**Set Theory (MATH 6730)**

## HOMEWORK 1

(First draft is due on February 8, 2021)

In all problems below, $\vdash$ refers to the proof system described on pp. 13–14 of the lecture notes "Background in Logic"; see also p. 17 for some metatheorems for that proof system. You may also use the results stated on p. 18. You are not allowed to use the Completeness Theorem to establish provability in the proof system $\vdash$.

In some of the problems below we use the following terminology.

**Definition.** Let $\Gamma \cup \{\varphi, \psi\}$ be a set of $\mathcal{L}_C$-formulas. We say that $\varphi$ and $\psi$ are *provably $\Gamma$-equivalent* if $\Gamma \vdash \varphi \leftrightarrow \psi$. Two formulas are *provably equivalent* if they are provably $\emptyset$-equivalent.

It is part of Problem 1 to show that $\Gamma \vdash \varphi \leftrightarrow \psi$ holds if and only if $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\psi\} \vdash \varphi$. You may use this fact in your solutions to Problems 2–7 by citing Problem 1.

**Problems:**

1. **Mateo**
   Let $\Gamma \cup \{\varphi, \psi\}$ be a set of $\mathcal{L}_C$-formulas.
   (i) Show that the following conditions on $\varphi$ and $\psi$ are equivalent:
      (a) $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\psi\} \vdash \varphi$;
      (b) $\Gamma \vdash \varphi \leftrightarrow \psi$.
   (ii) Prove that for any variables $x, y$,
      - $\forall x\, \forall y\, \varphi$ and $\forall y\, \forall x\, \varphi$ are provably equivalent, and
      - $\exists x\, \exists y\, \varphi$ and $\exists y\, \exists x\, \varphi$ are provably equivalent.

2. **Dale**
   Let $\varphi$ and $\psi$ be arbitrary $\mathcal{L}_C$-formulas.
   (i) Prove that for any variable $x$ that is not free in $\varphi$, the formulas $\varphi \to \forall x\, \psi$ and $\forall x\, (\varphi \to \psi)$ are provably equivalent.
   (ii) Find examples of $\varphi$, $\psi$ such that $x$ is free in $\varphi$, and the formula

   $$(\varphi \to \forall x\, \psi) \leftrightarrow \forall x\, (\varphi \to \psi)$$

   is not valid.

3. **Connor**
   Let $\varphi$ and $\psi$ be arbitrary $\mathcal{L}_C$-formulas.
   (i) Prove that for any variable $x$ that is not free in $\psi$, the formulas $\exists x \varphi \to \psi$ and $\forall x\, (\varphi \to \psi)$ are provably equivalent.

(ii) Use the statement in part (i) to prove that the formula

$$\mathsf{Uni} \qquad \forall \mathcal{A}\, \exists B\, \forall x\, \big(\exists A\, (x \in A \,\wedge\, A \in \mathcal{A}) \to x \in B\big)$$

for the Axiom of Union given on p. 4 of the lecture notes "The Axioms of Set Theory ..." and the formula

$$\mathsf{Uni}' \qquad \forall \mathcal{A}\, \exists B\, \forall A\, \forall x\, \big((x \in A \,\wedge\, A \in \mathcal{A}) \to x \in B\big)$$

given for the Axiom of Union on p. 68 of [LST] or p. 78 of [NST] (up to the letters used for the bound variables) are provably equivalent.

**4. Raymond**
Prove that $\{\mathsf{Cmpr}, \mathsf{Pair}\} \vdash \mathsf{Pair}^\sharp$ by formalizing our informal proof of this statement.

**5. Nick**
Prove that $\{\mathsf{Cmpr}, \mathsf{Pset}\} \vdash \mathsf{Pset}^\sharp$ by formalizing our informal proof of this statement.

**6. Chase**
Prove that $\{\mathsf{Pair}^\sharp, \mathsf{Fnd}\} \vdash \forall x\, \neg\, x \in x$ by formalizing our informal proof of this statement.

**7. Toby**
The sentences

$$\sigma \equiv \neg\exists v\, \forall x\, x \in v, \quad \text{and}$$
$$\rho \equiv \neg\exists s\, \forall x\, (x \in s \leftrightarrow \neg\, x \in x)$$

express that the class $\mathbf{V}$ of all sets is not a set and $\mathbf{S} = \{A \in \mathbf{V} \colon A \notin A\}$ from Russel's Paradox is not a set. Formalize our informal proofs of these statements (that do not rely on $\mathsf{Fnd}$) to show that

$$\vdash \rho, \quad \text{and hence} \quad \{\mathsf{Cmpr}\} \vdash \sigma.$$

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Please write your solution in TEX, include your name at the top of the first page, and email your solution to me (`szendrei@colorado.edu`) as a pdf file so that the file name includes the string

hw1pr$k$

where $k$ is the problem number.