Signal-Anticipating in Local Voltage Control in Distribution Systems

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***Abstract*—In this paper, we consider the signal-anticipating behavior in local volt/var control in distribution systems. We define a voltage control game, and show that the signal-anticipating voltage control is the best response algorithm of the voltage control game. We further show that the voltage control game has a unique Nash equilibrium, characterize it as the optimum of a global optimization problem, and establish its asymptotic global stability. We then introduce the notion of the price of signal-anticipating (PoSA) to characterize the impact of the signal-anticipating in local voltage control, and use the gap in cost between the network equilibrium in the signal-taking voltage control and the Nash equilibrium in the signal-anticipating voltage control as the metric for PoSA. We characterize how the PoSA scales with the size, topology, and heterogeneity of the power network for a few special cases. We find that the stronger the coupling between different buses is, the larger the PoSA is; the linear network gives the largest PoSA among all possible topologies, but as the size of the network increases, the PoSA will saturate.**

I. Introduction

We and our coauthors have studied in [1] a class of inverter-based local volt/var control schemes that are motivated by the proposed 1547.8 standard [2]. These schemes set the reactive power at the output of an inverter based only on the local voltage deviation from its nominal value at a bus. A linear branch flow model similar to the Simplified DistFlow equations introduced in [3] is used in [1], and together with the local volt/var control forms a closed loop dynamical system. We have shown in [1] that the dynamical system has a unique equilibrium point, and characterize it as the unique optimum of a certain convex optimization problem that has a simple interpretation: the local volt/var control tries to achieve an optimal tradeoff between minimizing the cost of voltage deviations and minimizing the cost of reactive power provisioning. Moreover, the objective of the optimization problem serves as a Lyapunov function for the dynamical system under local volt/var control, implying global asymptotic stability of the equilibrium. See Section II for a brief review of these results.

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The volt/var control considered in [1] takes the feedback signal, i.e., the voltage, as given, therefore the buses can be seen as being *signal-taking*. However, a bus may be able to learn or infer the impact of its own decision on the feedback signal, and take it into consideration when making the control decision on reactive power, which we call the *signal-anticipating* voltage control. In this paper, we study such a signal-anticipating behavior. Specifically, the signal-anticipating control makes the interaction among the buses a game, which we call the voltage control game. We show that the signal-anticipating voltage control is the best response algorithm of the voltage control game, and its fixed point or equilibrium is the Nash equilibrium of the voltage control game and vice versa. We further show that the voltage control game has a unique Nash equilibrium, characterize it as the optimum of a global optimization problem, and establish its asymptotic global stability.

The signal-taking and signal-anticipating behaviors in local voltage control are analogous to the price-taking and price-anticipating behaviors in economics. It is well-known that in a competitive market with price-taking customers the system achieves an efficient equilibrium and in an oligopolistic market with price-anticipating customers the system usually incurs efficiency loss. Similarly, we consider that the signal-taking behavior leads to an efficient equilibrium while the signal-anticipating behavior may result in efficiency loss. Specifically, we introduce the notion of the *price of signal-anticipating* (PoSA) to characterize the impact of the signal-anticipating in local voltage control in particular and in distributed control in general. We use the gap in cost between the network equilibrium in the signal-taking voltage control and the Nash equilibrium in the signal-anticipating voltage control as the metric for PoSA. We characterize how the PoSA scales with the size, topology, and heterogeneity of the power network for a few special cases. We find that the stronger the coupling between different buses is, the larger the PoSA is. The linear network gives the largest PoSA among all possible topologies, but as the size of the network increases, the PoSA will saturate.

To our knowledge, this paper is the first to study the signal-anticipating behavior and introduce the notion of the price of signal-anticipating in distributed control. Such results will be insightful to understanding strategic behaviors in distributed control and designing mechanisms to mitigate their impact if it is not desired.

Related work: Voltage control is a research area with a huge literature. Traditional approach to voltage control in distribution systems is via capacitor banks and under load tap changers; see, e.g., [4], [5], [6]. The new inverter-based approach that can control reactive power much faster and in a much finer granularity has been proposed and studied in, e.g., [7], [8], [9], [1], [10]. The local voltage control based real-time voltage measurement has also been proposed in transmission systems; see, e.g., [11].

The rest of the paper is organized as follows. Section II describes the network model and briefly reviews the signal-taking voltage control. Section III studies the signal-anticipating voltage control and the resulting game, and characterizes its equilibrium and dynamic properties. Section IV characterizes the price of signal-anticipating in voltage control, and Section V concludes the paper.

II. Network model and the signal-taking voltage control

Consider a tree graph $G = \{\mathcal{N} \cup \{0\}, \mathcal{L}\}$ that represents a radial distribution network consisting of $n+1$ buses and a set $\mathcal{L}$ of lines between these buses. Bus 0 is the substation bus and is assumed to have a fixed voltage. For each bus $i \in \mathcal{N}$, denote by $\mathcal{L}_i \subseteq \mathcal{L}$ the set of lines on the unique path from bus 0 to bus i, p_i^c and p_i^g the real power consumption and generation, and q_i^c and q_i^g the reactive power consumption and generation, respectively. Let v_i be the magnitude of the complex voltage at bus i. For each line $(i,j) \in \mathcal{L}$, denote by r_{ij} and x_{ij} its resistance and reactance, and P_{ij} and Q_{ij} the real and reactive power from bus i to bus j, respectively. Let ℓ_{ij} denote the squared magnitude of the complex branch current from bus i to bus j. These notations are summarized in Table I. A quantity without subscript is usually a vector with appropriate components defined earlier; e.g., $v := (v_i, i \in \mathcal{N}), q^g := (q_i^g, i \in \mathcal{N})$.

TABLE I
Notations

$\mathcal{N}$	Set of buses, excluding bus 0, labeled as {1, 2, ..., n}
$\mathcal{L}$	Set of all links representing the power lines
$\mathcal{L}_i$	Set of the links on the path form bus 0 to bus i
p_i^c, p_i^g	Real power consumption and generation at bus i
q_i^c, q_i^g	Reactive power consumption and generation at bus i
r_{ij}, x_{ij}	Resistance and reactance of line (i,j)
P_{ij}, Q_{ij}	Real and reactive power flows from i to j
v_i	Magnitude of complex voltage at bus i
ℓ_{ij}	Squared magnitude of complex current on (i,j)

A. Linearized branch flow model

We adopt the following branch flow model introduced in [4], [5] to model a radial distribution system:

$$P_{ij} = p_j^c - p_j^g + \sum_{k:(j,k)\in\mathcal{L}} P_{jk} + r_{ij}\ell_{ij}, \tag{1a}$$

$$Q_{ij} = q_j^c - q_j^g + \sum_{k:(j,k)\in\mathcal{L}} Q_{jk} + x_{ij}\ell_{ij}, \tag{1b}$$

$$v_j^2 = v_i^2 - 2\left(r_{ij}P_{ij} + x_{ij}Q_{ij}\right) + \left(r_{ij}^2 + x_{ij}^2\right)\ell_{ij}, \tag{1c}$$

$$\ell_{ij}v_i = P_{ij}^2 + Q_{ij}^2. \tag{1d}$$

Following [3], we have introduced in [1] a resistance matrix $R = [R_{ij}]_{n\times n}$ with $R_{ij} := \sum_{(h,k)\in\mathcal{L}_i\cap\mathcal{L}_j} r_{hk}$ and a reactance matrix $X = [X_{ij}]_{n\times n}$ with $X_{ij} := \sum_{(h,k)\in\mathcal{L}_i\cap\mathcal{L}_j} x_{hk}$, and derived from (1) a linearized branch flow model:

$$v = \overline{v}_0 + R(p^g - p^c) + X(q^g - q^c),$$

where $\overline{v}_0 = (v_0, \ldots, v_0)$ is an n-dimensional vector. We assume that $\overline{v}_0, p^c, p^g, q^c$ are given constants, and the only variables are (column) vectors $v := (v_1, \ldots, v_n)$ of voltage magnitudes and $q^g := (q_1^g, \ldots, q_n^g)$ of reactive powers. Let $\tilde{v} = \overline{v}_0 + R(p^g - p^c) - Xq^c$, which is a constant vector. For notational simplicity, in the rest of the paper we will ignore the superscript in q^g and write q instead. Then the linearized branch flow model reduces to the following:

$$v = Xq + \tilde{v}. \tag{2}$$

We have shown in [1] that the matrix X is positive definite.

B. The signal-taking volt/var control

The goal of volt/var control on a distribution network is to provision reactive power injections $q := (q_1, \ldots, q_n)$ in order to maintain the bus voltages $v := (v_1, \ldots, v_n)$ to within a tight range around their nominal values v_i^{nom}, $i \in \mathcal{N}$. In [1], we have considered a class of local volt/var control where each bus i makes an individual decision $q_i(t+1)$ based only on its own voltage $v_i(t)$:

$$q_i(t+1) = f_i(v_i(t) - v_i^{\text{nom}}), \quad \forall i \in \mathcal{N}, \tag{3}$$

where $f_i : \mathbb{R} \to \Omega_i$ with $\Omega_i = \left\{q_i \mid {q_i}^{\min} \le q_i \le {q_i}^{\max}\right\}$ the set of feasible reactive power injections at bus i. This leads to the following feedback dynamical system for the distribution network:

$$v(t) = Xq(t) + \tilde{v}, \tag{4}$$

$$q(t+1) = f(v(t) - v^{\text{nom}}), \tag{5}$$

where $f : \mathbb{R}^n \to \Omega$ denotes the collection of $(f_i, i \in \mathcal{N})$, with $\Omega = \prod_{i\in\mathcal{N}} \Omega_i$. A fixed point of the above dynamical system represents an equilibrium operating point of the network.

Definition 1. (Definition 2 in [1]) (v^*, q^*) *is called an* equilibrium point*, or a* network equilibrium*, if it is a fixed* *point of* (4)–(5)*, i.e.,*

$$\begin{aligned} v^* &= Xq^* + \tilde{v}, \\ q^* &= f(v^* - v^{nom}). \end{aligned}$$

Following [1], we assume for each bus $i \in \mathcal{N}$ a symmetric deadband $[-\delta_i/2, \delta_i/2]$ around the origin with $\delta_i \geq 0$. Let $\overline{v}_i := \min\left\{v_i | f_i(v_i) = q_i^{\min}\right\}$, i.e., the lowest voltage deviation from the nominal value that draws reactive power $q_i^{\min}$, and let $\underline{v}_i := \max\left\{v_i | f_i(v_i) = q_i^{\max}\right\}$, i.e., the highest voltage deviation that draws reactive power $q_i^{\max}$. We make the following assumptions:

- A1: The local volt/var control functions f_i are nonincreasing over $\mathbb{R}$ and strictly decreasing and differentiable in $(\underline{v}_i, -\delta_i/2)$ and in $(\delta_i/2, \overline{v}_i)$.
- A2: The derivative of the control function f_i is bounded, i.e., there exists a finite α_i such that $|f_i'(v_i)| \leq \alpha_i$ for all v_i in the appropriate domain, for all $i \in \mathcal{N}$.

Define a cost function for each bus $i \in \mathcal{N}$:

$$C_i(q_i) \quad := \quad -\int_0^{q_i} f_i^{-1}(q)\, dq,$$

which is convex since the inverse function f_i^{-1} is decreasing. Then, given any $v_i(t)$, $q_i(t+1)$ in (5) is the unique solution of a distributed decision problem:

$$q_i(t+1) = \arg\min_{q_i \in \Omega_i} C_i(q_i) + q_i\left(v_i(t) - v_i^{\text{nom}}\right),\ i \in \mathcal{N}, \quad (6)$$

i.e., (5) and (6) are equivalent specification of $q_i(t+1)$. Notice that, in the decision problem (6), each bus takes the feedback signal $v_i(t)$ as given and can be seen as being *signal-taking*. We thus call (5) and (6) the *signal-taking* voltage control.

We have the following results regarding the equilibrium and dynamic properties of the signal-taking voltage control.

Theorem 2. (Theorem 1 in [1]) *Suppose A1 holds. Then there exists a unique equilibrium point. Moreover a point* (v^*, q^*) *is an equilibrium if and only if* q^* *is the unique optimal solution of the following global optimization problem:*

$$\min_{q \in \Omega} \qquad F(q) = \sum_{i \in \mathcal{N}} C_i(q_i) + \frac{1}{2} q^T X q + q^T \Delta\tilde{v} \qquad (7)$$

with $\Delta\tilde{v} := \tilde{v} - v^{norm}$*, and* $v^* = Xq^* + \tilde{v}$.

Theorem 3. (Theorem 2 in [1]) *Suppose A1–A2 hold. if*

$$\mathbf{diag}\left(\frac{1}{\alpha_i}\right) \succ X, \qquad (8)$$

i.e., if the matrix diag$\left(\alpha_i^{-1}\right) - X$ *is positive definite, then the signal-taking volt/var control* (4)–(5) *converges to the unique equilibrium point* (v^*, q^*).

III. The price-anticipating voltage control

As discussed in Section II, in the equivalent decision problem (6), each bus takes the feedback signal $v_i(t)$ as given, so the local volt/var control (5) is a *signal-taking* control. However, a bus i may be able to learn or infer the impact of its own decision q_i on the feedback signal v_i (i.e., know v_i as a function of q_i; see equation (2)), and take it into consideration when making the control decision on reactive power, which we call the *signal-anticipating* voltage control. With the signal-anticipating control, bus $i \in \mathcal{N}$ will decide its reactive power output according to the following optimization problem:

$$\begin{aligned} &q_i(t+1) \\ &= \arg\min_{q_i \in \Omega_i} C_i(q_i) + q_i(X_{ii} q_i + \sum_{j \in \mathcal{N} \setminus \{i\}} X_{ij} q_j(t) + \Delta\tilde{v}_i). \quad (9) \end{aligned}$$

To see the difference from the signal-taking control, notice that (6) can be written as:

$$\begin{aligned} &q_i(t+1) \\ &= \arg\min_{q_i \in \Omega_i} C_i(q_i) + q_i(\sum_{j \in \mathcal{N}} X_{ij} q_j(t) + \Delta\tilde{v}_i). \quad (10) \end{aligned}$$

The signal-anticipating voltage control makes the interaction between the buses a game.

Definition 4. *A* voltage control game *is defined as a triple* $\mathcal{G} = \{\mathcal{N}, (\Omega_i)_{i \in \mathcal{N}}, (h_i)_{i \in \mathcal{N}}\}$*, with a set* $\mathcal{N}$ *of players (buses), bus* $i \in \mathcal{N}$ *strategy space* Ω_i*, and cost function* $h_i(q) = C_i(q_i) + q_i(\sum_{j \in \mathcal{N}} X_{ij} q_j + \Delta\tilde{v}_i)$.

Let $q_{-i} = (q_1, q_2, \cdots, q_{i-1}, q_{i+1}, \cdots, q_N)$ denotes the reactive powers at all buses other than i, and represent q as (q_i, q_{-i}). The signal-anticipating voltage control (9) can be written as

$$q_i(t+1) = \arg\min_{q_i \in \Omega_i} h_i(q_i, q_{-i}(t)),\ i \in \mathcal{N}, \qquad (11)$$

which is the best response algorithm for the voltage control game $\mathcal{G}$ [12].

A. Equilibrium

We now analyze the Nash equilibrium of the voltage control game [12]. A vector q^a of reactive powers is a Nash equilibrium if, for all buses $i \in \mathcal{N}$, $h_i(q_i^a, q_{-i}^a) \leq h_i(q_i, q_{-i}^a)$ for all $q_i \in \Omega_i$. We see that the Nash equilibrium is a set of reactive powers for which no bus has incentive to change unilaterally.

Lemma 5. *A Nash equilibrium* q^a *of the voltage control game* $\mathcal{G}$ *is a fixed point (or equilibrium) of the signal-anticipating voltage control* (9)*, and vice versa.*

Proof. The result follows from the fact that the signal-anticipating voltage control is the best response algorithm for the voltage control game $\mathcal{G}$; see equation (11) [12]. □

Consider the function $W : \Omega \to \mathbb{R}$:

$$W(q) = \sum_{i \in \mathcal{N}} \left(C_i(q_i) + \frac{1}{2} X_{ii} q_i^2\right) + \frac{1}{2} q^T X q + q^T \Delta\tilde{v}$$

and the global optimization problem:

$$\min_{q\in\Omega} \quad W(q). \tag{12}$$

Theorem 6. *Suppose A1 holds. Then there exists a unique Nash equilibrium for the voltage control game $\mathcal{G}$. Moreover a point q^a is a Nash equilibrium if and only if it is the unique optimum of* (12).

Proof. First, notice that the problem (12) is strictly convex. So, the first order optimality condition for (12) is both sufficient and necessary; and moreover, (12) has a unique optimum. Second, notice that the first oder optimality condition is just the fixed point condition of the best response algorithm (11). The existence and uniqueness of the optimum of (12) then implies that of the Nash equilibrium q^a. □

B. Dynamics

We now study the dynamic properties of the signal-anticipating voltage control (9), i.e., the best response algorithm (11).

Theorem 7. *Suppose A1-A2 hold. If*

$$\mathbf{diag}\left\{\frac{1}{\alpha_i}+3X_{ii}\right\} \succ X,$$

then the signal-anticipating voltage control (9) *converges to the unique Nash equilibrium of the voltage control game $\mathcal{G}$.*

Proof. The main idea of the proof is to show that $W(q)$ is a Lyapunov function of the discrete-time dynamical system (9).

Recall that $C(q)=\sum_i C_i(q_i)$. Its Hessain

$$\nabla^2 C(q)=\mathbf{diag}\left(-\frac{\partial f_i^{-1}(q_i)}{\partial q_i}\right).$$

By assumptions A1-A2, we have

$$\nabla^2 C(q)\succeq \mathbf{diag}\left(\frac{1}{\alpha_i}\right).$$

Now, let us decompose $W(q)=U(q)+V(q)$ where $U(q)=C(q)+q^TDq$ and $V(q)=\frac{1}{2}q^T(X-D)q+q^T\Delta\tilde{v}$, with $D=\mathbf{diag}\,(X_{ii})$ the diagonal part of X. By the Taylor's theorem, there exists $\tilde{q}$ such that

$$\begin{aligned}
& U(q(t)) \\
= \; & U(q(t+1))+\nabla U(q(t+1))^T(q(t)-q(t+1)) \\
& +\frac{1}{2}(q(t+1)-q(t))^T\nabla^2 U(\tilde{q})(q(t+1)-q(t)),
\end{aligned}$$

and $\nabla U(q)=\nabla C(q)+2Dq$ and $\nabla^2 U(q)=\nabla^2 C+2D$. Since $\nabla^2 C(\tilde{q})\succeq \mathbf{diag}\left(\frac{1}{\alpha_i}\right)$, we have

$$\begin{aligned}
& U(q(t+1)) \\
& \le U(q(t))+(\nabla C(q(t+1))+2Dq(t+1))^T(q(t+1)-q(t)) \\
& -\frac{1}{2}(q(t+1)-q(t))^T(2D+\mathbf{diag}\left(\frac{1}{\alpha_i}\right))(q(t+1)-q(t)).
\end{aligned}$$

For V we have,

$$\begin{aligned}
& V(q(t+1)) \\
= \; & V(q(t))+((X-D)q(t)+\Delta\tilde{v})^T(q(t+1)-q(t)) \\
& +\frac{1}{2}(q(t+1)-q(t))^T(X-D)(q(t+1)-q(t)).
\end{aligned}$$

By combining all these results, we have

$$\begin{aligned}
& W(q(t+1)) \\
& \le W(q(t))+z^T(q(t+1)-q(t)) \\
& -\frac{1}{2}(q(t+1)-q(t))^TQ(q(t+1)-q(t)),
\end{aligned}$$

where $z=\nabla C(q(t+1))+2Dq(t+1)+(X-D)q(t)+\Delta\tilde{v}$, and $Q=3D+\mathbf{diag}\left(\frac{1}{\alpha_i}\right)-X$. From the optimality condition in (9), $z^T(q(t+1)-q(t))\le 0$. Therefore

$$\begin{aligned}
& W(q(t+1)) \\
\le \; & W(q(t))-\frac{1}{2}(q(t+1)-q(t))^TQ(q(t+1)-q(t)).
\end{aligned}$$

Since $3D+\mathbf{diag}\left(\frac{1}{\alpha_i}\right)\succ X$, Q is positive definite. Therefore the quadratic term is strictly negative unless $q(t+1)=q(t)$. This shows that $W(q(t+1))<W(q(t))$ unless $q(t+1)=q(t)$. Since the Nash equilibrium is unique by Theorem 6, $q(t+1)=q(t)$ can only occur at the unique Nash equilibrium.

Therefore, we have shown the following:

1) $W(q)\ge W(q^a)$ with equality if and only if $q=q^a$,
2) $W(q(t+1))\le W(q(t))$ with equality if and only if $q(t+1)=q(t)=q^a$.

So, the function W is a discrete-time Lyapunov function for the signal-anticipating voltage control (9). By the Lyapunov stability theorem, we conclude that the equilibrium point q^a is an asymptomatically global stable point. □

IV. The Price of Signal-Anticipating

We have studied the signal-taking and signal-anticipating behaviors in local voltage control in [1] (reviewed in Section II-B) and Section III, respectively. They are analogous to the price-taking and price-anticipating behaviors in economics. It is well-known that in a competitive market with price-taking customers the system achieves an efficient equilibrium and in an oligopolistic market with price-anticipating (or strategic) customers the system usually incurs efficiency loss; see, e.g., [13], [14]. Similarly, the signal-taking behavior leads to an efficient equilibrium while the signal-anticipating behavior may result in efficiency loss, in term of the global cost function $F(q)$.[1]

We thus introduce the notion of the ***price of signal-anticipating*** (PoSA) to characterize the impact of the signal-anticipating in local voltage control in particular and in distributed control in general. Specifically, we will use the gap in

[1]Notice that function $F(q)$ is the summation of the cost of reactive power provisioning and the cost of voltage deviation, and the signal-taking voltage control tries to achieve an optimal tradeoff between minimizing the cost of voltage deviation and minimizing the cost of reactive power provisioning [1].

cost (or *efficiency loss*) between the network equilibrium q^* and the Nash equilibrium q^a

$$p = F(q^a) - F(q^*) \tag{13}$$

as a metric for the price of signal-anticipating. We aim to investigate how p scales with the size, topology, and heterogeneity of the power network. Such results will be insightful to understanding strategic behaviors in local voltage control and designing mechanisms to mitigate their impact.

A. The case with quadratic cost functions

In this paper we will focus on a special case where each node $i \in \mathcal{N}$ has a quadratic cost function $C_i(q_i) = \frac{1}{2} y_i q_i^2$ with $y_i > 0$. Quadratic cost functions are widely used in market models for the power system. We further assume for simplicity that there is no constraint in reactive power, i.e., $q_i^{min} = -\infty,\ q_i^{max} = \infty,\ i \in \mathcal{N}$. But the results expect to extend to more general settings.

Let $Y = \mathbf{diag}(y_i)$. The network equilibrium q^* arising from the signal-taking behavior solves

$$\min_q \quad F(q) = \frac{1}{2} q^T (X+Y) q + q^T \Delta\tilde{v}, \tag{14}$$

i.e., $q^* = -(X+Y)^{-1}\Delta\tilde{v}$; whereas the Nash equilibrium q^a arising from the signal-anticipating behavior solves

$$\min_q \quad W(q) = \frac{1}{2} q^T (X+D+Y) q + q^T \Delta\tilde{v} \tag{15}$$

with D the diagonal part of X, i.e., $q^* = -(X+D+Y)^{-1}\Delta\tilde{v}$. The PoSA can be written as

$$p = \frac{1}{2}\Delta\tilde{v}^T \Pi \Delta\tilde{v}, \tag{16}$$

where $\Pi = (X+D+Y)^{-1} D (X+Y)^{-1} D (X+D+Y)^{-1}$. Notice that $\Pi \succ 0$, so p is always positive for nonzero $\Delta\tilde{v}$ (which is actually the initial voltage deviation).

1) The worst-case PoSA: As the PoSA p is quadratic in $\Delta\tilde{v}$, without normalization p can be arbitrarily large. We therefore investigate a normalized, worst-case PoSA with respect to the norm of $\Delta\tilde{v}$:

$$p_{\max}(X,Y) = \sup_{\Delta\tilde{v}} \frac{p}{\Delta\tilde{v}^T\Delta\tilde{v}} = \frac{1}{2}\sup_{\Delta\tilde{v}} \frac{\Delta\tilde{v}^T \Pi \Delta\tilde{v}}{\Delta\tilde{v}^T\Delta\tilde{v}}. \tag{17}$$

It is obvious that $p_{\max}(X,Y) = \frac{1}{2}\lambda_{\max}\{\Pi\}$, where $\lambda_{\max}$ denotes the maximum eigenvalue, and can be achieved by the eigenvector of Π corresponding to the maximum eigenvalue.

Consider the subtrees of the network at bus 0. The voltage controls at these subtrees are independent. Mathematically, this can be seen from the fact that matrices D and Y are diagonal and X is block diagonal

$$X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_b \end{bmatrix},$$

where each block corresponds to a subtree and b is the number of subtrees. We thus consider, without loss of generality, only the network where bus 0 has only one direct child bus, i.e., $b = 1$.

Let $y = \min_{i\in\mathcal{N}}\{y_i\}$, $\underline{d} = \min_{i\in\mathcal{N}}\{X_{ii}\}$, $\overline{d} = \max_{i\in\mathcal{N}}\{X_{ii}\}$, and $\lambda_{\min}$ denotes the minimum eigenvalue of X. We have

$$\begin{aligned} Y + D + X &\succeq (y + \underline{d} + \lambda_{\min}) I, \\ Y + X &\succeq (y + \lambda_{\min}) I, \\ D &\preceq \overline{d} I. \end{aligned}$$

From this, we can conclude that

$$\Pi \preceq \frac{\overline{d}}{(y+\lambda_{\min}+\underline{d})^2 (y+\lambda_{\min})} I.$$

Therefore,

$$p_{\max}(X,Y) \le \frac{\overline{d}}{2(y+\lambda_{\min}+\underline{d})^2 (y+\lambda_{\min})}. \tag{18}$$

2) The impact of Y: From (18), we see that the larger Y is, the smaller the worst-case PoSA is; and when $y \to \infty$, $p_{\max}(X,Y) \to 0$. This is as expected, since the cost of provisioning dominates in the voltage control decision when it is large.

3) The impact of X: The analysis with X is challenging, as the relation between X and D is complicated. In order to obtain insights, we consider a relatively simple case with large Y, i.e., $Y >> X$. In this case, the matrix Π can be approximated as

$$\Pi \approx Y^{-1} D Y^{-1} D Y^{-1},$$

and the worst-case PoSA is approximately given by

$$\begin{aligned} p_{\max}(X,Y) &= \frac{1}{2}\lambda_{\max}(\Pi) \\ &\approx \max_i \frac{X_{ii}^2}{2 y_i^3}. \end{aligned} \tag{19}$$

Notice that X_{ii} is the summation of the reactances of all power lines along the unique path from bus 0 to bus i. So, the linear network, which has the largest X_{ii} value among all topologies, gives the largest worst-case PoSA. Intuitively, the linear network has the most "extensive" overlaps among the paths from bus 0 to a bus, and thus the "strongest" coupling among different buses. So, the stronger the coupling among different bus is, the larger the PoSA is.

4) Two-link Network: Here we numerically investigate a special case, in order to gain insights. Consider a two-link network consisting of 3 buses as shown in Fig. 1, and denote by x_1 and x_2 the reactance of the power lines $(0,1)$ and $(1,2)$, respectively. The corresponding reactance matrix is as follows:

$$X = \begin{bmatrix} x_1 & x_1 \\ x_1 & x_1 + x_2 \end{bmatrix}.$$

We set $Y = I$, and calculate the largest gap versus x_1 and x_2, shown in Fig. 2. We see that as x_1 becomes larger, the

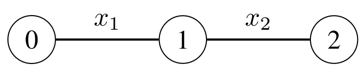

Fig. 1. Two-link network.

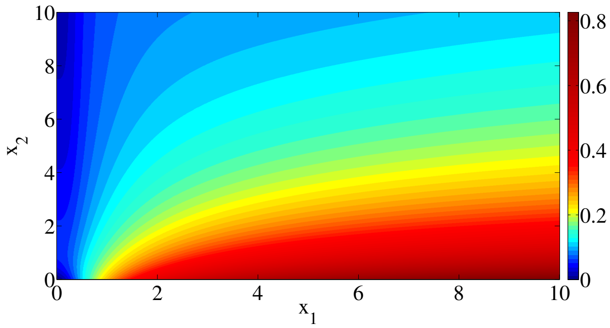

Fig. 2. The worst-case PoSA for the two-link network (blue to red representing small to large PoSA).

gap becomes larger, whereas as x_2 becomes larger, the gap becomes smaller. Notice that x_1 characterizes the coupling strength between buses 1 and 2. Again, the stronger the coupling among different buses is, the larger the PoSA is.

5) The linear network: As we have discussed in the above, the linear network gives the largest worst-case PoSA among all possible topologies. To see how the PoSA scales with the size of the network, we consider a linear network with all power lines having the same reactance x and with $Y = I$. Fig. 3 shows the scaling of the worst-case PoSA with the number N of power lines in the network. We see that the worst-case PoSA does not grow as fast as $\log N$, and will saturate as N becomes large. This means that the average gap per bus PoSA/N will become effectively zero for a large network. The saturation of the PoSA is very interesting. We are working on characterizing analytically the scaling of PoSA with the size of the network.

V. Conclusion

We have studied the signal-anticipating behavior in the local volt/var control. We define a voltage control game, and show that the signal-anticipating voltage control is the best response algorithm of the voltage control game. We further show that the voltage control game has a unique Nash equilibrium, characterize it as the optimum of a global optimization problem, and establish its asymptotic global stability. We then introduce the notion of the price of signal-anticipating to characterize the impact of the signal-anticipating in local voltage control, and use the gap in cost between the network equilibrium in the signal-taking voltage control and the Nash equilibrium in the signal-anticipating voltage control as the metric for PoSA. We characterize how the PoSA scales with the size, topology, and heterogeneity of the power network for a few special cases. We find that the stronger the coupling between different buses is, the larger the PoSA is. The linear network gives the largest PoSA among all possible topologies, but as the size of the network increases, the PoSA will saturate. As further work, we are characterizing the relation between the matrices X and D, and characterizing analytically the scaling of the price of signal-anticipating with the network size. We are also studying the signal-anticipating voltage control with exact branch flow model instead of the linearized model used in this paper.

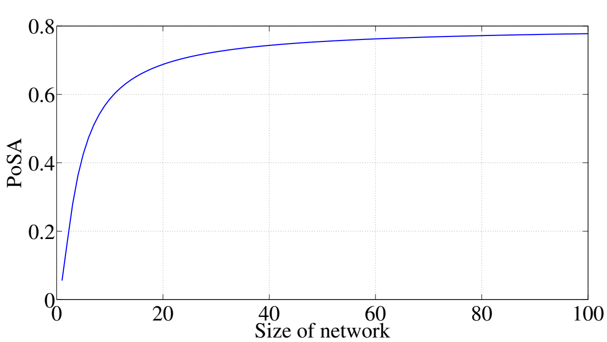

Fig. 3. The worst-case PoSA with the size of the linear network with $x = 1$.

References

[1] M. Farivar, L. Chen, and S. Low. Equilibrium and dynamics of local voltage control in distribution systems. In *Proc. of Conference on Decision and Control (CDC)*, 2013.

[2] IEEE P1547. Recommended practice for establishing methods and procedures that provide supplemental support for implementation strategies for expanded use of ieee standard 1547. Technical report, Draft D4.0, December 2012.

[3] M. E. Baran and F. F. Wu. Network reconfiguration in distribution systems for loss reduction and load balancing. *IEEE Trans. on Power Delivery*, 4(2):1401–1407, 1989.

[4] M.E. Baran and F.F. Wu. Optimal capacitor placement on radial distribution systems. *IEEE Trans. on Power Delivery*, 4(1):725–734, 1989.

[5] M. E. Baran and F. F. Wu. Optimal sizing of capacitors placed on a radial distribution system. *IEEE Trans. on Power Delivery*, 4(1):735–743, 1989.

[6] H-D. Chiang and M. E. Baran. On the existence and uniqueness of load flow solution for radial distribution power networks. *IEEE Trans. Circuits and Systems*, 37(3):410–416, 1990.

[7] K. Turitsyn, P. Sŭlc, S. Backhaus, and M. Chertkov. Options for control of reactive power by distributed photovoltaic generators. In *Proc. of the IEEE*, volume 99, pages 1063 –1073, 2011.

[8] J. W. Smith, W. Sunderman, R. Dugan, and B. Seal. Smart inverter volt/var control functions for high penetration of PV on distribution systems. In *Proc. of IEEE PES Power Systems Conference and Exposition (PSCE)*, 2011.

[9] M. Farivar, C. R. Clarke, S. H. Low, and K. M. Chandy. Inverter var control for distribution systems with renewables. In *Proc. of IEEE SmartGridComm Conference*, 2011.

[10] B. Zhang, A. D. Domnguez-Garcia, and D. Tse. A local control approach to voltage regulation in distribution networks. In *Proc. of NAPS*, 2013.

[11] M. Ilic-Spong, J. Christensen, and K. L. Eichorn. Secondary voltage control using pilot point information. *IEEE Trans. Power Systems*, 3(2):660–668, 1988.

[12] D. Fudenburg and J. Tirole. *Game Theory*. The MIT Press, 1991.

[13] A. Mas-Colell, Andreu, M. Whinston, and J. Green. *Microeconomic Theory*. Oxford, 1995.

[14] N. Nisan, T. Roughgarden, E. Tardos, Eva, and V. V. Vazirani. *Algorithmic Game Theory*. Cambridge University Press, 2007.