Demand Response Using Linear Supply Function Bidding

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***Abstract*— In this paper, we consider an abstract market model for demand response where a supply function bidding is applied to match power supply deficit or surplus. We characterize the resulting equilibria in competitive and oligopolistic markets and propose distributed demand response algorithms to achieve the equilibria. We further show that the equilibrium in competitive market maximizes social welfare, and the equilibrium in oligopolistic market has bounded efficiency loss under certain mild assumptions. We also propose distributed demand response algorithms to achieve the equilibria.**

***Index Terms*— Demand response, Supply function bidding, Competitive equilibrium, Nash equilibrium, Efficiency loss.**

I. Introduction

The usual practice in power networks is to match supply to demand. This is challenging because demand is highly time-varying. The utility company or generator needs to provision enough generation, transmission and distribution capacities for peak demand rather than the average. As a result, the power network has a low load factor and is underutilized most of the time, which is very costly. For example, the US national load factor is about 55%, and 10% of generation and 25% of distribution facilities are used less than 400 hours per year, i.e., 5% of the time [2]. To improve the load factor and the efficiency of electricity, demand response has been proposed to shape the demand to reduce the peak and smooth the variation [3], [4], [5], [6], [7]. An alternative strategy for improving efficiency and reducing cost is to match the demand to the supply. As the proportion of renewable sources such as solar and wind power steadily rises, power supply will also become highly time-varying. Matching demand to supply will become a more effective and common way to improve power system efficiency and reduce cost [8].

A key role in demand response is the appropriate market model which incentivizes users to coordinate. Many pricing schemes and market models have been proposed from different perspectives. To name a few, [5], [6], [9] develop autonomous demand response to reduce the peak using game theoretic approaches or real-time pricing, [10] studies the volatility of the power grids under real-time pricing, [11] proposes robust pricing schemes, [12] includes demand response to determine the optimal generation mix, and [13] models the pool-based demand response. However, there is a lack of work designing demand response to match the uncertain supply fluctuations. Moreover, because of the intermittency of the renewable energy, the self-interested nature of the customers, and the large scale of the system, it remains as a challenge to design a simple, flexible and scalable market to guarantee the individual benefit and the global system efficiency simultaneously.

To overcome this challenge, we consider one abstract retail market model for demand response to match the supply fluctuations using a linear supply function bidding. The supply function model has been used in the wholesale electricity market [14], [15], [16], [17], [18]. It assumes that each supplier submits a supply function to an auctioneer, who will set a uniform market clearing price. Supply function as a strategic variable allows to adapt better to changing market conditions (such as uncertain demand) than does a simple commitment to a fixed price or quantity. It also respects practical informational constraints in the power network, because a properly-chosen parameterized supply function "controls" information revelation. The seminal paper [14] studies the supply function equilibrium (SFE) and gives conditions for the existence and the uniqueness of the SFE under uncertain demand. Section II will discuss more advantages of using the (linear) supply function bidding.

Motivated by the application of SFE in the wholesale market for the supply side, we apply the supply function bidding method to the retail market for demand response. We adopt a special linear form of parameterized supply functions which is simple to be implemented as an effective demand response scheme. However, instead of merely focusing on the existence of SFE, we study how the cost functions affect the structure and efficiency (loss) of two different market equilibria (competitive and oligopolistic equilibria) regarding the global social welfare. Though [19], [20] also study the efficiency loss of a parameterized supply function bidding, we focus on a different group of supply functions which are more closely aligned with the electricity market literature and practice. We also propose distributed algorithms to reach the market equilibria. Lastly, to the best of our knowledge most of the existing analytic results on the SFE assumed the quadratic cost function, which we relax to be general increasing convex cost functions.

Specifically, we consider a situation where there is an inelastic supply deficit (or surplus) on electricity, and study a

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supply function bidding scheme for allocating load shedding (or load increasing) among different users to match the supply deficit (or surplus). Each user submits a parameterized linear supply function to the utility company and the utility company decides a market-clearing price based on the bids of users. Then users are committed to shed (or increase) their loads according to their bids and the market-clearing price and suffer disutility/cost from the load adjustment. Note that in order not to limit the applicability of the market model, we do not specify the type of users, which could be an aggregrated user, an industry company, a commercial building, or a residential household. The bidding process can also be applied in either forward market or spot market, as long as the objective is to clear supply deficit (or surplus).

Then we study customers' behavior in two different markets, competitive and oligopolistic. [1] In order to study the equilibria at the two markets, we reverse-engineer the conditions of the equilibria into optimality conditions of different optimization problems. Not only do those optimization problems allow us to characterize the existence of the equilibria, but also facilitate us to study the structure and efficiency of the equilibria. We show that in a competitive market the system achieves an efficient equilibrium that maximizes the social welfare, and in an oligopolistic market the system achieves a Nash equilibrium that maximizes another additive, global objective function. We further show that both of the two market equilibria guarantee the individual rationality and develop distributed supply function bidding algorithms to achieve those equilibria. Those algorithms require twoway communication between the utility company and each user, but they only require to communicate a very small amount of information and are scalable to large systems.

Due to customers' price-anticipating and strategic behavior, the Nash equilibrium is expected to be less efficient than competitive equilibrium. To analyze the efficiency loss, we then quantify the differences between the two market equilibria. Specifically, we find out that

- the set of customers who commit positive load adjustment at the Nash equilibrium is a superset of that at the competitive equilibrium;
- the market clearing price at the Nash equilibrium is higher than that at the competitive equilibrium but the ratio of the two prices are bounded;
- the total customers' disutility at the Nash equilibrium is larger than that at the competitive equilibrium but the ratio is bounded under certain mild assumptions.

An interesting result is that the differences between the two market equilibria depend on the heterogeneity of the customers' cost functions. If customers have relatively homogeneous cost functions, the differences between the two equilibria tend to be very small. If there are extremely heterogeneous customers, the quantification of the differences can serve as rules of thumb to guide the utility company to limit the market power of large customers in order to promote social welfare. For instance, splitting the largest customer or including another large customer to participate in the demand response will improve the efficiency of oligopolistic market.

Lastly we discuss a special case where each customer has a quadratic cost function and we demonstrate that the "optimal" bidding decision under the two markets are both independent of the value of supply fluctuations. [2] As a result, to determine the bidding decision, there is no need for the utility company and customers to know the precise value of the supply fluctuations. This is a very appealing property for the system operation (for both the utility company and the customers) because it is difficult to estimate the supply fluctuation precisely due to various uncertainties in power networks. For instance, it may be difficult to estimate or predict the power generation from the solar or wind farm accurately. Moreover, this property also implies that one bidding strategy can be used by the customers and the utility company over multi time periods instead of just for one event because the same bidding strategy is optimal at each time period.

The paper is organized as follows. Section II introduces the supply bid function model for matching the demand to the supply; Section III characterizes the competitive equilibrium where each customer is price taking; Section IV characterizes the game equilibrium where each customer is price anticipating; Section V studies the efficiency loss of the game equilibrium; Section VI discuss a special case where each customer has a quadratic cost function; and lastly Section VII provide case studies to complement the analysis.

II. SYSTEM MODEL: MATCHING THE SUPPLY

We consider a situation where there is a supply "deficit" or "surplus" on electricity. The deficit can be due to a decrease in power generation from, e.g., a wind or solar farm because of a change to worse weather condition, or an increase in power demand because of, e.g., a hot weather. The surplus can be due to an increase in power generation from, e.g., a wind or solar farm because of a change to better weather condition, or a decrease in power demand at, e.g., the late night time. We assume that it is very costly to increase the power supply in the case of a deficit or decrease the supply in the case of a surplus, i.e., the power supply is inelastic. If the utility company has good estimation of electricity deficit or surplus, the utility company can match the supply by customers shedding or increasing their loads. In the following we focus on the case with a supply deficit and consider a bidding scheme for the demand response over the distribution networks. The case with a supply surplus can be handled in the same way.

Consider a retail power market with a set $N := \{1, 2, \ldots, |N|\}$ of customers that are served by one utility company (or generator). Here a customer can be a single residential or commercial customer, or represent a group of customers that acts as a single demand response entity.

[1] In the competitive market, users are price taking and each one would like to maximize its net revenue under the given price. The competitive equilibrium is defined as the point at which each user maximizes its net revenue given the price and the price clears the market. Whereas, in the oligopolistic market, users are price anticipating, meaning that they know how their bids determine the price, and each one would like to maximize its net revenue given others bidding strategy. The game (Nash) equilibrium is defined as the point at which each user maximize its net revenue given others bidding profile.

[2] This result is consistent with the literature on supply function equilibrium [14].

Associated with each customer $i \in N$ is a load q_i that it is willing to shed in a demand response system. We assume that the total load shed needs to meet a specific amount $d > 0$ of electricity supply deficit, i.e.,

$$\sum_i q_i = d. \tag{1}$$

Assume that customer i incurs a cost (or disutility) $C_i(q_i)$ when it sheds a load of q_i. We assume that cost function $C_i(\cdot)$ is continuously differentiable, strictly increasing, convex, and with $C_i(0) = 0$.

We consider a market mechanism for the load shedding allocation, based on supply function bidding [14]. For simplicity of implementation of the demand response scheme, we assume that each customer's "supply" function (for load shedding) is parameterized by a single parameter $b_i \geq 0,\ i \in N$, and takes the form of

$$q_i(b_i, p) = b_i p,\ i \in N. \tag{2}$$

The supply function $q_i(b_i, p)$ gives the amount of load customer i is committed to shed when the price is p,[3] Note that the linear supply function form gives b_i a nature interpretation as being the sensitivity of customer i's decision to the market price.

The utility company will choose a price p that clears the market, i.e.,

$$\sum_i q_i(b_i, p) = \sum_i b_i p = d, \tag{3}$$

from which we get

$$p(b) = \frac{d}{\sum_i b_i}. \tag{4}$$

Here $b = (b_1, b_2, \cdots, b_{|N|})$ is the supply function profile.[4]

Before closing this section, we would like to comment on the linear supply function bidding.

A. *Why supply function bidding?*

Supply function as a strategic variable allows to adapt better to changing market conditions (such as uncertain and stochastic supply) than does a simple commitment to a fixed price or quantity [14], because no matter what the value of the supply deficit is, the utility company can use the supply function bid by the customers to clear the deficit. This is one reason why we use supply function bidding, as we will further study demand response under uncertain power network conditions. We will show later that if the cost function is quadratic, the "optimal" bidding strategy is independent of d for both competitive and oligopolistic market. Therefore this supply function bidding strategy can be applied to handle the uncertainty caused by renewable energy and also can be used over multi time periods. This property is consistent with the existing literature on supply function equilibrium in electricity market [14]. The other motivation to use supply function is to respect practical informational constraints in the power network. A customer might not want to reveal its cost function because of incentive or security concerns which means more communication. A properly-chosen parameterized supply function "controls" information revelation while requires less communication.

B. *Why linear supply function?*

Most existing literature on supply function bidding [14], [15], [16], [17], [18] used the general function as the bidding strategy. When the bidding action is changed from the linear form (represented by the single variable, b_i,) to a general form, the analysis of the strategic behavior of agents becomes much more complicated. To solve the general supply function equilibrium (SFE) requires solving a set of differential equations. To the best of our knowledge, there are only existence results about the SFE while assuming the agents are symmetric (i.e., with the same cost function) or assuming there are only two asymmetric agents. For practical applications, the asymmetric case is more interesting. The greatest advantage of using linear supply function over the general forms is the ability to handle asymmetric agents when there are more than two strategic agents. Moreover, as shown in our paper, the linear supply function allows us to get a closed form characterization for the structure and efficiency of the market equilibria, which could be impossible to get if using the general supply function. The closed form analysis is not only of academic interest, but also appeals to the practical applications because 1) it tells the utility company and customers what they expect if joining the demand response, helping them decide whether they should join the demand response, and 2) it can be used to guide the utility company to design demand response to improve the system efficiency.

Note that [19], [20] also study the efficiency loss of a parameterized supply function bidding in the form of $q_i = \frac{b_i}{p}$. This is motivated by resource allocation problems where each user bids total money b_i for competing the resources and p is the market clear price. The linear supply function is more closely aligned with the electricity market literature and practice. The advantage of linear supply function is the robustness to the system uncertainty as illustrated in Section VI. However, to ensure bounded efficiency loss of the linear supply function bidding, stronger assumptions need to be made compared to using the supply function in the form of $q_i = \frac{b_i}{p}$.

III. Competitive equilibrium: Optimal demand response

In this subsection, we consider a competitive market where customers are price taking. Given price p, each customer i

[3] Note that in practice a customer usually has a capacity limit, e.g. $\bar{q}_i$, on the load that she is able to shed. For the purpose of carrying out the closed form analysis, especially on the effect of cost functions on customer behaviors, we do not explicitly consider the capacity limit in this paper. As a result, our model only holds if we assume that the total supply deficit d is small enough and each customer represents an aggregate of users who has a large capacity limit. For the scenarios where the capacity limits are important, one way to incorporate the capacity limits is to translate the capacity constraint into a cost function, e.g., $c_i(q_i - \bar{q}_i)$ which is added into the original cost function $C_i(q_i)$. If the weight on the cost function $c_i(q_i - \bar{q}_i)$ is large enough, the demand response solution will satisfy the capacity limit constraint [21].

[4] We note that p is not well defined if $\sum_i b_i = 0$. Thus we assume that the utility company will reject the bid if $\sum_i b_i = 0$.

maximizes its net revenue

$$\max_{b_i \geq 0} \quad pq_i(b_i, p) - C_i(q_i(b_i, p)), \tag{5}$$

where the first term is the customer i's revenue when it sheds a load of $q_i(b_i, p)$ at a price of p and the second term is the cost incurred.

A. Competitive equilibrium

We now analyze the competitive equilibrium of the demand response system.

Definition 1 *A competitive equilibrium for the demand response system is defined as a tuple $\{(\bar{b}_i)_{i\in N}, \bar{p}\}$ such that $\bar{b}_i$ is optimal in (5) for each customer i given the price $\bar{p}$ and $\sum_i q_i(\bar{b}_i, \bar{p}) = d$.*

The following result shows the existence and uniqueness of such competitive equilibrium, and it also shows that the equilibrium is efficient in terms of maximizing social welfare.

Theorem 1 *The competitive equilibrium $\{(\bar{b}_i)_{i\in N}, \bar{p}\}$ for the demand response system exists. Moreover, the equilibrium is efficient, i.e., $(\bar{q}_i)_{i\in N} = \left(q_i(\bar{b}_i, \bar{q}_i)\right)_{i\in N}$ maximizes the social welfare:*

$$\max_{q_i \geq 0} \quad \sum_i -C_i(q_i) \tag{6a}$$

$$s.t. \quad \sum_i q_i = d. \tag{6b}$$

If the cost function $C_i(q_i)$ is strictly convex, then the competitive equilibrium is unique.

Proof: Definition 1 tells that $\{(\bar{b}_i)_{i\in N}, \bar{p}\}$ is a competitive equilibrium if and only if:

$$(C_i'(q_i(\bar{b}_i, \bar{p})) - \bar{p})(b_i - \bar{b}_i) \quad \geq \quad 0, \quad \forall b_i \geq 0, \tag{7a}$$

$$\sum_i q_i(\bar{b}_i, \bar{p}) \quad = \quad d. \tag{7b}$$

Here (7a) is from the optimality condition [21] of the convex optimization problem (5) and (7b) is directly from the definition 1.

Notice that $\bar{p} \geq 0$. Multiplying $\bar{p}$ to equation (7a), we get

$$(C_i'(\bar{q}_i) - \bar{p})(q_i - \bar{q}_i) \quad \geq \quad 0, \quad \forall q_i \geq 0, \tag{8a}$$

$$\sum_i \bar{q}_i \quad = \quad d. \tag{8b}$$

This is just the KKT optimality condition of optimization problem (6) [22]. Hence $(q_i)_{i\in N}$ maximize the social welfare. And if $\{(\bar{q}_i)_{i\in N}, \bar{p}\}$ is an optimal primal-dual solution of (6), $\{(\bar{b}_i := \bar{q}_i/\bar{p})_{i\in N}, \bar{p}\}$ satisfies (7) which tells that $\{(\bar{b}_i)_{i\in N}, \bar{p}\}$ is a competitive equilibrium.

If $C_i(q_i)$ is strictly convex for each customer i, then the social welfare maximization is a strictly convex problem. Thus there exists a unique optimal solution $(\bar{q}_i)_{i\in N}$. Moreover, (8a) tells that $\bar{p} = C_i'(\bar{q}_i)$ for any $\bar{q}_i > 0$, implying that $\bar{p}$ is unique as well. Thus we know that equilibrium is unique. ■

Based on this optimization problem characterization, we further study how a cost function affects a customer's demand response at a competitive equilibrium. For each customer i, define the base load shedding marginal cost as $C_i^0 := C_i'(0^+)$. Without loss of generality, we assume that $C_1^0 \leq C_2^0 \leq \ldots \leq C_{|N|}^0$. For later convenience, we also introduce one virtual parameter $C_{|N|+1}^0$ and define $C_{|N|+1}^0 := C_n'(d)$. Therefore we have $C_1^0 \leq C_2^0 \leq \ldots \leq C_{|N|}^0 \leq C_{|N|+1}^0$.

Theorem 2 *Let $\{(\bar{b}_i)_{i\in N}, \bar{p}\}$ be a competitive equilibrium and $\bar{q}_i = q_i(\bar{b}_i, \bar{p})$ be the corresponding load shed by $i \in N$. The set of customers that shed a positive load at the equilibrium, i.e. $\{i : \bar{q}_i > 0\}$, is $\bar{N} = \{1, 2, \cdots, \bar{n}\}$ with a $\bar{n}$ that satisfies:*

$$\sum_i^{\bar{n}} (C_i')^{-1}(C_{\bar{n}}^0) \leq d \leq \sum_i^{\bar{n}} (C_i')^{-1}(C_{\bar{n}+1}^0). \tag{9}$$

Moreover, the price $\bar{p}$ satisfies:

$$C_{\bar{n}}^0 \leq \bar{p} \leq C_{\bar{n}+1}^0 \tag{10}$$

and for any $i \in \bar{N}$, $\bar{p} = C_i'(\bar{q}_i)$.

Proof: From the proof of Theorem 1, we know that $\{\bar{p}, (\bar{q}_i)_{i\in N}\}$ satisfies Condition (8). From (8a), we know that, for any $i \in N$,

i) if $\bar{q}_i > 0$, then $\bar{p} = C_i'(\bar{q}_i) \geq C_i'(0)$;
ii) if $\bar{q}_i = 0$, then $\bar{p} \leq C_i'(\bar{q}_i) = C_i'(0)$

Thus we know that all the customers who shed a positive load have a smaller $C_i^0 := C_i'(0)$ than those who do not shed any load. Since C_i^0 is increasing in i, $\bar{N}$ takes the form of $1, 2, \cdots, \bar{n}$. If $\bar{n} < |N|$, Condition i) and ii) implies that

$$C_{\bar{n}}^0 \leq \bar{p} \leq C_{\bar{n}+1}^0.$$

If $\bar{n} = |N|$, $\bar{p} = C_{|N|}'(\bar{q}_{|N|}) \leq C_{|N|}'(d) = C_{\bar{n}+1}^0$, thus:

$$C_{\bar{n}}^0 \leq \bar{p} \leq C_{\bar{n}+1}^0.$$

Note that $C_i'(q_i)$ is an increasing function. Hence,

$$\sum_i^{\bar{n}} (C_i')^{-1}(C_{\bar{n}}^0) \leq \sum_i^{\bar{n}} (C_i')^{-1}(\bar{p}) \leq \sum_i^{\bar{n}} (C_i')^{-1}(C_{\bar{n}+1}^0),$$

which is,

$$\sum_i^{\bar{n}} (C_i')^{-1}(C_{\bar{n}}^0) \leq \sum_i^{\bar{n}} \bar{q}_i = d \leq \sum_i^{\bar{n}} (C_i')^{-1}(C_{\bar{n}+1}^0).$$

■

This theorem says that the competitive equilibrium has a water-filling structure. The base load shedding marginal cost $C_i'(0)$ determines whether the customer i sheds its load or not. The higher the marginal cost at zero, the less likely they will join the demand response. Moreover, those customers who join the demand response at competitive equilibrium have a same marginal cost.

Moreover, this theorem also implies the following corollary, which tells that the individual rationality is guaranteed at the competitive equilibrium.

Corollary 1 *Any customer who sheds a positive load receives positive net revenue at the competitive equilibrium, i.e. $\bar{p}\bar{q}_i - C_i(\bar{q}_i) \geq 0$ for all $i \in \bar{N}$.*

Proof: From Theorem 2, we know that for all $i \in \bar{N}$, $\bar{p} = C_i'(\bar{q}_i)$. Notice that $C_i(\cdot)$ is a convex function. Thus

$$C_i(\bar{q}_i) - C_i(0) \le C_i'(\bar{q}_i)\bar{q}_i.$$

As $C_i(0) = 0$, we have $C_i(\bar{q}_i) \le \bar{p}\bar{q}_i$. The statement follows. ■

B. Distributed supply function bidding

One way to find the competitive equilibrium is to solve the social welfare problem (6), requiring to know the cost function C_i of all customers. However, demand response typically involves a very large number of customers who are usually unwilling or unable to report their cost functions C_i.[5] These motivates the needs of *distributed* supply function bidding where the utility company sets the price and customers submit a bid based on the price. Such bidding schemes require only light communication and computation and can be easily scaled to large systems.

Note that the social welfare problem (6) can be easily solved in a distributed way by the dual gradient algorithm [22] or the alternative direction multiplier method (ADMM) [23]. This suggests an iterative, distributed supply function bidding scheme to achieve the market equilibrium for demand response given that customers are price-takers. As an example, we adopt the dual-gradient algorithm and design the following distributed supply function bidding scheme. Similar ideas can be applied to other distributed optimization methods.

Initially, the utility company randomly picks a price $p(0)$ and announces the price to each customer over the communication network. Set the iteration step $k = 0$. At k-th iteration:

- Upon receiving price $p(k)$ announced by the utility company, each customer i updates its supply function, i.e., $b_i(k)$, according to

$$b_i(k) = [\frac{(C_i')^{-1}(p(k))}{p(k)}]^+, \tag{11}$$

  and submits it to the utility company. Here '+' denotes the projection onto $\mathcal{R}^+$, the set of nonnegative real numbers.
- Upon gathering bids $b_i(k)$ from customers, the utility company updates the price according to

$$p(k+1) = [p(k) - \gamma(\sum_i b_i(k)p(k) - d)]^+, \tag{12}$$

  and announces the price $p(k+1)$ to the customers. Here $\gamma > 0$ is a constant stepsize.
- Set $k + 1 \leftarrow k$ and repeat.

It is straightforward to verify that the supply function bidding scheme is equivalent to the dual gradient algorithm of (6). Therefore, the bidding scheme possess all the convergence properties of the dual gradient algorithm. For example, when γ is small enough, the above algorithm converges exponentially to the competitive equilibrium under some assumptions of the cost function C_i. We refer readers to [21], [22] for more details regarding the choices of stepsizes, stopping criterion, and convergence speed. The numerical studies in Section VII also demonstrated that the algorithm converges very fast even for a very large network.

[5]In practice, the cost function may be a complicated function since it is related to the detailed appliance performance. Customers might be unwilling to report the cost function due to privacy and security concerns; even if they are willing to report them, collecting the cost function from all the customers takes a large amount of communication resources given the large number of customers.

IV. Strategic demand response in an oligopolistic market

In this section, we consider an oligopolistic market where customers know that price p is set according to (4) and behave strategically. Denote the supply function for all customers but i by $b_{-i} = (b_1, b_2, \cdots, b_{i-1}, b_{i+1}, \cdots, b_{|N|})$ and write (b_i, b_{-i}) for the supply function profile b. Each customer i chooses b_i to maximize its own net benefit $u_i(b_i, b_{-i})$ given others' bidding strategy b_{-i},

$$\begin{aligned} u_i(b_i, b_{-i}) &= p(b)q_i(p(b), b_i) - C_i(q_i(p(b), b_i)) \\ &= \frac{d^2 b_i}{(\sum_j b_j)^2} - C_i(\frac{d b_i}{\sum_j b_j}). \end{aligned} \tag{13}$$

Here the second equality is obtained by substituting the market clearing price $p(b) = \frac{d}{\sum_i b_i}$ and the linear supply bidding function $q_i(p(b), b_i) = b_i p(b)$ into the first equality. As a result, the net benefit functions $\{u_i(b_i, b_{-i})\}_{i \in N}$ define a demand response game among customers.

1) Game-theoretic equilibrium: We now analyze the equilibrium of the demand response game. The solution concept we use is the Nash equilibrium [24].

Definition 2 *A supply function profile b^* is a Nash equilibrium if for all customers $i \in N$,*

$$u_i(b_i^*, b_{-i}^*) \ge u_i(b_i, b_{-i}^*), \forall b_i \ge 0.$$

We see that the Nash equilibrium is a set of strategies for which no player has an incentive to change unilaterally. In the following, we will characterize the existence and structure of the Nash equilibrium.

Lemma 1 *If b^* is a Nash equilibrium of the demand response game, then $\sum_{j \neq i} b_j^* > 0$ for any $i \in N$.*

Proof: We prove the result by contradiction. Suppose that it does not hold, and without loss of generality, assume that $\sum_{j \neq i} b_j^* = 0$ for a customer i. Then, the payoff for the customer i is $u_i(b_i^*, b_{-i}^*) = 0$ if $b_i^* = 0$, and $u_i(b_i^*, b_{-i}^*) = d^2/b_i^* - C_i(d)$ if $b_i^* > 0$. We see that when $b_i^* = 0$, the customer i has an incentive to increase it, and when $b_i^* > 0$ the customer has an incentive to decrease it. Hence, there is no Nash equilibrium with $\sum_{j \neq i} b_j^* = 0$. ■

The above Lemma also implies the following lemma:

Lemma 2 *If b^* is a Nash equilibrium, then at least two customers have $b_i^* > 0$.*

Let $B_{-i} = \sum_{j \neq i} b_j$. We further have the following characterization for a Nash equilibrium.

Lemma 3 *If b^* is a Nash equilibrium of the demand response game, then $b_i^* < B_{-i}^* = \sum_{j \neq i} b_j^*$ for any $i \in N$, and* *thus, each customer will shed a load of less than $d/2$ at the equilibrium.*

Proof: From (13), we can derive:

$$\frac{\partial u_i(b_i, b_{-i})}{\partial b_i} = \frac{d^2(B_{-i} - b_i)}{(B_{-i} + b_i)^3} - \frac{dB_{-i}}{(B_{-i} + b_i)^2} C_i'(\frac{db_i}{B_{-i} + b_i}) = \frac{d^2}{(B_{-i} + b_i)^2}[\frac{B_{-i} - b_i}{B_{-i} + b_i} - \frac{B_{-i}}{d} C_i'(\frac{db_i}{B_{-i} + b_i})]. \quad (14)$$

The first term in the square bracket is no greater than 1 and strictly decreasing in b_i; the second term is increasing in b_i.

So, if $\frac{B_{-i}}{d} C_i'(0) \geq 1$, $\frac{\partial}{\partial b_i} u_i(b_i, b_{-i}) \leq 0$ for all b_i, and $b_i = 0$ maximizes the customer i payoff $u_i(b_i, b_{-i})$ for the given b_{-i}. If $\frac{B_{-i}}{d} C_i'(0) < 1$, $\frac{\partial}{\partial b_i} u_i(b_i, b_{-i}) = 0$ only at one point $b_i > 0$. Furthermore, note that $\frac{\partial}{\partial b_i} u_i(0, b_{-i}) > 0$ and $\frac{\partial}{\partial b_i} u_i(B_{-i}, b_{-i}) < 0$. So, this point b_i maximizes the customer i payoff $u_i(b_i, b_{-i})$ for the given b_{-i}.

Thus, at the Nash equilibrium for the demand response game, b^* satisfies

$$b_i^* = 0, \quad (15)$$

if $\frac{B_{-i}^*}{d} C_i'(0) \geq 1$; and otherwise,

$$\frac{B_{-i}^* - b_i^*}{B_{-i}^* + b_i^*} - \frac{B_{-i}^*}{d} C_i'(\frac{db_i^*}{B_{-i}^* + b_i^*}) = 0. \quad (16)$$

Given a Nash equilibrium, b^*, (i) if $b_i^* = 0$, then $b_i^* < B_{-i}^*$ from Lemman 1; (ii) otherwise, b_i^* satisifies (16). Note that the second term on the left hand side of equation (16) is positive. So the first term must be positive as well, which requires $B_{-i}^* > b_i^*$. Because for each customer i, $q_i^* = b_i^* \frac{d}{b_i^* + B_{-i}^*}$, each customer will shed a load of less than $\frac{d}{2}$ at the equilibrium. ■

The following result follows directly from Lemma 3.

Corollary 2 *No Nash equilibrium exists when $|N| = 2$.*

Using the conditions in (15) and (16), we can charaterize the existence and structure of the Nash equilibrium in the following theorem.

Theorem 3 *Assume $|N| \geq 3$. The demand response game has a unique Nash equilibrium. Moreover, the equilibrium solves the following convex optimization problem:*

$$\min_{0 \leq q_i < d/2} \quad \sum_i D_i(q_i) \quad (17a)$$

$$s.t. \quad \sum_i q_i = d, \quad (17b)$$

with

$$D_i(q_i) = (1 + \frac{q_i}{d - 2q_i}) C_i(q_i) - \int_0^{q_i} \frac{d}{(d - 2x_i)^2} C_i(x_i) dx_i.$$

Proof: First, note that

$$D_i'(q_i) = (1 + \frac{q_i}{d - 2q_i}) C_i'(q_i), \quad (18)$$

which is a positive, strictly increasing function in $q_i \in [0, d/2)$. So, $D_i(q_i)$ is a strictly increasing and strictly convex function in $[0, d/2)$. Because $D_i(q_i) = \int_0^{q_i} D_i'(x_i) dx_i \geq C_i'(0) \int_0^{q_i} (1 + \frac{x_i}{d - 2x_i}) dx_i = C_i'(0) \int_0^{q_i} (\frac{1}{2} + \frac{d/2}{d - 2x_i}) dx_i = C_i'(0)(\frac{1}{2} q_i - \frac{d}{4} \log(d - 2x_i))$. Therefore $\lim_{q_i \to \frac{d}{2}} D_i(q_i) = \infty$. Thus the optimization problem (17a)-(17b) is a strictly convex problem and has a unique optimal solution (see Appendix). Based on the optimality condition [21] and after a bit mathematical manipulation, the unique solution q^* is determined by

$$(p^* - (1 + \frac{q_i^*}{d - 2q_i^*}) C_i'(q_i^*))(q_i - q_i^*) \leq 0, \ \forall q_i, \quad (19)$$

$$\sum_i q_i^* = d, \quad (20)$$

$$p^* > 0. \quad (21)$$

Second, note that the Nash equilibrium condition (15)-(16) can be written compactly as

$$(\frac{d}{B_{-i}^* + b_i^*} - \frac{B_{-i}^*}{B_{-i}^* - b_i^*} C_i'(\frac{db_i^*}{B_{-i}^* + b_i^*}))(b_i - b_i^*) \leq 0, \ \forall b_i. \quad (22)$$

Recall that the (Nash) equilibrium price $p^* = d/\sum_i b_i^*$ and (Nash) equilibrium allocation $q_i^* = b_i^* p^*$. We can write equation (22) as

$$(p^* - (1 + \frac{q_i^*}{d - 2q_i^*}) C_i'(q_i^*))(b_i p^* - q_i^*) \leq 0. \quad (23)$$

Note that at the Nash equilibrium, $p^* > 0$ since $\sum_i b_i^* > 0$ by Lemma 1. So, the above equation is equivalent to equation (19). Thus, the Nash equilibrium of the demand response game satisfies the optimality condition (19)-(21), and solves the optimization problem (17a)-(17b). The existence and uniqueness of the Nash equilibrium is a result of the existence and uniqueness of the optimal solution of (17a)-(17b). ■

Remark 1 *Theorem 3 can be seen as reverse-engineering from the game-theoretic equilibrium into a global optimization problem. Define $\Delta C_i(q_i) := \frac{q_i}{d - 2q_i} C_i(q_i) - \int_0^{q_i} \frac{d}{(d - 2x_i)^2} C_i(x_i) dx_i = \int_0^{q_i} \frac{x_i}{(d - 2x_i)^2} C_i(x_i) dx_i$. Then $D_i(q_i) = C_i(q_i) + \Delta C_i(q_i)$. Thus $\Delta C_i(q_i)$ can be interpreted as the "false" information reported by customers in order to gain more benefits from the demand response through the self-interested and strategic bidding. Note that $\Delta_i C_i(q_i) > 0$ for all $q_i \in [0, d/2)$. The positiveness of $\Delta_i C_i(q_i)$ means that all the customers fake a higher cost function in order to increase the price. In Section V, we will leverage this theorem and the property of $D_i(q_i)$ to further study the efficiency loss of Nash equilibrium.*

Similarly as in Section III, we study how a cost function affects a customer's demand response at the Nash equilibrium using this optimization problem characterization. Recall that the based load shedding marginal cost C_i^0 is defined as $C_i^0 := C_i'(0^+)$ and without loss of generality, we assume that $C_1^0 \leq C_2^0 \leq \ldots \leq C_{|N|}^0$. Notice that $C_i'(0^+) = D_i'(0^+)$. For convenience, the virtual parameter $C_{|N|+1}^0$ is defined as $C_{|N|+1}^0 := \max_i D_{|N|}'(d/3)$ for the following theorem. Therefore we have $C_1^0 \leq C_2^0 \leq \ldots \leq C_{|N|}^0 \leq C_{|N|+1}^0$.

Theorem 4 *Assume $|N| \geq 3$. Let $\{(b_i^*)_{i\in N}\}$ be a Nash equilibrium, $p^* = d/\sum_i b_i^*$ be the Nash equilibrium price, and $q_i^* = b_i^* p^*)$ be the corresponding load shed by $i \in N$. The set of customers that shed a positive load at the Nash equilibrium, i.e. $\{i : q_i^* > 0\}$, is $N^* = \{1, 2, \cdots, n^*\}$ with a unique n^* that satisfies*

$$\sum_i^{n^*} (D_i')^{-1}(C_{n^*}^0) < d \leq \sum_i^{n^*} (D_i')^{-1}(C_{n^*+1}^0) \tag{24}$$

Moreover, the price p^* satisfies

$$C_{n^*}^0 < p^* \leq C_{n^*+1}^0. \tag{25}$$

and for any $i \in N^*$, $p^* = D_i'(q_i^*)$.

Proof: Noe that $D_i'(q_i)$ is an strictly increasing function of q_i and $D_i'(0) = C_i'(0)$. The proof follows the same argument as in Theorem 2. ∎

This theorem says that the Nash equilibrium also has a water-filling structure. The base load shedding marginal cost $C_i'(0)$ also determines whether a customer i sheds its load or not at the Nash equilibrium. The higher the marginal cost at zero, the less likely they will join the demand response. Moreover, those customers who join the demand response at competitive equilibrium have a same *modified* marginal cost.

This theorem also implies that the individual rationality is guaranteed at the Nash equilibrium as shown in the next corollary.

Corollary 3 *Any customer who shed a positive load at the Nash equilibrium receives positive net revenue, i.e. $p^* q_i^* - C_i(q_i^*) > 0$ for all $i \in N^*$.*

Proof: From Theorem 4, we know that for all $i \in \bar{N}$, $p^* = D_i'(q_i^*)$. Notice that $D_i(\cdot)$ is a strictly convex function. Thus

$$D_i(q_i^*) - D_i(0) < D_i'(q_i^*) q_i^*.$$

Because $D_i(0) = 0$, $D_i(q) > C_i(q)$, we have,

$$C_i(q_i^*) < p^* q_i^*.$$

The statement follows. ∎

Note that Corollary 1 and Corollary 3 justify the rationality of customers to join the demand response scheme because there is no revenue loss in both markets.

Though Theorem 4 and Corollary 3 tell that the Nash equilibrium and the competitive equilibrium share similar properties, we will show in the next section that the two equilibria are different in many perspectives especially in terms of system efficiency.

A. Distributed supply function bidding

Theorem 3 says that we can solve the Nash equilibrium of the demand response game by solving convex optimization problem (17a)-(17b). Therefore, we can apply the same idea used in III-B and design the following distributed supply function bidding scheme to reach the Nash equilibrium.

At k-th iteration:

- Upon receiving price $p(k)$ announced by the utility company, each customer i updates its supply function, i.e., $b_i(k)$, according to

$$b_i(k) = [\frac{(D_i')^{-1}(p(k))}{p(k)}]^+, \tag{26}$$

  and submits it to the utility company.
- Upon gathering bids $b_i(k)$ from customers, the utility company updates the price according to

$$p(k+1) = [p(k) - \gamma(\sum_i b_i(k)p(k) - d)]^+, \tag{27}$$

  and announces price $p(k+1)$ to customers.

All the discussions in Section III-B apply to the above algorithm. Thus we omit the duplication here. Note that the distributed convergence to the Nash equilibrium is a difficult problem in general, because of informational constraints in the system. Here we involve the utility company in mediating strategic interaction among customers (equation (27)) in order to achieve the equilibrium in a distributed manner. The strategic action of the customer is also partially encapsulated in equation (26).

V. Efficiency loss of the Nash equilibrium

We have shown that the competitive equilibrium maximizes the social welfare. By contrast, due to customers' price-anticipating and strategic behavior, the Nash equilibrium is expected to be less efficient. Though supply function bidding has been applied in electricity market in practice [15], [16], [17], [18], [20], there is a lack of work studying the efficiency loss of Nash equilibria. In this section, we demonstrate that if customers have extremely heterogeneous cost functions, the efficiency loss of the Nash equilibrium may be infinite. Here the efficiency loss is defined as the ratio of the total disutility at the Nash equilibrium to the minimum total disutility. However, under certain mild assumptions the efficiency loss is bounded. We provide a closed form characterization for the bound, which can serve as a guideline for the utility company to limit market power of certain customers in order to promote global social welfare. For instance, splitting the largest customer or including another large customer to participate in the demand response will improve the efficiency of the oligopolistic market.

Specifically, we will first use a 3 customer case to show that the efficiency loss may be unbounded if one customer has extremely low cost compared with others; and then we will provide a general characterization of the efficiency loss of the Nash equilibrium.

A. A special case with 3 customers

Consider a scenario where there are only 3 customers with cost functions being $c_1(q) = \frac{1}{2r}cq^2$, $c_2(q) = c_3(q) = \frac{1}{2}cq^2$ respectively. Here c and n are constant parameters.

Firstly, through Theorem 2, we can calculate the competitive equilibrium, which is given by $\bar{q}_1 = \frac{r}{r+2}d$, $\bar{q}_2 = \bar{q}_3 = \frac{1}{r+2}d$, $\bar{p} = c\bar{q} = \frac{r}{r+2}cd$.

Similarly, through Theorem 3, we can calculate the game Nash equilibrium, which is given by $q_1^* = \frac{-r+\sqrt{(16+9r)r}}{4(2+r)}d$, $q_2^* = q_3^* = \frac{8+5r-\sqrt{(16+9r)r}}{8(2+r)}d$, $p^* = \frac{D-q_1^*}{D-2q_1^*}q_1^*$.

Let $r \to \infty$, for the competitive equilibrium, we have $\bar{q}_1 \to d$, $\bar{q}_2, \bar{q}_3 \to 0$, $\bar{p} \to cd$ and total cost $\bar{C} \to 0$; for the Nash equilibrium, we have $q_1^* \to \frac{d}{2}$, $q_2^*, q_3^* \to \frac{d}{4}$, $p^* \to \infty$, and total cost $C^* \to \frac{cd^2}{4}$. Therefore we know that $\frac{p^*}{p} \to \infty$ and $\frac{C^*}{C} \to \infty$.

From this simple case, we see that if there exist customers with extremely heterogeneous cost functions, the efficiency loss of the Nash equilibrium may be unbounded.

B. General characterization

Now we provide a general characterization of the efficiency loss of the Nash equilibrium. Besides that, we also compare other differences between the competitive equilibrium and Nash equilibrium.

Theorem 5 *Let $\{(\bar{b}_i)_{i\in N}, \bar{p}\}$ be a competitive equilibrium, $\{(b_i^*)_{i\in N}\}$ be a Nash equilibrium and p^* be the corresponding price at the Nash equilibrium, we have:*

1) *$\bar{N} \subseteq N^*$, where $\bar{N} := \{i : \bar{q}_i := b_i\bar{p} > 0\}$ is the set of customers who shed a positive load at the competitive equilibrium; and $N^* := \{q_i^* := b_i^* p^* > 0\}$ is the set of customers who shed a positive load at the Nash equilibrium.*
2) *$\bar{p} \le p^* \le \frac{n-1}{n-2}\frac{M}{m}\bar{p}$, where $M := \max_{i\in N} C_i'(\frac{d}{n})$; $m := \min_{i\in N} C_i'(\frac{d}{n})$.*
3) *$\bar{C} \le C^*$, and if we further assume $\bar{q}_{max} := \max_i \bar{q}_i < \frac{d}{2}$, then we have*
$$C^* \le (1 + \frac{\bar{q}_{max}}{d - 2\bar{q}_{max}})\bar{C}.$$
Here $\bar{C} = \sum_i C_i(\bar{q}_i)$ be the total social cost at the competitive equilibrium and $C^* = \sum_i C_i(q_i^*)$ is the total cost at the Nash equilibrium.

Before providing a proof of the theorem, we interpret the three conditions as follows:

- Condition 1) means that the set of the customers who shed a positive load at the Nash equilibrium is a superset of that at the competitive equilibrium.
- Condition 2) means that the price at the Nash equilibrium is higher than that at the competitive equilibrium, but the ratio between the two prices are bounded.
- Condition 3) means that the total cost at the Nash equilibrium is higher than that at the competitive equilibrium, but the ratio between the two costs are bounded provided that no one shed more than half of the total load at the competitive equilibrium.

Proof: Notice that $D_i'(q_i), C_i'(q_i)$ are both strictly increasing function and $D_i'(q_i) \ge C_i'(q_i)$ for any $q_i \in [0, \frac{d}{2})$. For any $i \in \bar{N}$, $(D_i')^{-1}(\bar{p}) \le C_i'^{-1}(\bar{p})$.

Suppose $p^* < \bar{p}$. Because $C_{n^*}^0 \le p^* \le C_{n^*+1}^0$, $C_{\bar{n}}^0 \le \bar{p} \le C_{\bar{n}+1}^0$, and $C_1^0 \le C_2^0 \le \cdots \le C_n^0$, we have $n^* \le \bar{n}$. Therefore
$$\begin{aligned}\sum_i^{n^*}(D_i')^{-1}(p^*) &< \sum_i^{n^*}(D_i')^{-1}(\bar{p}) \\ &\le \sum_i^{n^*}(C_i')^{-1}(\bar{p}) \le \sum_i^{\bar{n}}(C_i')^{-1}(\bar{p}) = d.\end{aligned}$$
which contradicts that $\sum_i^{n^*}(D_i')^{-1}(p^*) = d$. Thus $p^* \le \bar{p}$. Therefore $\bar{n} \le n^*$, implying that $\bar{N} \subset N^*$.

If $n^* < n$, then $p^* \le D_{n^*+1}'(0) \le D_{n^*+1}'(\frac{d}{n}) = \frac{n-1}{n-2}C_{n^*+1}'(\frac{d}{n}) \le \frac{n-1}{n-2}M$. If $n^* = n$, then there exist one customer j such that $0 < q_j^* \le \frac{d}{n}$. Thus $p^* = D_j'(q_j^*) \le D_j'(\frac{d}{n}) \le \frac{n-1}{n-2}M$. In summary,
$$p^* \le \frac{n-1}{n-2}M. \tag{28}$$

On the other side, there exists at least one customer j such that $C_j'(\bar{q}_i) = \bar{p}$ and $\bar{q}_i \ge \frac{d}{n}$. Therefore,
$$\bar{p} \ge C_j'(\frac{d}{n}) \ge m. \tag{29}$$

Combining (28) and (29) gives
$$p^* \le \frac{n-1}{n-2}\frac{M}{m}\bar{p}.$$

Lastly $\bar{C} \le C^*$ comes from the fact that $(\bar{q}_i)_{i\in N}$ is an optimal solution of optimization problem (6a) as shown in Theorem 1.

If $\bar{q}_{\max} < \frac{d}{2}$, then $\sum_i D_i(q_i^*) \le \sum_i D_i(\bar{q}_i)$ since $(q_i^*)_{i\in N}$ is an optimal solution of problem (17). It is straightforward to check that $D_i(\bar{q}_i) \le (1 + \frac{\bar{q}_i}{d-2\bar{q}_i})C_i(q_i^*)$. Thus
$$\sum_i D_i(q_i^*) \le (1 + \frac{\bar{q}_{\max}}{d - 2\bar{q}_{\max}})\bar{C}.$$

On the other hand, for any $q_i < \frac{d}{2}$,
$$\begin{aligned}D_i(q_i) &= (1 + \frac{q_i}{d-2q_i})C_i(q_i) - \int_0^{q_i}\frac{d}{(d-2x_i)^2}C_i(x_i)dx_i \\ &\ge (1 + \frac{q_i}{d-2q_i})C_i(q_i) - C_i(q_i)\int_0^{q_i}\frac{d}{(d-2x_i)^2}dx_i \\ &\ge (1 + \frac{q_i}{d-2q_i})C_i(q_i) - C_i(q_i)\frac{q_i}{d-2q_i} \\ &\ge C_i(q_i).\end{aligned}$$
therefore,
$$C^* = \sum_i C_i(q_i^*) \le \sum_i D_i(q_i^*) \le (1 + \frac{\bar{q}_{\max}}{d - 2\bar{q}_{\max}})\bar{C}.$$

■

From this theorem, we see that as long as there are no one shed more than one third of the total loads at the competitive equilibrium, the efficiency loss $\frac{C^*}{C}$ is bounded by $3/2$. The former condition can be guaranteed if there are at least 3 customers having comparably low cost.

Another interesting result which can be derived from Theorem 2 and Theorem 4 is that if the customers are homogeneous (i.e., symmetric), the differences between the two market equilibria are small. This is shown in the following corollary.

Corollary 4 *Assume that all the customers have a same cost function. Then we have*

1) $p^* = \frac{n-1}{n-2}\bar{p}$. *As* $n \to \infty$, $p^* \to \bar{p}$
2) $C^* = \bar{C}$. *As* $n \to \infty, C^* \to \bar{C}$.

Proof: Using Theorem 2 and Theorem 4, we know that if all the customers have a same cost function, then $\bar{q}_i = q_i^* = \frac{d}{n}$. Therefore $C^* = \bar{C}$ and $p^* = (1 + \frac{\frac{d}{n}}{d - \frac{2d}{n}})C'(\frac{d}{n}) = \frac{n-1}{n-2}\bar{p}$. ■

Note that customers are relatively homogeneous in demand response, especially residential demand response. Therefore, applying the supply function bidding scheme will guarantee the efficiency of the demand response no matter whether the customers are price-taking or price-anticipating. In Section VII we will further discuss the effects of the heterogeneity of cost functions on the efficiency loss of Nash equilibrium using concrete case studies.

VI. A SPECIAL CASE WITH QUADRATIC COST FUNCTION

A common cost function that has been used in existing literature and even in practice is the quadratic cost function [14], [15], [16], [17], [18], i.e. $C_i(q) = \frac{1}{2}c_i q^2$. Existing literature has focused on the existence of supply function equilibirum(SFE) given an uncertain d or a time sequence of $d(t)$ where $t \in \mathcal{T}$ and $\mathcal{T}$ has multi time periods. It is shown that if the cost function is in quadratic form, then a supply function equilibrium exists and it is independent of d. However, those literature often require relevant complicated analysis to show the existence of such SFE and it is more complicated to characterize the properties of such SFE. We have adopted a different approach to study the strategic behavior among customers and obtained an optimization characterization for the Nash equilibrium. As SFE is defined in a similar way to Nash equilibrium, we would expect that our analysis will lead to a same property of the Nash equilibrium given quadratic cost functions. In the following, we will show that the bidding strategy at the Nash equilibrium is independent of d if every customer's cost function is quadratic.

Now suppose $C_i(q) = \frac{1}{2}c_i q^2$ for each i. The following theorem characterizes the competitive equilibrium and the Nash equilibrium.

Theorem 6 *Suppose each customer has a quadratic cost function, i.e.* $C_i(q) = \frac{1}{2}c_i q^2$ *for each* i. i

1) $\{(\bar{b}_i)_{i\in N}, \bar{p}\}$ *is a competitive equilibrium if and only if* $\bar{b}_i = \frac{1}{c_i}$.
2) $\{(b_i^*)_{i\in N}\}$ *is a Nash equilibrium if and only if* $\{(b_i^*)_{i\in N}\}$ *satisfies the following equalities,*

$$b_i^* = (1 - c_i b_i^*) B_{-i}^*, \forall i \in N. \tag{30}$$

Proof: It is straightforward to get the first statement. The second statement follows by substituting $C_i'(q_i) = c_i q_i$ into equation (16). ■

This theorem tells that given quadratic cost functions, the customers' optimal bidding strategy is independent of d either in competitive market or oligopolistic market. As a result, to determine the bidding decision, there is no need for the utility company and customers to know the precise value of the supply fluctuations. This is a very appealing property for the system operations (both the utility company and the customers) because it is difficult to estimate the supply deficit precisely due to various uncertainties in power networks, e.g., it may be difficult to estimate or predict the power generation from the solar or wind farm accurately. Moreover, this property also implies that one bidding strategy can be applied by the customers and the utility company over multi time periods instead of just for one event because the same bidding strategy is optimal at each time period.

VII. NUMERICAL STUDIES

In this section, we provide numerical examples to complement the analysis in previous sections. We first show the convergence of the algorithms proposed in Section III-B and Section IV-A, and then we compare the two market equilibria.

A. Distributed algorithms

We first consider a demand response system with 30 customers. We assume that each customer i has a cost function $C_i(q_i) = a_i q_i + h_i q_i^2$ with $a_i \geq 0$ and $h_i > 0$. The electricity supply deficit is normalized to be 10, and a_i and h_i are randomly drawn from $[1, 2]$ and $[0.5, 4.5]$, respectively.

Figure 1 shows the evolution of the price and 5 customers' supply functions with stepsize $\gamma = 0.1$ for optimal supply function bidding and for strategic supply function bidding, respectively. We see that the price and supply functions approach the market equilibria very quickly, within less than 10 iterations. Moreover, the price at the Nash equilibrium is higher than the price at competitive equilibrium, which is consistent with Theorem 5. Compared to the supply bidding value b_i at competitive equilibrium, the customers (e.g. customer 10, 20, 25) who have low bids at the competitive equilibrium tend to bid a higher value at the Nash equilibrium; whereas the customers (e.g. customer 15) who have high bids at the competitive equilibrium tend to bid a lower value at the Nash equilibrium. Loosely speaking, if a customer bids a very low value at the competitive equilibrium, this customer has an incentive to increase the bid at the Nash equilibrium because the price at the Nash equilibrium is higher and this customer might gain more benefit by increase the bid; in contrast, if a customer bids a very high value at the competitive equilibrium, this customer might have incentive to decrease the bid at the Nash equilibrium because this customer might gain more benefit by reducing the load shedding amount but collecting the same amount of payment due to the higher price at the Nash equilibrium.

Lastly, in order to show the scalability of the algorithm, we also simulated a large network with 300 customers. Figure 2 shows the evolution of the price and 5 customers' supply function with stepsize $\gamma = 0.05$ for optimal supply function bidding and for strategic supply function bidding respectively. We see that the algorithm converges very fast.

B. Comparison between the competitive equilibrium and the Nash equilibrium

To amplify the effect of cost functions on the efficiency loss of the Nash equilibrium, we simulate three scenarios: 1)

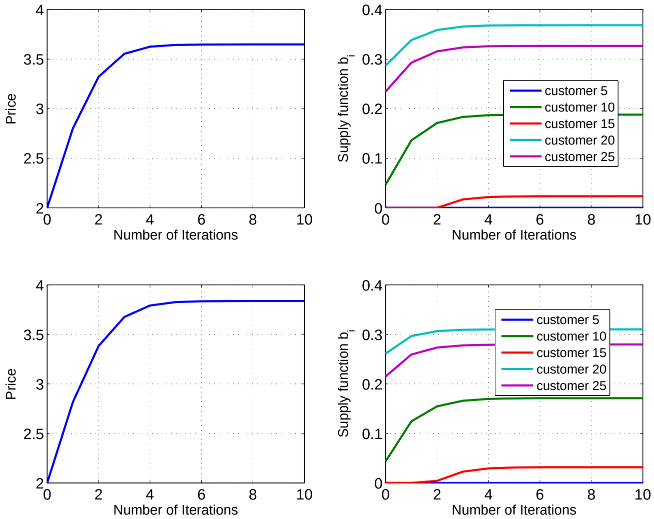

Fig. 1. 30 customers: Price and supply function evolution of optimal supply function bidding (upper panels) and strategic supply function bidding (lower panels) for demand response.

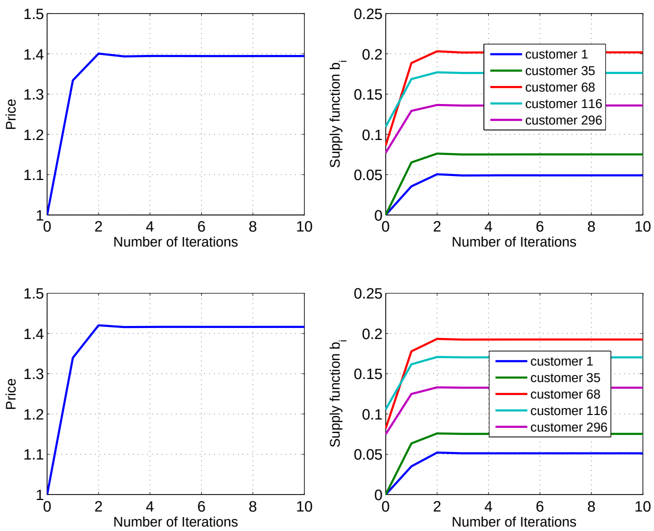

Fig. 2. 300 customers: Price and supply function evolution of optimal supply function bidding (upper panels) and strategic supply function bidding (lower panels) for demand response.

customers are homogeneous (i.e., symmetric), 2) one customer has a extremely low cost function and other customers have the same cost functions, and 3) two customers have extremely low cost functions and other customers have the same cost functions. In all scenarios, each customer has a quadratic cost function in the form of $C_i(q_i) = a_i q_i + h_i q_i^2$. In scenario 1, $a_i = 1$ and $h_i = 2$ for all customer i. In scenario 2, the first customer has $a_1 = 0.1$ and $h_1 = 0.2$ and all other customers have $a_i = 1$, and $h_i = 2$. In scenario 3, the first two customers have $a_1 = 0.1$ and $h_1 = 0.2$ and all other customers have $a_i = 1$, and $h_i = 2$. We simulate different sizes of systems, where $n = 3, 5, \ldots, 99$. Figure 3 plots the ratio of

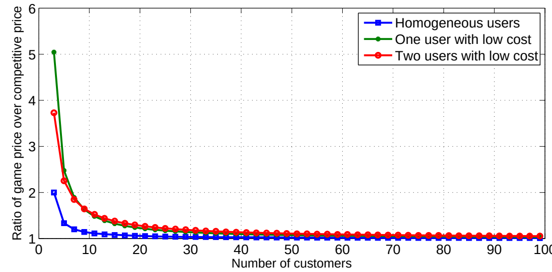

Fig. 3. Price ratio between the Nash equilibrium and the competitive equilibrium under different scenarios with different number of users.

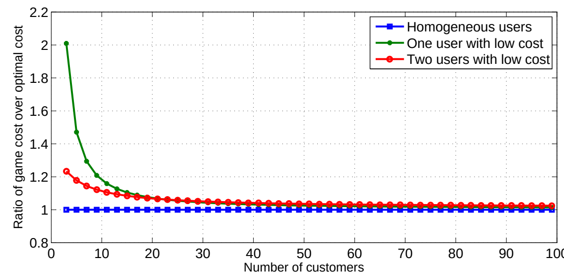

Fig. 4. Cost ratio between the Nash equilibrium and the competitive equilibrium under different scenarios with different number of users.

price at the Nash equilibrium to the price at the competitive equilibrium for the three scenarios and Figure 4 plots the ratio of the total cost at the Nash equilibrium to the total cost at the competitive equilibrium for the three scenarios. Figure 5 and Figure 6 plot the load shed by the low cost user and high cost user respectively at the competitive equilibrium and Nash equilibrium. From the plots, we know that:

1) if all customers are symmetric (scenario 1), the differences between the two market prices are small and the social welfare of the two market equilibria are the same. This is consistent with the theoretical results in Corollary 4.
2) in all the three scenarios, the differences between the two market equilibria decreases quickly as the system size increases. Intuitively speaking, when the system size increases, the market power (a.k.a the ability of affecting the price) of each individual customer decreases and thus the oligopolistic market tends to behavior more similar to the competitive market.
3) when the system size is small, if there is only one customer has extremely low cost function, the differences between the two market equilibria are large. We say this customer has large market power in this case. However, as long as there is one more customer who also has extremely low cost, the differences between the two market equilibrium decreases by a large degree. This is because the market power of the low cost customer is limited by the other low cost customer. This behavior suggests that if the utility company needs to limit market power of certain customers, he can either split the customer into two customers who have relatively the same cost or include another customer into the demand

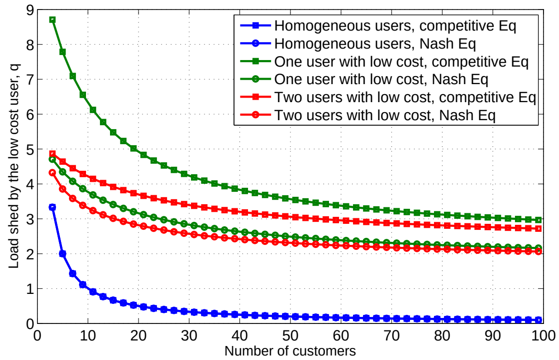

Fig. 5. Load shed by the low cost user under the different scenarios with different number of users.

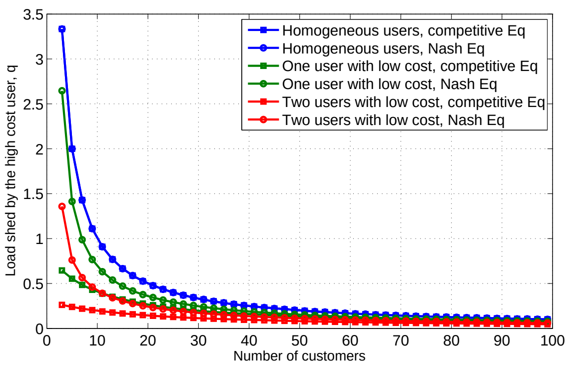

Fig. 6. Load shed by the high cost user under the different scenarios with different number of users.

response who has relatively the same cost.

4) when the system size is large, the differences between the two markets equilibria are small for all the three scenarios. But one interesting phenomena is that scenario 3, where there are two users with low cost functions, has both a larger price ratio and a larger cost ratio than scenario 2. This is opposite to the case where the system size is small (as discussed in the preceding point). One intuitive explanation for this is as follows. When the system size is large, though each individual high cost customer does not shed much load, all the high cost customers together contribute a large amount to the demand response, which thus limit the market power of the low cost customer(s). However, for scenario 3 there are two customers with low cost and for scenario 2 there is only one user with low cost. Therefore, given a fixed large system size, the low cost customers in scenario 3 will have a larger market power than the low cost customer in scenario 2, causing a large price ratio and cost ratio.
5) customers with low cost shed less load at Nash equilibrium than competitive equilibrium, whereas customers with high cost shed more load at Nash equilibrium than competitive equilibrium. This is because at Nash equilibrium customers have market power to raise the price. For the customers with low cost, they gain more net revenue by decreasing the amount of load shedding; whereas for the customers with high cost, they have incentives to shed more load because of the increased price.

VIII. Conclusion

We have studied a market model for demand response in power networks. We characterize the equilibria in competitive and oligopolistic markets, and propose distributed demand response schemes and algorithms to match electricity supply and to shape electricity demand accordingly. We also characterize the efficiency loss of the game-theoretic equilibria.

This paper serves as a starting point for designing practical demand response schemes and algorithms for smart power grids. We will further bring in the power network constraints and detailed dynamics of the demand response appliances. We expect that these new constraints will not change the general structure of our models (in terms of, e.g, equilibrium characterization, and distributed decomposition structure, etc), but they will complicate the analysis and the computation as we come to the scheduling of individual electronic appliances over the power network.

Appendix

Here we prove the existence and uniqueness of the optimal solution of optimization problem (17). Firstly, we pick $\hat{d} < \frac{d}{2}$ such that $|N|\hat{d} > d$ and solve the following optimization problem:

$$\min_{0\le q_i<\hat{d}} \quad \sum_i D_i(q_i) \tag{31a}$$

$$s.t. \quad \sum_i q_i = d, \tag{31b}$$

Denote the optimal value as $D^*_{\hat{d}}$. For each i, find such ϵ_i such that $D_i(q_i) \ge D^*_{\hat{d}}$ for all $q_i \in [\frac{d}{2}-\epsilon_i, \frac{d}{2})$. Such ϵ_i exists because $D_i(q_i)$ is a strictly increasing function and $\lim_{q_i\to\frac{d}{2}} D_i(q_i) = \infty$. Therefore, we know that the optimization problem (17) is equivalent to the following problem:

$$\min_{0\le q_i\le d/2-\epsilon_i} \quad \sum_i D_i(q_i) \tag{32a}$$

$$s.t. \quad \sum_i q_i = d, \tag{32b}$$

which has a unique optimal solution. Therefore we know that the optimal solution of (17) exists. The uniqueness follows from the strictly convexity of $D_i(q_i)$.

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