Abstract Algebra 1 (MATH 3140)

Practice Problems on Groups

Problem 1.
Let S be a nonempty set with an associative operation $\cdot$ such that

(i) there exists an identity element e in S for $\cdot$,
(ii) for every element $a \in S$, left multiplication by a is a bijection $L_a\colon S \to S$, $x \mapsto ax$, and
(iii) for every element $a \in S$, right multiplication by a is a bijection $R_a\colon S \to S$, $x \mapsto xa$.

Show that

(a) S with $\cdot$ is a group;
(b) in fact, even if we only assume that conditions (i)–(ii) hold, it follows that S with $\cdot$ is a group.[1]
(c) Give an example of a non-associative operation $\cdot$ on a 5-element set S such that conditions (i)–(iii) hold.

Problem 2. Let G be a group, let S be a subset of G, and let $a, b \in G$.

(a) Prove that $C_G(S) = \{g \in G : sg = gs \text{ for all } s \in S\}$ is a subgroup of G. ($C_G(S)$ is called the *centralizer* of S in G.)
(b) Use the result in part (a) to verify that if $ab = ba$, then $a^m b^n = b^n a^m$ holds for all integers m, n.
(c) Prove that if $ab = ba$, then $(ab)^n = a^n b^n$ for all integers n.

Problem 3. Find the order of the subgroup of S_{20} generated by the ten transpositions $(1\ 2)$, $(3\ 4)$, $\ldots$, $(19\ 20)$.

Problem 4. Let $G = \langle a \rangle$ be a cyclic group of order $4095 = 3^2 \cdot 5 \cdot 7 \cdot 13$, and consider the subgroups $H = \langle a^{231} \rangle$ and $K = \langle a^{182} \rangle$ of G. Find a generator a^d for the subgroup $H \cap K$ of G such that $d \mid 4095$, and express a^d as a power of a^{231} and as a power of a^{182}.
Hint: First, find generators a^{d_H} for H and a^{d_K} for K such that $d_H, d_K \mid 4095$, and determine which powers of a are in $H \cap K$.

[1] This part of the problem is harder than the rest.