Abstract Algebra 1 (MATH 3140)

Worksheet 6: Groups of small order: Orders 8 and 12

1. Let G be a nonabelian group of order 12. Let P denote a Sylow 2-subgroup of G and Q a Sylow 3-subgroup of G.

(a) Use Sylow's 3rd theorem to show that $(n_2, n_3) = (1, 4)$, $(3, 1)$, or $(3, 4)$.[1]

(b) Argue that if $n_3 = 4$, then G has $4 \cdot 2 = 8$ elements of order 3, therefore it must be that $n_2 = 1$.

Note: (a)–(b) imply that $(n_2, n_3) = (1, 4)$ or $(3, 1)$, i.e., either $P \trianglelefteq G$ or $Q \trianglelefteq G$.

(c) Now assume that $P \trianglelefteq G$.

(i) Verify that G is a semidirect product of $P \trianglelefteq G$ and $Q \leq G$.

(ii) Consider the homomorphism $\varphi\colon Q \to \mathrm{Aut}(P)$, $b \mapsto c_b$, where c_b is conjugation by b on P; that is, for every $b \in Q$, $c_b\colon P \to P$, $a \mapsto bab^{-1}$.
Use the assumption that G is nonabelian to conclude that P cannot be cyclic, so it must be that $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

By continuing this analysis it can be shown that

- if $P \trianglelefteq G$, then $G \cong A_4$;
- if $Q \triangleleft G$ and P is not cyclic, then $G \cong D_6$; and
- if $Q \triangleleft G$ and P is cyclic, then G is isomorphic to a third group ($\ncong A_4$ or D_6).

[1] Recall that for any finite group G and any prime p dividing $|G|$, n_p denotes the number of Sylow p-subgroups of G.

2. Let G be a nonabelian group of order 8. Clearly, G has no element of order 8. Also, is not the case that every nonidentity element of G has order 2 (see Lec 3/3, Pr 5). Thus,

- ◇ G has a cyclic subgroup $P = \langle x \rangle$ of order 4, and every element $g \in G \setminus P$ has order 2 or 4. Since $[G : P] = 2$, we know that $P \trianglelefteq G$ (see Wsh 3, Pr 3).

By the same argument that we used in class to prove that every nonabelian group of order $2p$ ($p > 2$ prime) is isomorphic to D_p, one can show the following:

- ◇ if P is the only cyclic subgroup of G of order 4 (i.e., $Q = \langle g \rangle$ is a cyclic group of order 2 for each $g \in G \setminus P$), then $G \cong D_4$.

(∗) Assume now that G has at least two different cyclic subgroups of order 4: $P = \langle x \rangle$ above and $R = \langle y \rangle$.

(a) Show that $G = P \cup Py$, $x^2 = y^2$, and every element of G can be written uniquely in the form $x^k y^u$ with $0 \le k \le 3$, $0 \le u \le 1$.

(b) Consider the homomorphism $\psi \colon Q \to \mathrm{Aut}(P)$, $b \mapsto c_b$, where c_b is conjugation by b on P; that is, for every $b \in Q$, $c_b \colon P \to P$, $a \mapsto bab^{-1}$.
Use the assumption that G is nonabelian to conclude that $\psi(c_y)$ is inversion in $P = \langle x \rangle$, that is, $yx^k y^{-1} = x^{-k}$ for all $k \in \mathbb{Z}$.

(c) Show that the operations of G are uniquely determined by assumption (∗), and therefore there is at most one nonabelian group G, up to isomorphism, which satisfies (∗).

In fact, there is a nonabelian group G satisfying assumption (∗), the *quaternion group* Q_8 with elements $1, -1, i, -i, j, -j, k, -k$ and multiplication defined (using the convention $-(-a) = a$) by

- $1a = a$, $a1 = a$, $(-1)a = -a$, $a(-1) = -a$ for all $a \in \{1, -1, i, -i, j, -j, k, -k\}$,
- $i^2 = -1$, $j^2 = -1$, $k^2 = -1$, $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$, $ik = -j$,
- $(-a)b = -ab$, $a(-b) = -ab$, $(-a)(-b) = ab$ for all $a, b \in \{i, j, k\}$.

Hence, every nonabelian group of order 8 is isomorphic to D_4 or Q_8.