \in \theta \cap \theta^{\cup}$, which is the equality relation. Hence $b = p_i(b, b, a) = p_{i+1}(b, a, a)$. By induction, $p_i(b, a, a) = b$ holds for all $i \geq 1$, and so $a = p_k(b, a, a) = b$. □

Combining Lemma 18 and Corollary 10, we can now easily derive the following

Theorem 19. *For topological algebras in a k-permutable variety, the conditions H_j and sH_j coincide.*

Proof. Just note that Δ_j is a reflexive compatible relation, H_j means Δ_j is the identity relation, and sH_j means Δ_j is antisymmetric. □

We can now state and prove the main theorem of this section.

Theorem 20. *Let $k \geq 1$. For topological algebras in a k-permutable variety,*

$$T_0 \implies \mathrm{H}_{\lfloor \frac{k}{2} \rfloor}$$

Proof. Let $\mathcal{A}$ be a T_0 topological algebra in a k-permutable variety. By the previous theorem, it is enough to show $\mathcal{A}$ is $\mathrm{sH}_{\lfloor \frac{k}{2} \rfloor}$. Suppose $a, b \in A$ with $a \,\Delta_{\lfloor \frac{k}{2} \rfloor}\, b$ and $b \,\Delta_{\lfloor \frac{k}{2} \rfloor}\, a$. By Lemma 17, we have

$$b = p_{\lfloor \frac{k}{2} \rfloor}(b, a, a) = p_{1+\lfloor \frac{k}{2} \rfloor}(b, a, a)$$

and

$$a = p_{k-1-\lfloor \frac{k}{2} \rfloor}(b, b, a)$$

Thus, by the Hagemann-Mitschke identities,

$$a = p_{k-1-\lfloor \frac{k}{2} \rfloor}(b, b, a) = p_{k-\lfloor \frac{k}{2} \rfloor}(b, a, a) = p_{\lceil \frac{k}{2} \rceil}(b, a, a) = b$$

since $\lceil \frac{k}{2} \rceil$ is equal to either $\lfloor \frac{k}{2} \rfloor$ or $1 + \lfloor \frac{k}{2} \rfloor$ (depending on whether k is even or odd). □

We cannot improve the subscript in Theorem 20 since we have:

Theorem 21. *For each $k \geq 2$, there exists a k-permutable variety containing a topological algebra which satisfies $\mathrm{H}_{\lfloor \frac{k}{2} \rfloor}$ but not $\mathrm{H}_{\lfloor \frac{k}{2} \rfloor - 1}$.*

The proof of this theorem will occupy the next section.

5. Not All k-Permutable Varieties Satisfy $T_0 \implies \mathrm{H}_{\lfloor \frac{k}{2} \rfloor - 1}$

Definition 22. For each $k \geq 2$, $\mathcal{P}_k$ denotes the variety defined with $k+1$ ternary fundamental operations $p_0, \ldots, p_k$ obeying the identities of Theorem 4.

From Theorem 4 we have that $\mathcal{P}_k$ is k-permutable.

It is sometimes convenient to allow for extra operations p_j with $j > k$: we will do so, with the assumption that such operations always act as the third projection; we also allow for p_j with $j < 0$, these always acting as the first projection. Note that with these conventions the identities $p_j(x, x, z) = p_{j+1}(x, z, z)$ still hold for all j and each $\mathcal{P}_k$ may be construed as the subvariety of $\mathcal{P}_{k+1}$ defined by the extra identity $p_k(x, y, z) \approx z$.

We need a few relatively easy syntactic facts about $\mathcal{P}_k$. We do not offer detailed proofs here, as these can be derived from knowledge about the word problem for $\mathcal{P}_k$, which is easily solvable.

Lemma 23. *Let $k \geq 2$, let X be a set and let $\mathcal{F}$ be a $\mathcal{P}_k$-algebra, freely generated by X. Let $x \in X$. Then*

(i) *$p_i(a, b, c) = x$ if and only if one of the following conditions holds:*
  a) *$i \leq 0$ and $a = x$;*
  b) *$i = 1$, $b = c$ and $a = x$;*
  c) *$1 < i < k - 1$ and $a = b = c = x$;*
  d) *$i = k - 1$, $a = b$ and $c = x$;*
  e) *$i \geq k$ and $c = x$.*

(ii) *$F \setminus \{ x \}$ is a subuniverse of $\mathcal{F}$.*

Sketch of proof. (i) can be derived from the study of the word problem for $\mathcal{P}_k$ (see, for example, [3] or [16]). (ii) follows immediately from (i). □

We will need the following result of Taylor [19], which is an application of a previous result of Świerczkowski [17]:

Lemma 24. *Let (X, d) be a metric space and $\mathcal{V}$ be a nontrivial variety. There exists a metric $\hat{d}$ on $\mathcal{F} = F_{\mathcal{V}}(X)$, extending d and such that the $\mathcal{V}$-operations are continuous with respect to $\hat{d}$ (This $\hat{d}$ is called the* Świerczkowski metric *on $\mathcal{F}$).*

To prove Theorem 21, we will make use of the following construction, which is also present in [4] (although our definition is slightly different — see [4, Definition 2.3 (1)b]).

Definition 25. Let A and B be topological spaces, and let $b \in B$, such that $\{ b \}$ is closed in B. We denote by $A \rightsquigarrow_b B$ the space with underlying set $A \mathbin{\dot\cup} (B \setminus \{ b \})$ and such that a subset $U \subseteq A \rightsquigarrow_b B$ is open if and only if:

a) $U \cap A$ is A-open;
b) $U \cap B$ is B-open;
c) if $U \cap A \neq \emptyset$, then $(U \cap B) \cup \{ b \}$ is B-open.

The following lemma is implicit in [4], although it is proved there only for a particular case.

Lemma 26. *Let $k \geq 1$. Suppose A is a topological space which satisfies H_k, but not H_{k-1}. Suppose B is Hausdorff, and $b \in B$ is such that $\{ b \}$ is* not *open. Then $A \rightsquigarrow_b B$ satisfies H_{k+1}, but not H_k. The same holds if we replace each H_j by sH_j.*

Proof. Suppose A and B satisfy the hypotheses of the Lemma and let $X := A \rightsquigarrow_b B$. For each $j \geq 0$, let Δ_j^A denote Δ_j as calculated on the space A. Consider any $x \in X$. If $x \in B \setminus \{ b \}$, then we have $\Delta_1(x) = \{ x \}$ and thus, as $k + 1 \geq 1$, also $\Delta_{k+1}(x) = \{ x \}$. If $x \in A$, then $\Delta_1(x) = A = \Delta_0^A(x)$, from which we easily obtain $\Delta_{j+1}(x) = \Delta_j^A(x)$ for all $j \geq 0$. Thus, since A is H_k, $\Delta_{k+1}(x) = \{ x \}$ for all $x \in A$, so X is H_{k+1}. Also, as A is not H_{k-1}, there exists $x \in A \subseteq X$ such that $\Delta_k(x) = \Delta_{k-1}^A(x) \neq \{ x \}$, so X is not H_k. □

This lemma will be instrumental in our construction. We will use it to prove the next lemma, from which Theorem 21 easily follows.

Lemma 27. *Let $j, k \geq 1$, and suppose there exists a topological algebra in $\mathcal{P}_k$, which satisfies H_j, but not H_{j-1}. Then there exists a topological algebra in $\mathcal{P}_{k+2}$ which satisfies H_{j+1} but not H_j.*

Proof. Let $\mathcal{A}$ be a topological $\mathcal{P}_k$-algebra which satisfies H_j but not H_{j-1}. Let $\mathcal{B} = F_{\mathcal{P}_{k+2}}(\mathbb{R})$ be endowed with the topology induced by the Świerczkowski metric, and let $b = 0 \in B$. It is easy to see that the hypotheses of Lemma 26 are satisfied, and thus that $X := A \rightsquigarrow_0 B$ becomes a topological space which satisfies H_{j+1} but not H_j. To complete the proof of the lemma, we will presently define operations $p_0, \ldots, p_{k+2}$ on X, and show them to be continuous and obey the identities of $\mathcal{P}_{k+2}$. We will denote by $p_i^{\mathcal{A}}$, $p_j^{\mathcal{B}}$ the operations defined in $\mathcal{A}$ and $\mathcal{B}$. We define $\phi : X \to B$ by

$$\phi(x) = \begin{cases} 0 & \text{if } x \in A \\ x & \text{if } x \in B \end{cases}$$

and let $p_0(x, y, z) = x$, $p_{k+2}(x, y, z) = z$, and

$$p_1(x, y, z) = \begin{cases} x & \text{if } x, y, z \in A \\ x & \text{if } x \in A,\ y = z \in B \\ p_1^{\mathcal{B}}(\phi(x), \phi(y), \phi(z)) & \text{otherwise} \end{cases}$$

$$p_{k+1}(x, y, z) = \begin{cases} z & \text{if } x, y, z \in A \\ z & \text{if } z \in A,\ x = y \in B \\ p_{k+1}^{\mathcal{B}}(\phi(x), \phi(y), \phi(z)) & \text{otherwise} \end{cases}$$

and, for $1 < i < k + 1$,

$$p_i(x, y, z) = \begin{cases} p_{i-1}^{\mathcal{A}}(x, y, z) & \text{if } x, y, z \in A \\ p_i^{\mathcal{B}}(\phi(x), \phi(y), \phi(z)) & \text{otherwise} \end{cases}$$

Although 0 appears in the definition of these operations, the reader can check, using lemma 23, that the above definitions never produce 0 as the value of $p_i(x, y, z)$, and thus that the above do indeed define operations on X. To complete the proof of the lemma, we will establish the following two claims.

Claim 28. *The operations $p_0, \ldots, p_{k+2}$ defined on X satisfy the identities for $\mathcal{P}_{k+2}$.*

Claim 29. *The operations $p_0, \ldots, p_{k+2}$ defined on X are continuous.*

□

Proof of Claim 28.

- $p_1(x, z, z) = x$: if $x \in A$, this follows from one of the first two clauses, depending on whether $z \in A$ or $z \in B$; if $x \in B$, then $x = \phi(x)$ and the equality follows from the third clause by the identities of $\mathcal{B}$.
- $p_{k+1}(x, x, z) = z$: this is entirely analogous to the previous identity.
- $p_i(x, x, z) = p_{i+1}(x, z, z)$: again, this is clear in case x and z both belong to A or to B; in the other two possible cases, we always have

$$p_i(x, x, z) = p_i^{\mathcal{B}}(\phi(x), \phi(x), \phi(z)) = p_{i+1}^{\mathcal{B}}(\phi(x), \phi(z), \phi(z)) = p_{i+1}(x, z, z)$$

□

Proof of Claim 29. We will adhere to the following notations throughout this proof: for an (X-)open set U, we will let U_A denote $U \cap A$, U_B denote $U \cap B$ and U_B^0 denote $U_B \cup \{ 0 \}$.

- p_0 and p_{k+2} are continuous.

- p_1 is continuous: suppose first that $x_1, x_2, x_3 \in A$; let U be open such that $x_1 = p_1(x_1, x_2, x_3) \in U$. Choose A-open sets $A_i \ni x_i$ and B-open sets $B'_i \ni 0$ such that

$$A_1 \subseteq U_A$$
$$p_1^{\mathcal{B}}(B'_1, B'_2, B'_3) \subseteq U_B^0$$

Define $B_i := B'_i \setminus \{0\}$; note that since 0 is one of the free generators of the free algebra B, it follows from Lemma 23 that

$$p_1^{\mathcal{B}}(B_1, B_2, B_3) \subseteq U_B$$

Define $U_i := A_i \cup B_i$. Then $p_1(U_1, U_2, U_3) \subseteq U$. Thus p_1 is continuous on A^3.

Next, suppose $x_1 \in A$, $x_2 = x_3 \in B$ and U is open such that

$$x_1 = p_1(x_1, x_2, x_3) \in U$$

Take $A_1 := U_A$ and choose B-open sets B'_i such that $0 \in B'_1$, $x_2 = x_3 \in B'_2, B'_3$ and $p_1^{\mathcal{B}}(B'_1, B'_2, B'_3) \subseteq U_B^0$ (we're just using the continuity of $p_1^{\mathcal{B}}$ at $(0, x_2, x_3)$). Letting $B_i := B'_i \setminus \{0\}$, it follows as above that $p_1^{\mathcal{B}}(B_1, B_2, B_3) \subseteq U_B$. Letting $U_1 := A_1 \cup B_1$, $U_2 := B_2$, $U_3 := B_3$ we have $p_1(U_1, U_2, U_3) \subseteq U$. Finally, we prove continuity at those triples where p_1 is defined by the third clause. Take such $x_1, x_2, x_3 \in X$ and an open set U such that $p_1(x_1, x_2, x_3) = p_1^{\mathcal{B}}(\phi(x_1), \phi(x_2), \phi(x_3)) \subseteq U$. By continuity of $p_1^{\mathcal{B}}$, we can choose B-open sets $B'_i \ni \phi(x_i)$ such that $p_1^{\mathcal{B}}(B'_1, B'_2, B'_3) \subseteq U_B$. Since B is Hausdorff, we may as well require that $B'_i \cap B'_j = \emptyset$ when $\phi(x_i) \neq \phi(x_j)$. For each i, let $B_i := B'_i \setminus \{0\}$ and, in case $x_i \in A$, choose an A-open set $A_i \ni x_i$; then let

$$U_i := \begin{cases} A_i \cup B_i & \text{if } x_i \in A \\ B_i & \text{if } x_i \in B \end{cases}$$

From our assumptions, it is not hard to check that in evaluating $p_1(U_1, U_2, U_3)$ the third clause of the definition is always used and that $p_1(U_1, U_2, U_3) \subseteq U$, as required. Thus p_1 is continuous.
- p_{k+1} is continuous: this follows from an entirely analogous argument.
- p_j is continuous for each $1 < j < k+1$: first suppose $x_1, x_2, x_3 \in A$, U is open and $p_j(x_1, x_2, x_3) = p_{j-1}^{\mathcal{A}}(x_1, x_2, x_3) \in U$. Choose A-open sets $A_i \ni x_i$ and B-open sets $B'_i \ni 0$ such that

$$p_{j-1}^{\mathcal{A}}(A_1, A_2, A_3) \subseteq U_A$$
$$p_j^{\mathcal{B}}(B'_1, B'_2, B'_3) \subseteq U_B^0$$

Again, let $B_i := B'_i \setminus \{0\}$; let $U_i := A_i \cup B_i$. Then $p_j(U_1, U_2, U_3) \subseteq U$. Next, we consider those triples (x_1, x_2, x_3) for which p_j is defined by the second clause. Let U be open and

$$p_j(x_1, x_2, x_3) = p_j^{\mathcal{B}}(\phi(x_1), \phi(x_2), \phi(x_3)) \in U$$

As before, choose open sets $B'_i \ni \phi(x_i)$, let $B_i := B'_i \setminus \{0\}$ and, in case $x_i \in A$, also choose an A-open set $A_i \ni x_i$. Let

$$U_i := \begin{cases} A_i \cup B_i & \text{if } x_i \in A \\ B_i & \text{if } x_i \in B \end{cases}$$

Then $p_j(U_1, U_2, U_3) \subseteq U$. Thus p_j is continuous.

□

Proof of Theorem 21. There certainly exist 1-step Hausdorff algebras in $\mathcal{P}_2$ and $\mathcal{P}_3$ whose base set has more than one point: take, for example the real numbers with the usual topology and $p_1(x, y, z) := x - y + z$, $p_2(x, y, z) := z$. Such algebras therefore satisfy H_1 but not H_0. The theorem now follows from Lemma 27 by induction. □

6. Congruence Modular, k-Permutable Varieties Satisfy $T_0 \implies T_2$

Theorem 30. *For topological algebras in a congruence modular, k-permutable variety,*

$$T_0 \implies T_2$$

Proof. We prove the theorem by contradiction. Assume that $\mathcal{A} \in \mathcal{V}$ is a T_0 topological algebra that is not T_2. According to Theorem 20 there is some $j > 1$ such that $\mathcal{A}$ is H_j but not H_{j-1}. For this value of j we have that Δ_j is the equality relation on A, but Δ_{j-1} is different from equality. If $\theta = \Delta_{j-1} \cap \Delta_{j-1}^{\cup}$, then Lemma 18 ensures that θ is not the equality relation. Choose distinct $a, b \in A$ such that $a \;\theta\; b$. Then $(a, b), (b, a) \in \Delta_{j-1}$.

Let $q_0, \ldots, q_{n-1}, p$ be terms satisfying the conditions of Theorem 6. Without loss of generality, we may assume n is even (otherwise we could add $q_n := q_{n-1}$). Consider the sequence of elements: $q_1(a, a, b)$, $q_1(a, b, b)$, $q_2(a, b, b)$, $q_2(a, a, b)$, $q_3(a, a, b)$, $q_3(a, b, b)$, $\ldots$, $q_{n-1}(a, a, b)$, $q_{n-1}(a, b, b)$, $p(a, b, b)$, $p(a, a, b)$. According to Theorem 6, the first element of this sequence is a and the last element of the sequence is b. Moreover, the last element of the form $q_i(-, -, -)$ equals the first element of the form $q_{i+1}(-, -, -)$, and $q_{n-1}(a, b, b) = p(a, b, b)$. Thus, since $a \neq b$, one of the following cases must occur:

(*a*) $a = q_i(a, a, b) \neq q_i(a, b, b)$ for some odd i;
(*b*) $a = q_i(a, b, b) \neq q_i(a, a, b)$ for some even i; or
(*c*) $a = p(a, b, b) \neq p(a, a, b) = b$.

We will explain why each of these cases leads to a contradiction.

Assume that we are in Case (*a*): $a = q_i(a, a, b) \neq q_i(a, b, b) = q$. Since Δ_{j-1} is a compatible reflexive relation containing (b, a), it contains $q_i((a, a), (b, a), (b, b)) = (q, a)$, so $q \in \Delta_{j-1}(a) \subseteq \Delta_1(a)$. Therefore every open set containing q has nonempty intersection with every open set containing a. But because $q \notin \Delta_j(a) = \{a\}$ we can find open sets U and V such that $a \in U, q \in V$, and

$$U \cap V \cap \Delta_{j-1}(a) = \emptyset.$$

Since $q_i(a, a, b) = a \in U$ and $q_i(a, x, b)$ is continuous there is an open set U' containing a such that $q_i(a, U', b) \subseteq U$. Similarly, since $q_i(a, b, b) = q \in V$ and $q_i(a, x, b)$ is continuous, there is an open set V' containing b such that $q_i(a, V', b) \subseteq V$. Since $a \;\Delta_1\; b$ there must exist $c \in U' \cap V'$. For this c we have $q_i(a, c, b) \in U \cap V$. According to the last displayed line this forces $q_i(a, c, b) \notin \Delta_{j-1}(a)$. But this is impossible, since $q_i(a, c, b) \;\Delta_{j-1}\; q_i(a, c, a) = a$. This contradiction shows that Case (*a*) cannot occur.

The argument for Case (*b*) is essentially the same as Case (*a*), since both $(a, b), (b, a) \in \Delta_{j-1}$.

In Case (c) we have $a = p(a, b, b)$ and $p(a, a, b) = b$. According to Theorem 6 the latter equation can be strengthened to $p(x, x, y) \approx y$. Now, since $U = A \setminus \{b\}$ is open, the fact that $p(a, b, b) = a \neq b$ implies that we can find open sets U an V containing a and b respectively such that $p(U, V, V) \subseteq A \setminus \{b\}$. Since $a \; \Delta_1 \; b$ there exists $c \in U \cap V$, and for this element we have $b = p(c, c, b) \in p(U, V, V) \subseteq A \setminus \{b\}$, which is the contradiction we need for Case (c). □

Corollary 31. [5, Theorem 4.4] *If $\mathcal{V}$ is a weakly regular variety, then the topological algebras in $\mathcal{V}$ satisfy $T_0 \Longrightarrow T_2$.*

Proof. Weakly regular varieties are congruence modular and k-permutable for some k. (Refer to [5] for the definition of weakly regular and for Coleman's proof of this result.) □

We now prove a result that may be viewed as a partial converse to Theorem 30.

Theorem 32. *Let ε be a lattice identity and let $k \geq 2$ be a fixed integer. If it is true that all topological algebras in congruence-ε, k-permutable varieties satisfy $T_0 \implies T_2$, then either*

(1) *every congruence-ε variety is congruence modular, or*
(2) *every k-permutable variety is congruence modular.*

In other words, this theorem asserts that if the implication $T_0 \Longrightarrow T_2$ can be characterized by congruence identities and k-permutability for some k, then the characterization is the one suggested by Theorem 30.

Proof. Assume that the theorem statement is false, and that its falsity is witnessed by some fixed integer k and some fixed lattice identity ε. The falsity of the theorem implies that there is some nonmodular, k-permutable, congruence-ε variety. Since this nonmodular variety is k-permutable, it must be that $k \geq 4$. Now, since Polin's variety [15] is k-permutable for all $k \geq 4$, and satisfies every congruence identity that fails to entail modularity (see [7]), the falsity of this theorem would force topological algebras in Polin's variety satisfy $T_0 \implies T_2$. But they do not, as we now show.

First, recall from [7] that a typical algebra $\mathcal{P}(\mathcal{S}, \mathcal{A})$ in Polin's variety may be specified in terms of an "external" Boolean algebra $\mathcal{A}$, a family of "internal" Boolean algebras $\mathcal{S}(a)$ $(a \in A)$, and for each pair of elements $a, b \in A$ with $a \geq b$ a Boolean algebra homomorphism $\xi^a_b : \mathcal{S}(a) \to \mathcal{S}(b)$ satisfying

(i) $\xi^b_c \circ \xi^a_b = \xi^a_c$ if $a \geq b \geq c$, and
(ii) $\xi^a_a = \mathrm{id}_{\mathcal{S}(a)}$.

The universe of $\mathcal{P}(\mathcal{S}, \mathcal{A}) = \langle P; \wedge, ', ^+, 1\rangle$ is

$$P = \bigcup_{a \in A} \{a\} \times \mathcal{S}(a)$$

and the operations are defined by:

(i) $(a, u) \wedge (b, v) = (a \wedge b, \xi^a_{a \wedge b}(u) \wedge \xi^b_{a \wedge b}(v))$,
(ii) $(a, u)' = (a, u')$,
(iii) $(a, u)^+ = (a', 1)$,
(iv) $1_P = (1_{\mathcal{A}}, 1_{\mathcal{S}(1_{\mathcal{A}})})$,

where the right hand sides are computed using the operations of the internal and external Boolean algebras.

In this proof we will work only with the algebra $\mathcal{P} = \mathcal{P}(\mathcal{S}, \mathcal{A})$ where $\mathcal{A}$ is an arbitrary but fixed infinite Boolean algebra, $\mathcal{S}(a)$ is the 1-element Boolean algebra for all $a < 1$, $\mathcal{S}(1)$ is the 2-element Boolean algebra, and ξ^a_b is constant whenever $a \neq b$. $\mathcal{P}$ is subdirectly irreducible, the smallest nonzero congruence μ on $\mathcal{P}$ is the equivalence relation generated by $\langle (1,0), (1,1) \rangle$, and $\mathcal{P}/\mu$ is definitionally equivalent to the Boolean algebra $\mathcal{A}$.

$\mathcal{P}/\mu$ has a natural Hausdorff topology, which makes it into a topological algebra. We can describe this topology as follows: since $\mathcal{P}/\mu$ is definitionally equivalent to a Boolean algebra it can be embedded in $\mathbf{2}^X$ where X is the Stone space of $\mathcal{P}/\mu$. We can give $\mathbf{2}$ the discrete topology, $\mathbf{2}^X$ the product topology, and $\mathcal{P}/\mu$ the induced topology. We can now make $\mathcal{P}$ a topological algebra using the natural homomorphism $\nu : \mathcal{P} \to \mathcal{P}/\mu$, by letting each $\nu^{-1}(U)$ be open in P if U is open in $\mathcal{P}/\mu$. (For an alternate description of the same topology on $\mathcal{P}$, take as a basis of open sets all classes of congruences of finite index on $\mathcal{P}$.)

The topology we have defined on $\mathcal{P}$ is not T_0, since $\mathrm{cl}((1,0)) = \mathcal{S}(1) = \mathrm{cl}((1,1))$. Hence we refine the topology by adding two new subbasis elements: $U = P \setminus \{(1,0)\}$ and $V = P \setminus \{(1,1)\}$. The open sets of the new topology, τ, are the same as those of the original topology, except that deleting either $(1,0)$ or $(1,1)$ from an open set leaves it open.

We leave the verification of the following facts to the reader.

(a) $\langle P; \tau \rangle$ is T_0. (In fact, this space is naturally homeomorphic to $\{0,1\} \rightsquigarrow_1 P/\mu$ where $\{0,1\}$ is the 2-element discrete space and P/μ has the Hausdorff topology described above. Hence $\langle P; \tau \rangle$ is even H_2.)

(b) $\mathcal{P}$ is a topological algebra. (It suffices to check that if f is one of the basic operations, and $f(c,d) \in U$ or $f(c,d) \in V$, then there exist open C, D with $c \in C, d \in D$ and $f(C,D) \subseteq U$ or V.)

(c) $(1,0)\ \Delta_1\ (1,1)$.

Once the reader completes these verifications he will see that Polin's variety fails the implication $T_0 \Longrightarrow T_2$. (It even fails the weaker implication $\mathrm{H}_2 \Longrightarrow T_2$.) □

Definition 33. [2] For each $k \geq 2$, let $\mathcal{W}_k$ be the variety with ternary operations $d_1, d_2, d, p_1, \ldots, p_{k-1}$ satisfying the following equations:

$$\begin{aligned} x &\approx d_1(x,y,y) \\ d_1(x,x,y) &\approx d_2(x,x,y) \\ d_2(x,y,x) &\approx x \\ d_2(x,y,y) &\approx d(x,y,y) \\ d(x,x,y) &\approx y \end{aligned} \qquad \begin{aligned} x &\approx p_1(x,y,y) \\ p_1(x,x,y) &\approx p_2(x,y,y) \\ &\vdots \\ p_{k-2}(x,x,y) &\approx p_{k-1}(x,y,y) \\ p_{k-1}(x,x,y) &\approx y \end{aligned}$$

Bentz [2] showed that all $\mathcal{W}_k$ satisfy $T_0 \Longrightarrow T_2$. Since W_2 and W_3 are 2- and 3-permutable respectively, they are modular. The terms d_1, d_2, d "almost" satisfy the conditions for congruence modularity given by Theorem 6 (the identity "$d_1(x,y,x) \approx x$" is missing). Bentz asked whether or not there is a large value of k such that $\mathcal{W}_k$ provides an example of a *nonmodular* variety satisfying $T_0 \Longrightarrow T_2$. We answer this question in the negative; all $\mathcal{W}_k$ are modular. (Hence Theorem 30 provides a second proof that these varieties satisfy $T_0 \Longrightarrow T_2$.)

Theorem 34. *For each $k \geq 2$, $\mathcal{W}_k$ is congruence modular.*

Proof. Define the following terms in $\mathcal{W}_k$:

$$\begin{aligned}
m_0(x,y,z,w) &:= x\\
m_1(x,y,z,w) &:= x\\
m_2(x,y,z,w) &:= d_1(x, d_2(x,z,y), d(x,y,z))\\
m_3(x,y,z,w) &:= d_2(x,y,w)\\
m_4(x,y,z,w) &:= d_2(x,z,w)\\
m_5(x,y,z,w) &:= d(y,z,w)\\
m_6(x,y,z,w) &:= w
\end{aligned}$$

These terms satisfy the identities of Theorem 5. We check that $m_2(x,x,w,w) \approx m_3(x,x,w,w)$ holds. The other required identities can be just as easily verified.

$$\begin{aligned}
m_2(x,x,w,w) &= d_1(x, d_2(x,w,x), d(x,x,w))\\
&= d_1(x,x,w)\\
&= d_2(x,x,w)\\
&= m_3(x,x,w,w)
\end{aligned}$$

□

7. Concluding Remarks

The following diagram describes the relations among the separation conditions discussed in this paper.

$$\begin{array}{ccccccccccccccc}
T_2 & \Leftrightarrow & \mathrm{H}_1 & \Rightarrow & \mathrm{H}_2 & \Rightarrow & \dots & \Rightarrow & \mathrm{H}_j & \Rightarrow & \dots & & & & \\
 & & \Updownarrow & & \Downarrow & & & & \Downarrow & & & & & & \\
 & & \mathrm{sH}_1 & \Rightarrow & \mathrm{sH}_2 & \Rightarrow & \dots & \Rightarrow & \mathrm{sH}_j & \Rightarrow & \dots & \Rightarrow & T_1 & \Rightarrow & T_0
\end{array}$$

The only implications that need to be justified are those of the form $\mathrm{sH}_j \Longrightarrow T_1$. For these implications, note that

(i) ($\mathrm{sH}_j \Longrightarrow T_0$): If $a \neq b$ either $\Delta_j(a)$ is a closed set containing a and not b or $\Delta_j(b)$ is a closed set containing b and not a.

(ii) ($T_0 \wedge \neg T_1 \Longrightarrow \neg\,\mathrm{sH}_j$): A T_0 space X that fails to be T_1 has a subspace $\{a,b\}$ with induced topology $\{\emptyset, \{a\}, \{a,b\}\}$. For these a and b we have $a\,\Delta_j\,b$ and $b\,\Delta_j\,a$ for all j, thus X fails to satisfy sH_j for any j.

For topological algebras in some k-permutable variety, all vertical arrows may be reversed (Theorem 19), as can all but a finite number of horizontal arrows (Theorem 20). And for varieties that are k-permutable and modular, then all these conditions are equivalent, by Theorem 30.

For topological spaces in general, the situation is quite different. In fact, none of the unidirectional arrows in the above diagram can be reversed. In view of the remarks after Definition 12, we need only show that sH_j does not imply H_j for $j \geq 2$. Take $X := \mathbb{R} \cup \{p\}$, for some $p \notin \mathbb{R}$. Topologize X in the following way: the open subsets not containing p are just the usual open sets in $\mathbb{R}$; the open sets containing p are those which are cofinite. Then it is not hard to check that X satisfies sH_j for each $j \geq 2$, but does not satisfy H_j, for any j.

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