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On Asymptotic Normality of the Local Polynomial Regression Estimator with Stochastic Bandwidths

CARLOS MARTINS-FILHO[1,2] AND PAULO SARAIVA[1]

[1]Department of Economics, University of Colorado, Boulder, Colorodo, USA
[2]IFPRI, Washington, D.C., USA

Nonparametric density and regression estimators commonly depend on a bandwidth. The asymptotic properties of these estimators have been widely studied when bandwidths are non stochastic. In practice, however, in order to improve finite sample performance of these estimators, bandwidths are selected by data driven methods, such as cross-validation or plug-in procedures. As a result, nonparametric estimators are usually constructed using stochastic bandwidths. In this article, we establish the asymptotic equivalence in probability of local polynomial regression estimators under stochastic and nonstochastic bandwidths. Our result extends previous work by Boente and Fraiman (1995) and Ziegler (2004).

Keywords Asymptotic normality; Local polynomial estimation; Mixing processes; Stochastic bandwidth.

Mathematics Subject Classification 62G05; 62G08; 62G20.

1. Introduction

Currently, there exist several papers that establish the asymptotic properties of kernel based nonparametric estimators. For the case of density estimation, Parzen (1962), Robinson (1983), and Bosq (1998) established the asymptotic normality of Rosenblatt's density estimator under independent and identically distributed (IID) and stationary strong mixing data generating processes. For the case of regression, Fan (1992), Masry and Fan (1997), and Martins-Filho and Yao (2009) established asymptotic normality of local polynomial estimators under IID, stationary and non stationary strong mixing processes. All of these asymptotic approximations are obtained for a sequence of non stochastic bandwidths $0 < h_n \to 0$ as the sample size $n \to \infty$.

In practice, to improve estimators' finite sample performance, bandwidths are normally selected using data-driven methods (Ruppert et al., 1995; Xia and Li, 2002). As such, bandwidths are in practical use generally stochastic. Therefore, it is

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Address correspondence to Carlos Martins-Filho, Department of Economics, University of Colorado, Boulder, CO 80309, USA; E-mail: carlos.martins@colorado.edu

desirable to obtain the aforementioned asymptotic results when h_n is data dependent and consequently stochastic.

There have been previous efforts in establishing asymptotic properties of nonparametric estimators constructed with stochastic bandwidths. Consider, for example, the local polynomial regression estimator proposed by Fan (1992). Dony et al. (2006) proved that such estimator, when constructed with a stochastic bandwidth, is uniformly consistent. More precisely, suppose $\{(Y_t, X_t)\}_{t=1}^n$ is a sequence of random vectors in $\mathbb{R}^2$ with regression function $m(x) = E(Y_t \mid X_t = x)$ for all t. The local polynomial regression estimator of order p is defined by $m_{LP}(x; h_n) \equiv \hat{b}_{n0}(x; h_n)$ where

$$(\hat{b}_{n0}(x; h_n), \ldots, \hat{b}_{np}(x; h_n)) = \underset{b_0,\ldots,b_p}{\operatorname{argmin}} \sum_{t=1}^{n} \left(Y_t - \sum_{j=0}^{p} b_j (X_t - x)^j \right)^2 K\left(\frac{X_t - x}{h_n}\right)$$

and $K : \mathbb{R} \to \mathbb{R}$ is a kernel function. If the sequence $\{(Y_t, X_t)\}_{t=1}^n$ is IID, then it follows from Dony et al. (2006) that

$$\limsup_{n\to\infty} \sup_{h_n \in [a_n, b_n]} \frac{\sqrt{nh_n}\, \sup_{x\in G} |m_{LP}(x; h_n) - m(x)|}{\sqrt{|\log h_n| \vee \log\log n}} = O_{a.s.}(1),$$

where (a_n, b_n) is a nonstochastic sequence such that $0 \le a_n < b_n \to 0$ as $n \to \infty$, G is a compact set in $\mathbb{R}$ and $|\log h_n| \vee \log\log n = \max\{|\log h_n|, \log\log n\}$. If there exists a stochastic bandwidth $\hat{h}_n$ such that $\frac{\hat{h}_n}{h_n} - 1 = o_p(1)$ and we define $a_n = rh_n$ and $b_n = sh_n$ with $0 < r < 1 < s$. Then it follows that

$$\sup_{x\in G} |m_{LP}(x; \hat{h}_n) - m(x)| = o_p(1).$$

When $p = 1$ and the sequence $\{(Y_t, X_t)\}_{t=1}^n$ is IID, if $\hat{h}_n$ is obtained by a cross validation procedure, Li and Racine (2004) showed that

$$\sqrt{n\hat{h}_n} \left(m_{LP}(x; \hat{h}_n) - m(x) - \frac{\hat{h}_n^2}{2} m^{(2)}(x) \int K(u)u^2 du \right) \xrightarrow{d} N\left(0, \frac{\sigma^2(x)}{f_X(x)} \int K^2(u)du\right),$$

where X is a random variable that has the same distribution of X_t, f_X is the density function of X and $\sigma^2(x) = Var(Y_t \mid X_t = x)$. Xia and Li (2002) established that, if $\hat{h}_n$ is obtained through cross validation, $\frac{\hat{h}_n}{h_n} - 1 = o_p(1)$ for strong mixing and strictly stationary sequences $\{(Y_t, X_t)\}_{t=1}^n$.

When $p = 0$, the case of a Nadaraya–Watson regression estimator $m_{NW}(x; h_n)$, and the sequence $\{(Y_t, X_t)\}_{t=1}^n$ is a strictly stationary strong mixing random process, Boente and Fraiman (1995) showed that if $\frac{\hat{h}_n}{h_n} - 1 = o_p(1)$, then

$$\sqrt{n\hat{h}_n} \left(m_{NW}(x; \hat{h}_n) - E\left(m_{NW}(x; \hat{h}_n) \mid \vec{X} \right) \right) \xrightarrow{d} N\left(0, \frac{\sigma^2(x)}{f_X(x)} \int K^2(u)du\right),$$

where $\vec{X}' = (X_1, \ldots, X_n)$. Since independent processes are strong mixing, their result encompasses the case where $\{(Y_t, X_t)\}_{t=1}^n$ is IID, which is treated in the otherwise broader article by Ziegler (2004).

In this article, we expand the result of Boente and Fraiman (1995) by obtaining that local polynomial estimators for the regression and derivatives of orders $j = 1, \ldots, p$ constructed with a stochastic bandwidth $\hat{h}_n$ are asymptotically normal. We do this for processes that are strong mixing and strictly stationary. Our proofs build and expand on those of Boente and Fraiman (1995) and Masry and Fan (1997).

2. Preliminary Results and Assumptions

Define the vector $b_n(x; h) = (\hat{b}_{n0}(x; h), \ldots, \hat{b}_{np}(x; h))'$ and the diagonal matrix $H_n = diag\{h_n^j\}_{j=0}^p$. Given that Masry and Fan (1997) established the asymptotic normality of $\sqrt{nh_n}(H_n b_n(x; h_n) - E(H_n b_n(x; h_n) \mid \vec{X}))$, it suffices for our purpose to show that

$$\sqrt{nh_n}\left(H_n(h_n)b_n(x; h_n) - E(H_n b_n(x; h_n) \mid \vec{X})\right) - \sqrt{n\hat{h}_n}\left(\widehat{H}_n b_n(x; \hat{h}_n) - E(\widehat{H}_n b_n(x; \hat{h}_n) \mid \vec{X})\right) = o_p(1), \quad (1)$$

where $\hat{h}_n$ is a bandwidth that satisfies $\frac{\hat{h}_n}{h_n} - 1 = o_p(1)$ and $\widehat{H}_n = diag\{\hat{h}_n^j\}_{j=0}^p$.

Lemma 2.1 simplifies condition (1) further. It allows us to use a nonstochastic normalization in order to obtain the asymptotic properties of the local polynomial estimator constructed with stochastic bandwidths. Throughout the article, for an arbitrary stochastic vector W_n, all orders in probability are taken element-wise.

Lemma 2.1. *Define* $\Delta_n(h) = Hb_n(x; h) - E(Hb_n(x; h) \mid \vec{X})$. *Suppose that* $\sqrt{nh_n}\left(\Delta_n(h_n) - \Delta_n(\hat{h}_n)\right) = o_p(1)$ *and* $\sqrt{nh_n}\Delta_n(h_n) \xrightarrow{d} W$ *a suitably defined random variable. Then it follows that* $\sqrt{nh_n}\Delta_n(h_n) - \sqrt{n\hat{h}_n}\Delta_n(\hat{h}_n) = o_p(1)$ *provided that* $\frac{\hat{h}_n}{h_n} - 1 = o_p(1)$.

Our subsequent results depend on the following assumptions.

A1. 1. The process $\{(Y_t, X_t)\}_{t=1}^n$ is strictly stationary. 2. For some $\delta > 2$ and $a > 1 - \frac{2}{\delta}$ we assume that $\sum_{l=1}^{\infty} l^a \alpha(l)^{1-\frac{2}{\delta}} < \infty$, where $\alpha(l)$ is a mixing coefficient which is defined below. 3. $\sigma^2(x) \equiv Var(Y_t \mid X_t = x)$ is a continuous and differentiable function at x. 4. The pth-order derivative of the regressions, $m^{(p)}(x)$, exists at x.

The mixing coefficient $\alpha(j)$ is defined as $\alpha(j) \equiv \sup_{A \in \mathcal{F}_{-\infty}^t, B \in \mathcal{F}_{t+j}^{\infty}} |P(A \cap B) - P(A)P(B)|$ where for a sequence of strictly stationary random vectors $\{(Y_t, X_t)\}_{t \in \mathbb{Z}}$ defined on the probability space $(\Lambda, \mathcal{A}, P)$ we define $\mathcal{F}_a^b$ as the σ-algebra induced by $((Y_a, X_a), \ldots, (Y_b, X_b))$ for $a \le b$ (Doukhan, 1994). If $\alpha(j) = O(j^{-a-\epsilon})$ for $a \in \mathbb{R}$ and some $\epsilon > 0$, α is said to be of size $-a$. Condition A1.2 is satisfied by a large class of stochastic processes. In particular, if $\{(Y_t, X_t)\}_{t=1,2,\ldots}$ is α-mixing of size -2, i.e., $\alpha(l) = O(l^{-2-\epsilon})$ for some $\epsilon > 0$ then A1.2 is satisfied. Since Pham and Tran (1985) showed that finite dimensional stable vector ARMA process are α-mixing with $\alpha(l) \to 0$ exponentially as $l \to \infty$, we have that these ARMA processes have size $-a$ for all $a \in \Re^+$, therefore satisfying A1.2.[1]

[1]Linear stochastic processes also satisfy A1.2 under suitable restrictions. See Pham and Tran (1985) Theorem 2.1.

A2. 1. The bandwidth $0 < h_n \to 0$ and $nh_n \to \infty$ as $n \to \infty$. 2. There exists a stochastic bandwidth $\hat{h}_n$ such that $\frac{\hat{h}_n}{h_n} - 1 = o_p(1)$ holds.

A3. 1. The kernel function $K : \mathbb{R} \to \mathbb{R}$ is a bounded density function with support $supp(K) = [-1, 1]$. 2. $u^{2p\delta+2}K(u) \to 0$ as $|u| \to \infty$ for $\delta > 2$. 3. The first derivative of the kernel function, $K^{(1)}$, exists almost everywhere with $K^{(1)}$ uniformly bounded whenever it exists.

A4. The density $f_X(x)$ for X_t is differentiable and satisfies a Lipschitz condition of order 1, i.e., $|f_X(x) - f_X(x')| \le C|x - x'|$, $\forall x, x' \in \mathbb{R}$.

A5. 1. The joint density of (X_t, X_{t+s}), $f_s(u, v)$, is such that $f_s(u, v) \le C$ for all $s \ge 1$ and $u, v \in [x - h_n, x + h_n]$. 2. $|f_s(u, v) - f_X(u)f_X(v)| \le C$ for all $s \ge 1$.

A6. $E(Y_1^2 + Y_l^2 \mid X_1 = u, X_l = v) < \infty$, $\forall l \ge 1$ and $E(|Y_t|^\delta \mid X_t = u) < \infty$, $\forall t$, for all $u, v \in [x - h_n, x + h_n]$ and some $\delta > 2$.

A7. There exists a sequence of natural numbers satisfying $s_n \to \infty$ as $n \to \infty$ such that $s_n = o(\sqrt{nh_n})$ and $\alpha(s_n) = o\left(\sqrt{\frac{h_n}{n}}\right)$.

Assumption A7 places a restriction on the speed at which the mixing coefficient decays to zero relative to h_n. Specifically, since the distributional convergence in Eq. (5) below is established using the large-block/small-block method in Bernstein (1927), the speed at which the small-block size evolves as $n \to \infty$ is related to speed of decay for α. In fact, as observed by Masry and Fan (1997), if $h_n \sim n^{-1/5}$ and $s_n = (nh_n)^{1/2}/logn$ it suffices to have $n^g\alpha(n) = O(1)$ for $g > 3$ to satisfy A7 (and A1.2).

A8. The conditional distribution of Y given $X = u$, $f_{Y\mid X=u}(y)$ is continuous at the point $u = x$.

Let

$$s_{n,l}(x; h_n) = \frac{1}{nh_n}\sum_{t=1}^{n} K\left(\frac{X_t - x}{h_n}\right)\left(\frac{X_t - x}{h_n}\right)^l, \tag{2}$$

$$g_{n,l}(x; h_n) = \frac{1}{nh_n}\sum_{t=1}^{n} K\left(\frac{X_t - x}{h_n}\right)\left(\frac{X_t - x}{h_n}\right)^l Y_t \quad \text{and} \tag{3}$$

$$g^*_{n,l}(x; h_n) = \frac{1}{nh_n}\sum_{t=1}^{n} K\left(\frac{X_t - x}{h_n}\right)\left(\frac{X_t - x}{h_n}\right)^l (Y_t - m(X_t)) \quad \text{for } l = 1, \dots, 2p. \tag{4}$$

Then $b_n(x; h_n) = H_n^{-1}S_n^{-1}(x; h_n)G_n(x; h_n)$ where $S_n(x; h_n) = \{s_{n,i+j-2}(x; h_n)\}_{i,j=1}^{p+1,p+1}$ and $G_n(x; h_n) = \{g_{n,l}(x; h_n)\}_{l=0}^{p}$. Masry and Fan (1997) showed that under assumption A1–A8

$$\sqrt{nh_n}(H_n b_n(x; h_n) - E(S_n(x; h_n)^{-1}G_n(x; h_n) \mid \vec{X})) \xrightarrow{d} N\left(0,\ \frac{\sigma^2(x)}{f_X(x)}S^{-1}\widetilde{S}S^{-1}\right), \tag{5}$$

where $S = \{\mu_{i+j-2}\}_{i,j=1}^{p+1,p+1}$, $\widetilde{S} = \{v_{i+j-2}\}_{i,j=1}^{p+1,p+1}$ with $\mu_l = \int \psi^l K(\psi)d\psi$ and $v_l = \int \psi^l K^2(\psi)d\psi$.

Equation (5) gives us $\sqrt{nh_n}\Delta_n(h_n) \xrightarrow{d} W$ in Lemma 2.1. In particular, $W \sim N(0,\ f_X^{-1}(x)\sigma^2(x)S^{-1}\widetilde{S}S^{-1})$. Consequently, it suffices to show that

$$\sqrt{nh_n}\Delta_n(h_n) - \sqrt{nh_n}\Delta_n(\hat{h}_n) = o_p(1). \tag{6}$$

As will be seen in Theorem 3.1, the key to establish (6) resides in obtaining asymptotic uniform stochastic equicontinuity of $\sqrt{nh_n}\Delta_n(x;\tau h_n)$ with respect to τ. To this end, we establish the following auxiliary lemmas.

Lemma 2.2. *Let* $Z_n(x;l,\tau) = \left|\frac{d}{d\tau}s_{n,l}(x;\tau h_n)\right|$, *for some* τ *finite and* $l = 0,\ldots,2p$. *If* A1.1, A2.1, A3.1, A3.3, *and* A4 *hold, then* $\sup_{\tau\in[r,s]} Z_n(x;l,\tau) = O_p(1)$ *where* $r, s > 0$ *and* $r < s$.

Lemma 2.3. *Let* $B_n(x;l,\tau) = \sqrt{nh_n}\frac{d}{d\tau}g^*_{n,l}(x;\tau h_n)$, *for* $l = 0,\ldots,2p$. *If* A1–A6 *hold, then* $\int_r^s B_n^2(x,l,\tau)d\tau = O_p(1)$ *where* $r, s > 0$ *and* $r < s$.

3. Main Results

The following theorem and corollary establish $\sqrt{n\hat{h}_n}$-normality of the local polynomial estimator constructed with stochastic bandwidths. As in Masry and Fan (1997), we are able to obtain asymptotic normality for the regression estimator as well as for the estimators of the regression derivatives.

Theorem 3.1. *Suppose* A1–A8 *hold, then it follows that*

$$\sqrt{n\hat{h}_n}\Delta_n(\hat{h}_n) \xrightarrow{d} N\left(0, \frac{\sigma^2(x)}{f_X(x)}S^{-1}\widetilde{S}S^{-1}\right).$$

With the following corollary we also obtain the asymptotic bias for local polynomial estimators with stochastic bandwidths.

Corollary 3.1. *Let* $m^{(j)}$ *denote the jth-order derivative of m. Suppose* A1–A8 *hold, then*

$$\sqrt{n\hat{h}_n}\left(\widehat{H}_n(b_n(x;\hat{h}_n) - b(x)) - \frac{\hat{h}_n^{p+1}m^{(p+1)}(x)}{(p+1)!} + \hat{h}_n^{p+1}o_p(1)\right) \xrightarrow{d} N\left(0, \frac{\sigma^2(x)}{f_X(x)}S^{-1}\widetilde{S}S^{-1}\right),$$

where $\widehat{H}_n = diag\{\hat{h}_n^j\}_{j=0}^p$ *and* $b(x) = \left(m(x), m^{(1)}(x), \ldots, \frac{1}{p!}m^{(p)}(x)\right)'$.

4. Monte Carlo Study

In this section, we investigate some of the finite sample properties of the local linear regression and derivative estimators constructed with bandwidths selected by cross validation and a plug in method proposed by Ruppert et al. (1995) for data generating processes (DGP) exhibiting dependence. In our simulations two regression functions are considered, $m_1(x) = \sin(x)$ and $m_2(x) = 3(x-0.5)^3 + 0.25x_0 + 1.125$ with first derivatives given, respectively, by $m_1^{(1)}(x) = -\cos(x)$ and $m_2^{(1)}(x) = 9(x-0.5)^2 + 0.25$.

We generate $\{\epsilon_t\}_{t=1}^n$ by $\epsilon_t = \rho\epsilon_{t-1} + \sigma U_t$, where $\{U_t\}_{t\geq 1}$ is a sequence of IID standard normal random variable and $(\rho, \sigma^2) = (0, 0.01), (0.5, 0.0075)$, and $(0.9, 0.0019)$. This implies that for $\rho \neq 0$ $\{\epsilon_t\}_{t=1}^n$ is a normally distributed AR(1) process with mean zero and variance equal to 0.01.

For m_1 we draw IID regressors $\{X_t\}_{t=1}^n$ from a uniform distribution that takes value on $[0, 2\pi]$. For m_2 we draw IID regressors $\{X_t\}_{t=1}^n$ from a beta distribution with parameters $\alpha = 2$ and $\beta = 2$ given by

$$f_X(x; \alpha, \beta) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du} & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

The regressands are constructed using $Y_t = m_i(X_t) + \epsilon_t$, where $i = 1, 2$.

Two sample sizes are considered $n = 200, 600$, and 1,000 repetitions are performed. We evaluate the regression and regression derivative estimators at $x = 0.5\pi, \pi, 1.5\pi$, and $x = 0.25$, 0.5, 0.75 for m_1 and m_2, respectively. These estimators are constructed with a nonstochastic optimal regression bandwidth h_{AMISE} and with two data dependent bandwidths: a cross-validated bandwidth h_{CV} and a plug in bandwidth h_{ROT}. The nonstochastic bandwidth is given by $h_{AMISE} = \left(\frac{\lambda_1}{n\lambda_2}\right)^{1/5}$, where $\lambda_1 = Var(\epsilon_t) \int K^2(u)du \int \mathbf{1}(f_X(x) \neq 0)dx$ and $\lambda_2 = \int u^2K(u)du \int (m^{(2)}(x))^2 f_X(x)dx$ (Ruppert et al., 1995). The cross-validated bandwidth is given by $h_{CV} = \text{argmin}_h \sum_{t=1}^n (m_{LP,t}(X_t; h) - Y_t)^2$, where $m_{LP,t}(x; h)$ is the local linear regression estimator constructed with the exclusion of observation t (Xia and Li, 2002). The h_{ROT} bandwidth is calculated as described in Ruppert et al. (1995). Specifically, we estimate $Var(\epsilon_t)$, $\int \mathbf{1}(f_X(x) \neq 0)dx$ and $\int (m^{(2)}(x))^2 f_X(x)dx$ which appear the expression for h_{AMISE}. First, we approximate $m(x)$ by $m(x) \approx \beta_0 + \beta_1 x + \beta_2 x^2/2 + \beta_3 x^3/3! + \beta_4 x^4/4!$ and obtain $m^{(2)}(x) \approx \beta_2 + \beta_3 x + \beta_4 x^2/2$. Second, the vector $(\beta_0, \ldots, \beta_4)'$ is estimated by $(\hat{\beta}_0, \ldots, \hat{\beta}_4)' = (\sum_{i=1}^n R_{i\cdot}'R_{i\cdot})^{-1} \sum_{i=1}^n R_{i\cdot}'Y_i$, where $R_{i\cdot} = (1 \;\; X_i \;\; X_i^2/2 \cdots X_i^4/4!)$. Third, we estimate $Var(\epsilon_t)$ and $\int (m^{(2)}(x))^2 f_X(x)dx$ by $n^{-1}\sum_{i=1}^n \tilde{e}_i^2$ and $n^{-1}\sum_{i=1}^n [\hat{\beta}_2 + \hat{\beta}_3 X_i + \hat{\beta}_4 X_i^4/2]^2$ respectively, where $\tilde{e}_i = y_i - R_{i\cdot}(\hat{\beta}_0, \ldots, \hat{\beta}_4)'$. The estimator used for $\int \mathbf{1}(f_X(x) \neq 0)dx$ is given by $\max_i X_i - \min_i X_i$.

The results of our simulations are summarized in Tables 1–2 and Figs. 1–2. Tables 1 and 2 provide the bias ratio and mean squared error (MSE) ratio of estimators constructed with h_{CV}, h_{ROT}, and h_{AMISE} for m_1 and m_2, respectively. These ratios are constructed with estimators using the data dependent bandwidth h_{CV} or h_{ROT} in the numerator and h_{AMISE} in the denominator. Figure 1 shows the estimated densities of the difference between the estimated regression constructed with h_{CV} and h_{AMISE} (panels (a) and (c)) and h_{ROT} and h_{AMISE} (panels (b) and (d)), for $m_1(\pi)$ and $m_2(0.5)$, with $n = 200$ and $\rho = 0, 0.9$. Similarly, Fig. 2 shows the estimated density of the difference between the estimated regression first derivative constructed with h_{CV} and h_{AMISE} (panels (a) and (c)) and h_{ROT} and h_{AMISE} (panels (b) and (d)) for $m_1^{(1)}(\pi)$ and $m_2^{(1)}(0, 5)$, with $n = 200$ and $\rho = 0, 0.9$.

As expected from the asymptotic results, the bias and MSE ratios are in general close to 1, especially for the regression estimators. Ratios that are farther form 1 are more common in the estimation of the regression derivatives. This is consistent with the asymptotic results since the rate of convergence of the regression estimator is $\sqrt{nh_n}$, whereas regression first derivative estimators have rate of convergence $\sqrt{nh_n^3}$.

Hence, for fixed sample sizes we expect regression estimators to outperform those associated with derivatives.

Note that most bias and MSE ratios given in Tables 1 and 2 are positive values larger than 1. Since we constructed both bias and MSE ratios with estimators constructed with h_{CV} or h_{ROT} in the numerator and estimators constructed with h_{AMISE} in the denominator, the results indicate that bias and MSE are larger for estimators constructed with h_{CV} and h_{ROT}. This too was expected, since h_{AMISE} is the true optimal bandwidth for the regression estimator. Positive bias ratios indicate that the direction of the bias is the same for estimators constructed with h_{AMISE} and h_{CV} or h_{ROT}. It is also important to note that the bias and MSE for estimators of both regression and derivatives are generally larger when calculated using h_{CV} compared to the case when h_{ROT} is used.

We note that in general the estimators for the function m_1 outperformed those for function m_2. We observe that m_2 takes value on [0.75, 1.75] and ϵ_t on $\mathbb{R}$. Thus, although the variance of ϵ_t was chosen to be small, 0.01, estimating the bandwidth was made difficult due to the fact that ϵ_t had a large impact on Y_t in terms of its relative magnitude. The regression function m_1 also took values on a bounded interval, however this interval had a larger range. In fact the standard deviation of h_{CV} for $n = 200$ and $\rho = 0.5$ was 0.0422 and 11.552 for the DGP's associated with m_1 and m_2, respectively.

The kernel density estimates shown in Figs. 1 and 2 were calculated using the Gaussian kernel and bandwidths were selected using the *rule-of-thumb* procedure of Silverman (1986). We observe that the change from IID ($\rho = 0$) to dependent DGP ($\rho \neq 0$) did not yield significantly different results in terms of estimator performance under h_{CV} or h_{ROT}. In fact, our results seem to indicate that for $\rho = 0.9$ the estimators had slightly better general performance than for the case where $\rho = 0$. As expected from our asymptotic results, Figs. 1 and 2 show that the difference between derivative estimates using h_{CV} and h_{AMISE} and h_{ROT} and h_{AMISE} were more disperse around zero than those associated with regression estimates, especially for the DGP using m_2. Even though the DGP for m_1 provided better results, the estimators of m_2 and $m_2^{(1)}$, as seen in Figs. 1 panels (c) and (d) and 2 panels (c) and (d) performed well, in the sense that such estimators produced estimated densities with fairly small dispersion around zero. Another noticeable result from the Monte Carlo is that the estimated densities associated with estimators calculated using h_{ROT} are much less dispersed than those calculated using h_{CV}. Overall, as expected from asymptotic theory, estimators calculated with h_{CV} and h_{ROT} performed fairly well in small samples, however our results seem to indicate better performance when a plug in bandwidth is used.

5. Final Remarks

We have established the asymptotic properties of the local polynomial regression estimator constructed with stochastic bandwidths. Our results validate the use of the normal distribution in the implementation of hypotheses tests and interval estimation when bandwidths are data dependent. Most assumptions that we have imposed, were also explored by Masry and Fan (1997). The assumptions we place on $\hat{h}_n$ coincides with the properties of the bandwidths proposed by Ruppert et al. (1995) and Xia and Li (2002) under IID and strong mixing, respectively.

Appendix 1: Proofs

Proof of Lemma 2.1. Since $\sqrt{nh_n}(\Delta_n(h_n) - \Delta_n(\hat{h}_n)) = o_p(1)$, we have that

$$\sqrt{nh_n}\Delta_n(h_n) - \sqrt{n\hat{h}_n}\Delta_n(\hat{h}_n) = o_p(1) + \sqrt{nh_n}\Delta_n(\hat{h}_n) - \sqrt{n\hat{h}_n}\Delta_n(\hat{h}_n).$$

$\sqrt{nh_n}\Delta_n(h_n) \xrightarrow{d} W$ and $\sqrt{nh_n}\Delta_n(h_n) - \sqrt{nh_n}\Delta_n(\hat{h}_n) = o_p(1)$ imply that $\Delta_n(\hat{h}_n) = O_p\left((nh_n)^{-1/2}\right)$. Consequently,

$$\sqrt{nh_n}\Delta_n(\hat{h}_n) - \sqrt{n\hat{h}_n}\Delta_n(\hat{h}_n) = \left(1 - \sqrt{\frac{\hat{h}_n}{h_n}}\right) O_p(1) = o_p(1)$$

since $\left(1 - \sqrt{\frac{\hat{h}_n}{h_n}}\right) = o_p(1)$. □

Proof of Lemma 2.2. For any $\epsilon > 0$, we must find $M_\epsilon < \infty$ such that

$$P\left(\sup_{\tau\in[r,s]} Z_n(x; l, \tau) > M_\epsilon\right) \leq \epsilon. \tag{7}$$

By Markov's inequality, we have that

$$P\left(\sup_{\tau\in[r,s]} Z_n(x; l, \tau) > \frac{1}{\epsilon}\right) \leq E\left(\sup_{\tau\in[r,s]} Z_n(x; l, \tau)\right)\epsilon. \tag{8}$$

Thus, it suffices to show that $E\left(\sup_{\tau\in[r,s]} Z_n(x; l, \tau)\right) = O(1)$. Let $\widetilde{K}_l(x) = K(x)x^l(1 + l) + K^{(1)}(x)x^{l+1}$ and write

$$Z_n(x; l, \tau) = \frac{1}{nh_n\tau^2}\left|\sum_{t=1}^{n}\widetilde{K}_l\left(\frac{X_t - x}{h_n\tau}\right)\right|. \tag{9}$$

By strict stationarity, we write

$$E\left(\sup_{\tau\in[r,s]} Z_n(x; l, \tau)\right) \leq \frac{1}{h_n r^2} E\left(\sup_{\tau\in[r,s]}\left|\widetilde{K}_l\left(\frac{X_t - x}{h_n\tau}\right)\right|\right). \tag{10}$$

Now, note that

$$\begin{aligned}
&E\left(\sup_{\tau\in[r,s]}\left|\widetilde{K}_l\left(\frac{X_t - x}{h_n\tau}\right)\right|\right)\\
&= h_n \int \sup_{\tau\in[r,s]} |\tau\widetilde{K}_l(\phi)||f_X(x + h_n\tau\phi) - f_X(x) + f_X(x)|d\phi\\
&\leq h_n^2 C \int |\widetilde{K}_l(\phi)\phi|d\phi \sup_{\tau\in[r,s]} \tau^2 + h_n f_X(x) \int |\widetilde{K}_l(\phi)|d\phi \sup_{\tau\in[r,s]} \tau\\
&\leq h_n^2 s^2 C \int |\widetilde{K}_l(\phi)\phi|d\phi \sup_{\tau\in[r,s]} + h_n f_X(x) s \int |\widetilde{K}_l(\phi)|d\phi
\end{aligned}$$

$$
\begin{aligned}
&\leq h_n^2 s^2 C \int_{-1}^{1} (|1+l||K(\phi)||\phi^{l+1}| + |K^{(1)}(\phi)||\phi^{l+2}|) d\phi \\
&\quad + h_n f_X(x) s \int_{-1}^{1} (|1+l||K(\phi)||\phi^{l}| + |K^{(1)}(\phi)||\phi^{l+1}|) d\phi \\
&\leq (h_n^2 s^2 C + s h_n f_X(x)) \int_{-1}^{1} (|1+l||K(\phi)| + |K^{(1)}(\phi)|) d\phi. \qquad (11)
\end{aligned}
$$

Hence,

$$
E\left(\sup_{\tau\in[r,s]} Z_n(x; l\tau)\right) \leq \frac{1}{r^2}(h_n C + f_X(x) C) \qquad (12)
$$

$$
= O(1) \qquad (13)
$$

as $h_n \to 0$ and $n \to \infty$. □

Proof of Lemma 2.3. Using Markov's inequality it suffices to establish that

$$
E\left(\int_r^s B_n^2(x; l, \tau) d\tau\right) = \int_r^s E\left(B_n^2(x; l, \tau)\right) d\tau = O(1). \qquad (14)
$$

Note that

$$
B_n^2(x; l, \tau) = \frac{1}{n h_n \tau^4}\left\{\sum_{t=1}^{n} \widetilde{K}_l^2\left(\frac{X_t - x}{h_n \tau}\right)\epsilon_t^2 + 2\sum_{t=1}^{n}\sum_{i\neq t} \widetilde{K}_l\left(\frac{X_t - x}{h_n\tau}\right)\epsilon_t \widetilde{K}_l\left(\frac{X_i - x}{h_n\tau}\right)\epsilon_i\right\}, \qquad (15)
$$

where $\epsilon_t = Y_t - m(X_t)$. Thus, by the law of iterated expectations and strict stationarity, we obtain

$$
\begin{aligned}
E\left(B_n^2(x; l, \tau)\right) &= \Bigg| E\left(\frac{1}{h_n\tau^4}\widetilde{K}_l^2\left(\frac{X_t - x}{h_n\tau}\right)\sigma^2(X_t)\right) \\
&\qquad + 2\frac{1}{h_n\tau^4}\sum_{t=2}^{n}\left(1-\frac{t}{n}\right) E\left(\widetilde{K}_l\left(\frac{X_1 - x}{h_n\tau}\right)\widetilde{K}_l\left(\frac{X_t - x}{h_n\tau}\right)\epsilon_1\epsilon_t\right)\Bigg| \\
&\leq E\left(\frac{1}{h_n\tau^4}\widetilde{K}_l^2\left(\frac{X_t - x}{h_n\tau}\right)\sigma^2(X_t)\right) \\
&\qquad + 2\frac{1}{h_n\tau^4}\sum_{t=2}^{n}\left(1-\frac{t}{n}\right)\left| E\left(\widetilde{K}_l\left(\frac{X_1 - x}{h_n\tau}\right)\widetilde{K}_l\left(\frac{X_t - x}{h_n\tau}\right)\epsilon_1\epsilon_t\right)\right|
\end{aligned} \qquad (16)
$$

Notice that

$$
\begin{aligned}
E\left(\frac{1}{h_n\tau^3}\widetilde{K}_l^2\left(\frac{X_t - x}{h_n\tau}\right)\sigma^2(X_t)\right) &= \int \frac{1}{\tau^3}\widetilde{K}_l^2(\phi)\sigma^2(x + h_n\tau\phi) f_X(x + h_n\tau\phi) d\phi \\
&= \int \tau^{-3}\widetilde{K}_l^2(\phi)\left\{\sigma^2(x) f_X(x) + \frac{dw(x^*)}{dx} h_n\tau\phi\right\} d\phi
\end{aligned}
$$

$$\leq \sigma^2(x) f_X(x)\tau^{-3}\int \widetilde{K}_l^2(\phi)d\phi + \tau^{-2}O(h_n) = O(1) \tag{17}$$

where $w(x) = f_X(x)\sigma^2(x)$. Let $\xi_t = \widetilde{K}_l(\frac{X_t - x}{h_n \tau})$, and without loss of generality take $s \geq 1$.[2] Then,

$$\begin{aligned}
&|E(\xi_1\xi_t\epsilon_1\epsilon_t)|\\
&\quad = |E\left(E\left(\epsilon_1\epsilon_t|X_1, X_t\right)\xi_1\xi_t\right)|\\
&\quad \leq E\left(\sup_{X_1, X_t\in[x-sh_n, x+sh_n]} E(|\epsilon_1\epsilon_t||X_1, X_t)|\xi_1\xi_t|\right)\\
&\quad \leq E\left(\sup_{X_1, X_t\in[x-sh_n, x+sh_n]} E\left((|Y_1| + B)(|Y_t| + B)|X_1, X_t\right)|\xi_1\xi_t|\right)\\
&\quad \leq E\left(\sup_{X_1, X_t\in[x-sh_n, x+sh_n]} \left\{E\left((|Y_1| + B)^2|X_1, X_t\right)E\left((|Y_t| + B)^2|X_1, X_t\right)\right\}^{\frac{1}{2}}|\xi_1\xi_t|\right)\\
&\quad \leq CE(|\xi_1\xi_t|)\\
&\quad = C\iint\left|\widetilde{K}_j\left(\frac{u-x}{\tau h_n}\right)\widetilde{K}_j\left(\frac{v-x}{\tau h_n}\right)\right| f_t(u, v)dudv\\
&\quad \leq C\iint\left|\widetilde{K}_j\left(\frac{u-x}{\tau h_n}\right)\widetilde{K}_j\left(\frac{v-x}{\tau h_n}\right)\right| dudv\\
&\quad \leq h_n^2\tau^2 C\left(\int\left|\widetilde{K}_l(\phi)\right|d\phi\right)^2,
\end{aligned} \tag{18}$$

where $B = \sup_{X\in[x-sh_n, x+sh_n]}|m(X)|$. Let $\{d_n\}_{n\geq 1}$ be a sequence of positive integers, such that $d_n \to \infty$ as $n \to \infty$. Then we can write

$$\sum_{t=2}^{n}|E\left(\xi_1\xi_t\epsilon_1\epsilon_t\right)| = \sum_{t=2}^{d_n+1}|E\left(\xi_1\xi_t\epsilon_1\epsilon_t\right)| + \sum_{t=d_n+2}^{n}|E\left(\xi_1\xi_t\epsilon_1\epsilon_t\right)| \tag{19}$$

and note that

$$\begin{aligned}
\sum_{t=2}^{d_n+1}|E\left(\xi_1\xi_t\epsilon_1\epsilon_t\right)| &\leq \sum_{t=2}^{d_n+1}\tau^2 h_n^2 C\left(\int\left|\widetilde{K}_l(\phi)\right|d\phi\right)^2\\
&= d_n h_n^2\tau^2 C.
\end{aligned} \tag{20}$$

Then using the fact that $E(\xi_t\epsilon_t) = 0$ and Davydov's Inequality we obtain,

$$|E\left(\xi_1\xi_{t+1}\epsilon_1\epsilon_{t+1}\right)| \leq 8[\alpha(t)]^{1-2/\delta}\left(E|\xi_1\epsilon_1|^{\delta}\right)^{2/\delta}. \tag{21}$$

[2]Let $s* = \max\{1, s\}$ and note that this proof follows with $s = s^*$.

Note also that

$$\begin{aligned}
E\,|\xi_1\epsilon_1|^{\delta} &= E\left|(Y_1 - m(X_1))\widetilde{K}_l\left(\frac{X_1 - x}{h_n\tau}\right)\right|^{\delta} \\
&\le E\left(\sup_{X_1\in[x-sh_n,x+sh_n]} E\{(|Y_1| + B)^{\delta}|X_1\}\left|\widetilde{K}_l\left(\frac{X_1 - x}{h_n\tau}\right)\right|^{\delta}\right) \\
&\le CE\left(\left|\widetilde{K}_l\left(\frac{X_1 - x}{h_n\tau}\right)\right|^{\delta}\right) = C\int\left|\widetilde{K}_l\left(\frac{u - x}{h_n\tau}\right)\right|^{\delta} f_X(u)du \\
&= C\tau h_n\int|\widetilde{K}_l(v)|^{\delta} f_X(v + \tau h_n x)dv \le C\tau h_n \qquad (22)
\end{aligned}$$

which leads to

$$\begin{aligned}
\sum_{t=d_n+2}^{n}|E(\xi_1\epsilon_1\xi_{t+1}\epsilon_{t+1})| &\le \sum_{t=d_n+2}^{n} 8\alpha(t)^{1-2/\delta}\left(E|\xi_1\epsilon_1|^{\delta}\right)^{2/\delta} \\
&\le h_n^{2/\delta}\tau^{2/\delta}C\sum_{t=d_n+2}^{n}\alpha(t)^{1-2/\delta} \\
&\le h_n^{2/\delta}\tau^{2/\delta}C\sum_{t=d_n+2}^{n}\frac{t^a}{d_n^{1-2/\delta}}\alpha(t)^{1-2/\delta} \\
&= Ch_n^{2/\delta}d_n^{-1+2/\delta}\tau^{2/\delta}\sum_{t=d_n+2}^{n}t^a\alpha(t)^{1-2/\delta} \\
&= Ch_n^{2/\delta}d_n^{-1+2/\delta}\tau^{2/\delta}o(1) = C\tau^{2/\delta}o(h_n) \qquad (23)
\end{aligned}$$

given that d_n is chosen as the integer part of h_n^{-1} and $a > 1 - \frac{2}{\delta}$.

Consequently,

$$E(B_n^2(x; l, \tau)) = \left(\tau^{-3}O(1) + \tau^{-2}O(h_n)\right) + \left(\tau^{-2}O(1) + \tau^{-4+2/\delta}o(1)\right). \qquad (24)$$

Proof of Theorem 3.1. From Masry and Fan (1997) and Lemma 2.1, it suffices to show that

$$\sqrt{nh_n}\Delta_n(h_n) - \sqrt{nh_n}\Delta_n(\hat{\tau}_n h_n) = o_p(1),$$

where $\hat{\tau}_n = \frac{\hat{h}_n}{h_n}$. It suffices to show that all elements of the vector $\sqrt{nh_n}\Delta_n(\tau h_n)$ are stochastically equicontinuous on τ.

For any $\epsilon > 0$, given that $\hat{\tau}_n = O_p(1)$ there exists $r, s \in (0, \infty)$ with $r < s$ such that $P(\hat{\tau}_n \notin [r, s]) \le \epsilon/3$, $\forall n$. For $\delta > 0$, let $w_n(i, \delta) = \sup_{\{(\tau_1,\tau_2)\in[r,s]\times[r,s]:|\tau_1-\tau_2|<\delta\}} d_n(i, \tau_1, \tau_2)$ where $d_n(x; i, \tau_1, \tau_2) = |e_i\Delta_n(\tau_2 h_n) - e_i\Delta_n(\tau_1 h_n)|$, e_i is a row vector with ith component equal to 1, and 0 elsewhere. Then, for $\eta > 0$

$$P(d_n(i, 1, \hat{\tau}_n) \ge \eta) \le P(\mathbf{1}(|\hat{\tau}_n - 1| \le \delta)d_n(i, 1, \hat{\tau}_n) \ge \eta) + P(|\hat{\tau}_n - 1| > \delta),$$

where $1(A)$ is the indicator function for the set A.

By assumption, there exists $N_{\epsilon,1}$ such that $P(|\hat{\tau}_n - 1| > \delta) \leq \frac{\epsilon}{3}$, $\forall\, n \geq N_{\varepsilon,1}$. Also,

$$\begin{aligned} &P(\mathbf{1}(|\tau_n - 1| \leq \delta) d_n(i, 1, \hat{\tau}_n) \geq \eta) \\ &\quad \leq P(\mathbf{1}(\hat{\tau}_n \in [1-\delta, 1+\delta] \cap [r, s]) d_n(i, 1, \hat{\tau}_n) \geq \eta) + P(\hat{\tau}_n \notin [r, s]) \\ &\quad \leq P\left(w_n(i, \delta) \geq \eta\right) + P(\hat{\tau}_n \notin [r, s]), \end{aligned} \tag{25}$$

where, as mentioned before $P(\hat{\tau}_n \notin [r, s]) \leq \frac{\epsilon}{3}$. Furthermore, if $\sqrt{nh_n} e_i \Delta_n(\tau h_n)$ is asymptotically stochastically uniformly equicontinuous with respect to τ on $[r, s]$, then there exists $N_{\epsilon,2}$ such that

$$P\left(w_n(i, \delta) \geq \eta\right) \leq \frac{\epsilon}{3}$$

whenever $n \geq N_{\epsilon,2}$. Setting $N_\epsilon = \max\{N_{\epsilon,1}, N_{\epsilon,2}\}$ we obtain that with stochastic equicontinuity we have $\sqrt{nh_n}\Delta_n(h_n) - \sqrt{nh_n}\Delta_n(\hat{\tau}_n h_n) = o_p(1)$. Now, since τ_1, τ_2, r, and s are nonstochastic, then

$$\begin{aligned} w_n(i, \delta) = \sqrt{nh_n} \sup_{\{(\tau_1,\tau_2)\in[r,s]\times[r,s]:|\tau_1-\tau_2|<\delta\}} &|e_i S_n(x; \tau_1 h_n)^{-1} G_n^*(x; \tau_1 h_n) \\ &- e_i S_n(x; \tau_2 h_n)^{-1} G_n^*(x; \tau_2 h_n)|, \end{aligned}$$

where $G_n^*(x; h_n) = (g_{n,0}^*(x; h_n), \ldots, g_{n,p}^*(x; h_n))'$. Thus, if

$$\sup_{\{(\tau_1,\tau_2)\in[r,s]\times[r,s]:|\tau_1-\tau_2|<\delta\}} |s_{n,l}(x; \tau_1 h_n) - s_{n,l}(x; \tau_2 h_n)| = o_p(1) \tag{26}$$

and

$$\sup_{\{(\tau_1,\tau_2)\in[r,s]\times[r,s]:|\tau_1-\tau_2|<\delta\}} |\sqrt{nh_n} g_{n,l}^*(x; \tau_1 h_n) - \sqrt{nh_n} g_{n,l}^*(x; \tau_2 h_n)| = o_p(1) \tag{27}$$

the desired result is obtained.

By the Mean Value Theorem of Jennrich (1969),

$$\sup_{\{(\tau_1,\tau_2)\in[r,s]^2:|\tau_1-\tau_2|<\delta\}} |s_{n,l}(x; \tau_1 h_n) - s_{n,l}(x; \tau_2 h_n)| \leq \sup_{\tau\in[r,s]} \left| \frac{ds_{nl}(x; \tau h_n)}{d\tau} \right| \delta \quad a.s.$$

Lemma 2.2 and Theorem 21.10 in Davidson (1994) imply that Eq. (26) holds.

Furthermore, by the Mean Value Theorem of Jennrich (1969) and by Cauchy–Schwarz Inequality, we have that

$$\begin{aligned} &|\sqrt{nh_n} g_{n,l}^*(x; \tau_1 h_n) - \sqrt{nh_n} g_{n,l}^*(x; \tau_2 h_n)| \\ &\quad = \left| \int_{\tau_2}^{\tau_1} \left(\sqrt{nh_n} \frac{dg_{nl}^*(x; \tau h_n)}{d\tau} \right) d\tau \right| \\ &\quad \leq \int_{\tau_2}^{\tau_1} \left| \sqrt{nh_n} \frac{dg_{nl}^*(x; \tau h_n)}{d\tau} \right| d\tau \\ &\quad \leq \left| \int_{\tau_2}^{\tau_1} 1 d\tau \right|^{1/2} \left(\int_{\tau_2}^{\tau_1} \left(\sqrt{nh_n} \frac{dg_{nl}^*(x; \tau h_n)}{d\tau} \right)^2 d\tau \right)^{1/2} \end{aligned}$$

$$= |\tau_1 - \tau_2|^{1/2} \left(\int_{\tau_2}^{\tau_1} \left(\sqrt{nh_n} \frac{dg^*_{nl}(x; \tau h_n)}{d\tau} \right)^2 d\tau \right)^{1/2}$$

$$\leq |\tau_1 - \tau_2|^{1/2} \left(\int_{r}^{s} \left(\sqrt{nh_n} \frac{dg^*_{nl}(x; \tau h_n)}{d\tau} \right)^2 d\tau \right)^{1/2}. \quad (28)$$

Once again, Theorem 21.10 in Davidson (1994) and Lemma 2.3 imply that Eq. (27) holds. □

Proof of Corollary 3.1. For any $\epsilon > 0$, given that $\hat{\tau}_n = O_p(1)$ there exists $r, s \in (0, \infty)$ with $r < s$ such that $P(\hat{\tau}_n \notin [r, s]) \leq \epsilon/3,\ \forall\ n$. From Theorem 3.1, it suffices to show that

$$\sqrt{nh_n} \left(H_{n\tau}(b_n(x; \tau h_n) - b(x)) - \frac{(\tau h_n)^{p+1} m^{(p+1)}(x)}{(p+1)!} + (\tau h_n)^{p+1} o_p(1) \right)$$

is stochastic equicontinuous with respect to τ on $[r, s]$ with $H_{n\tau} = diag\{(\tau h_n)^j\}_{j=0}^{p}$.

Masry and Fan (1997) showed that

$$H_{n\tau}(b_n(x; \tau h_n) - b(x)) - \frac{(\tau h_n)^{p+1} m^{(p+1)}(x)}{(p+1)!} + (\tau h_n)^{p+1} o_p(1) = S_n^{-1}(x; \tau h_n) G_n^*(x; \tau h_n),$$

thus, from Theorem 3.1, the result follows. □

Appendix 2: Tables and Graphs

Table 1
Bias and MSE ratios for $m_1(x)$ and $m_1^{(1)}(x)$ data driven h and h_{AMISE}

	n		$x = 0.5\pi$ h_{CV}	$x = 0.5\pi$ h_{ROT}	$x = \pi$ h_{CV}	$x = \pi$ h_{ROT}	$x = 1.5\pi$ h_{CV}	$x = 1.5\pi$ h_{ROT}
				$m_1(x)$				
$\rho = 0$	200	Bias	1.014	0.858	1.267	1.021	1.022	0.854
		MSE	1.160	0.974	1.058	1.085	1.082	0.972
	600	Bias	1.015	0.858	1.196	1.113	1.015	0.862
		MSE	1.039	0.972	1.047	1.077	1.050	0.964
$\rho = 0.5$	200	Bias	1.010	0.856	1.158	1.115	1.008	0.859
		MSE	1.088	0.974	1.050	1.073	1.107	0.979
	600	Bias	0.988	0.861	1.782	0.807	1.010	0.856
		MSE	1.035	0.975	1.045	1.065	1.051	0.971
$\rho = 0.9$	200	Bias	0.980	0.794	1.104	1.033	0.977	0.835
		MSE	1.030	0.988	1.039	1.038	1.025	0.982
	600	Bias	0.981	0.851	0.922	1.013	0.986	0.872
		MSE	1.027	0.981	1.018	1.035	1.014	0.979

(continued)

Table 1
Continued

	n		$x = 0.5\pi$		$x = \pi$		$x = 1.5\pi$	
			h_{CV}	h_{ROT}	h_{CV}	h_{ROT}	h_{CV}	h_{ROT}
			$m_1^{(1)}(x)$					
$\rho = 0$	200	Bias	0.973	1.192	1.000	1.000	0.661	1.124
		MSE	1.676	1.237	1.002	1.002	1.709	1.253
	600	Bias	0.825	1.202	0.999	1.000	1.120	1.137
		MSE	1.178	1.240	1.000	1.001	1.174	1.237
$\rho = 0.5$	200	Bias	1.922	1.141	0.998	1.001	3.651	1.156
		MSE	1.417	1.250	0.998	1.003	1.514	1.245
	600	Bias	1.077	1.043	1.000	1.000	0.956	1.008
		MSE	1.000	1.001	1.001	1.001	1.341	1.242
$\rho = 0.9$	200	Bias	0.705	0.852	1.000	1.001	1.594	1.176
		MSE	1.434	1.272	1.001	1.004	1.434	1.290
	600	Bias	0.786	1.070	1.000	1.001	0.965	1.180
		MSE	1.205	1.248	1.002	1.002	1.233	1.256

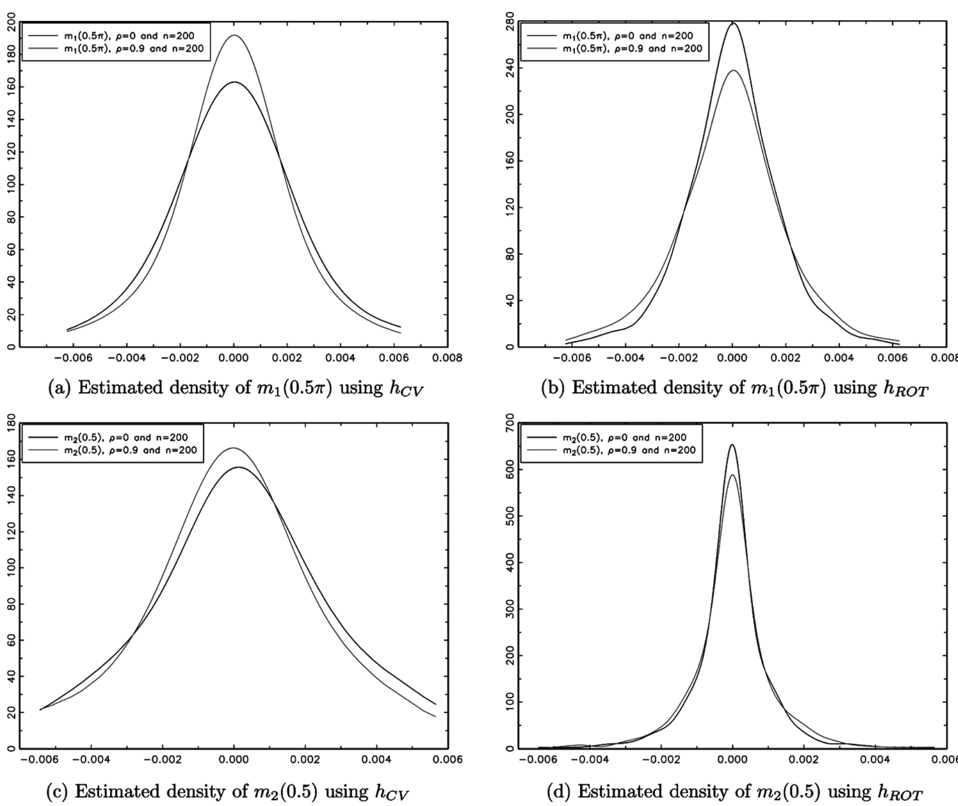
Table 2
Bias and MSE ratios for $m_2(x)$ and $m_2^{(1)}(x)$ data driven h and h_{AMISE}

	n		$x = 0.25$		$x = 0.5$		$x = 0.75$	
			h_{CV}	h_{ROT}	h_{CV}	h_{ROT}	h_{CV}	h_{ROT}
			$m_2(x)$					
$\rho = 0$	200	Bias	1.477	0.902	0.207	1.194	1.383	0.924
		MSE	1.291	1.045	0.983	1.050	1.298	1.029
	600	Bias	1.348	0.962	0.772	1.414	1.329	0.957
		MSE	1.231	1.020	1.000	1.019	1.229	1.016
$\rho = 0.5$	200	Bias	1.499	0.916	0.853	1.058	1.422	0.922
		MSE	1.226	1.022	0.971	1.033	1.228	1.017
	600	Bias	1.291	0.966	0.914	1.025	1.352	0.963
		MSE	1.178	1.012	1.007	1.012	1.189	1.016
$\rho = 0.9$	200	Bias	1.368	0.877	0.368	1.163	1.294	0.898
		MSE	1.075	0.997	1.006	1.011	1.061	1.006
	600	Bias	1.447	0.941	0.972	1.017	1.285	0.955
		MSE	1.070	1.006	1.002	1.001	1.082	1.003
			$m_2^{(1)}(x)$					
$\rho = 0$	200	Bias	2.119	0.943	1.607	0.878	2.909	0.796
		MSE	1.514	1.133	1.871	1.133	1.481	1.163
	600	Bias	1.619	1.018	1.562	0.969	1.269	0.937
		MSE	1.751	1.067	2.413	1.067	1.422	1.050

(continued)

Table 2
Continued

	n		$x = 0.25$		$x = 0.5$		$x = 0.75$	
			h_{CV}	h_{ROT}	h_{CV}	h_{ROT}	h_{CV}	h_{ROT}
$\rho = 0.5$	200	Bias	1.808	0.836	1.517	0.950	3.076	0.908
		MSE	1.765	1.120	1.643	1.120	1.573	1.122
	600	Bias	2.272	0.918	1.172	0.940	3.139	0.832
		MSE	1.492	1.052	1.715	1.052	1.667	1.050
$\rho = 0.9$	200	Bias	1.681	0.915	1.538	0.904	2.035	0.870
		MSE	1.972	1.207	2.695	1.207	1.629	1.182
	600	Bias	1.622	0.948	1.206	0.969	2.285	1.007
		MSE	2.103	1.066	1.486	1.066	1.691	1.076



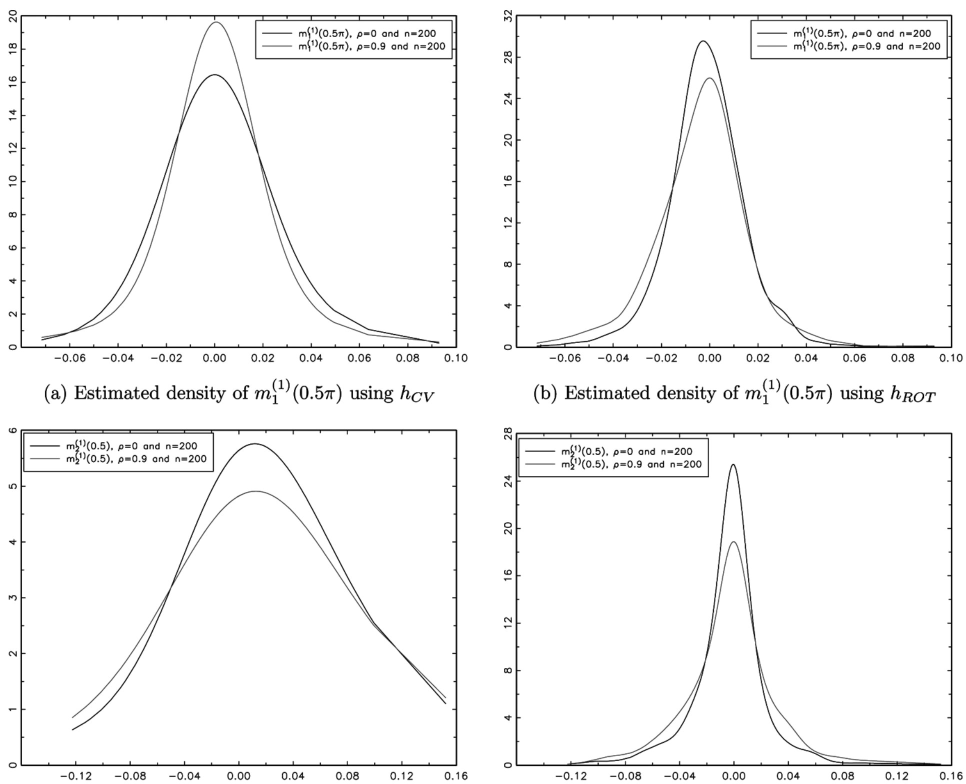
(a) Estimated density of $m_1(0.5\pi)$ using h_{CV}

(b) Estimated density of $m_1(0.5\pi)$ using h_{ROT}

(c) Estimated density of $m_2(0.5)$ using h_{CV}

(d) Estimated density of $m_2(0.5)$ using h_{ROT}

Figure 1. Estimated density of regression.



(a) Estimated density of $m_1^{(1)}(0.5\pi)$ using h_{CV}

(b) Estimated density of $m_1^{(1)}(0.5\pi)$ using h_{ROT}

(c) Estimated density of $m_2^{(1)}(0.5)$ using h_{CV}

(d) Estimated density of $m_2^{(1)}(0.5)$ using h_{ROT}

Figure 2. Estimated density of regression derivative.

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References

Bernstein, S. (1927). Sur l'extension du theorem du calcul des probabilites aux sommes de quantites dependantes. *Mathematische Annalen* 97:1–59.

Boente, G., Fraiman, R. (1995). Asymptotic distribution of data-driven smoothers in density and regression estimation under dependence. *Can. J. Statist.* 23:383–397.

Bosq, D. (1998). *Nonparametric Statistics for Stochastic Processes: Estimation and Prediction*. No. 110 in Lecture Notes in Statistics. New York: Springer Verlag.

Davidson, J. (1994). *Stochastic Limit Theory*. New York: Oxford University Press.

Dony, J., Einmahl, U., Mason, D. (2006). Uniform in bandwidth consistency of local polynomial regression function estimators. *Aust. J. Statist.* 35:105–120.

Doukhan, P. (1994). *Mixing*. New York: Springer-Verlag.

Fan, J. (1992). Design-adaptive nonparametric regression. *J. Amer. Statist. Assoc.* 87:998–1004.

Jennrich, R. I. (1969). Asymptotic properties of non-linear least squares estimators. *Ann. Mathemat. Statist.* 40(2):633–643.

Li, Q., Racine, J. (2004). Cross-validated local linear nonparametric regression. *Statistica Sinica* 14:485–512.

Martins-Filho, C., Yao, F. (2009). Nonparametric regression estimation with general parametric error covariance. *J. Multivariate Anal.* 100:309–333.

Masry, E., Fan, J. (1997). Local polynomial estimation of regression functions for mixing processes. *Scand. J. Statist.* 24:1965–1979.

Parzen, E. (1962). On estimation of a probability density and mode. *Ann. Mathemat. Statist.* 33:1065–1076.

Pham, T. D., Tran, L. T. (1985). Some mixing properties of time series models. *Stochastic Process. Applic.* 19:297–303.

Robinson, P. M. (1983). Nonparametric estimators for time series. *J. Time Ser. Anal.* 4:185–207.

Ruppert, D., Sheather, S., Wand, M. P. (1995). An effective bandwidth selector for local least squares regression. *J. Amer. Statist. Assoc.* 90:1257–1270.

Silverman, B. W. (1986). *Density Estimation for Statistics and Data Analysis*. London: Chapman and Hall.

Xia, Y., Li, W. K. (2002). Asymptotic behavior of bandwidth selected by the cross-validation method for local polynomial fitting. *J. Multivariate Anal.* 83:265–287.

Ziegler, K. (2004). Adaptive kernel estimation of the mode in nonparametric random design regression model. *Probab. Mathemat. Statist.* 24:213–235.