Abstract Algebra 1 (MATH 3140)

Worksheet 5: Direct Product and Finite Abelian Groups

1. In how many ways can the 4-element abelian group $V = \{\mathrm{id},\ (1\,2)(3\,4),\ (1\,3)(2\,4),\ (1\,4)(2\,3)\}$ be written as an internal direct product of two normal subgroups of order < 4 ?

Hint: Count the possibilities by counting the unordered pairs of normal subgroups that yield an internal direct product decomposition of V.

2. How many nonisomorphic abelian groups G of order $2160 = 2^4 3^3 5$ have the property that $a^{180} = e$ for all $a \in G$?

3. Show that $\mathbb{Z}_{16} \not\cong H \times K$ for any groups H, K of order < 16. (Do not use the Fundamental Theorem of Finite Abelian Groups.)

Hint: Assume that an isomorphism $\psi \colon H \times K \to \mathbb{Z}_{16}$ exists. Use the fact that $H \times K$ is the internal direct product of its normal subgroups $H \times \{e_K\}$ and $\{e_H\} \times K$ to derive a contradiction.

4. Let G, H be arbitrary groups, and let $M \trianglelefteq G$, $N \trianglelefteq H$. Use the Homomorphism Theorem to show that $M \times N \trianglelefteq G \times H$, and $(G \times H)/(M \times N) \cong (G/M) \times (H/N)$.
Hint: Find a surjective homomorphism $G \times H \to (G/M) \times (H/N)$ with kernel $M \times N$.

5. Recall from the Chinese remainder theorem that if $n = ab$ for some nonzero natural numbers n, a, b such that $\gcd(a, b) = 1$, then the map

$$\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_a \times \mathbb{Z}_b, \quad [x]_n \mapsto ([x]_a, [x]_b)$$

is a ring isomorphism. Recall (see Lec.Notes 2/19) that for any ring R which has a multiplicative identity element 1, R^* denotes *the group of units of* R, defined by

$$R^* = \{r \in R : r \text{ has a multiplicative inverse}\}.$$

(a) Show that for every integer $x \in \mathbb{Z}$, we have $[x]_n \in \mathbb{Z}_n^*$ if and only if $\varphi([x]_n) = ([x]_a, [x]_b) \in \mathbb{Z}_a^* \times \mathbb{Z}_b^*$.

(b) Deduce that $\mathbb{Z}_n^* \cong \mathbb{Z}_a^* \times \mathbb{Z}_b^*$.