Set Theory (MATH 6730)
Mateo Muro

HOMEWORK 1

Problems:

1. Let $\Gamma \cup \{\varphi, \psi\}$ be a set of $\mathcal{L}_C$-formulas.

(i) Show that the following conditions on φ and ψ are equivalent:

(a) $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\psi\} \vdash \varphi$;

(b) $\Gamma \vdash \varphi \leftrightarrow \psi$.

(ii) Prove that for any variables x, y,

- $\forall x\, \forall y\, \varphi$ and $\forall y\, \forall x\, \varphi$ are provably equivalent, and
- $\exists x\, \exists y\, \varphi$ and $\exists y\, \exists x\, \varphi$ are provably equivalent.

Proof. To prove (i) first, assume that $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\psi\} \vdash \varphi$. The Deduction Theorem tells us that $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \varphi$. For brevity, we will write $\alpha \equiv \varphi \to \psi$ and $\beta \equiv \psi \to \varphi$. We show that $\Gamma \cup \{\alpha, \beta\} \vdash \alpha \wedge \beta$.

(1) $\alpha \to (\beta \to (\alpha \wedge \beta))$ Ax1
(2) α hypothesis
(3) β hypothesis
(4) $\beta \to (\alpha \wedge \beta)$ MP(1)(2)
(5) $\alpha \wedge \beta$ MP(3)(4)

Let $\Delta = \{\alpha, \beta\}$. We have $\Gamma \cup \Delta \vdash \alpha \wedge \beta$ and $\Gamma \vdash \delta$ for every $\delta \in \Delta$. So by Metatheorem(ii) we have $\Gamma \vdash \alpha \wedge \beta$. Remember that $\alpha \wedge \beta$ is a short notation for $(\varphi \to \psi) \wedge (\psi \to \varphi)$, which in turn is abbreviated as $\varphi \leftrightarrow \psi$.

Now we prove the converse. Assume that $\Gamma \vdash (\varphi \to \psi) \wedge (\psi \to \varphi)$. Since $\Gamma \subset \Gamma \cup \{\varphi\}$, by Metatheorem(i), we have $\Gamma \cup \{\varphi\} \vdash (\varphi \to \psi) \wedge (\psi \to \varphi)$. To conclude that $\Gamma \cup \{\varphi\} \vdash \psi$, it suffices to verify (by Metatheorems(i)-(ii)) that

$$\{\varphi, (\varphi \to \psi) \wedge (\psi \to \varphi)\} \vdash \psi,$$

which can be done as follows.

(1) $((\varphi \to \psi) \wedge (\psi \to \varphi))$ hypothesis
(2) $((\varphi \to \psi) \wedge (\psi \to \varphi)) \to (\varphi \to \psi)$ (Ax1)
(3) $\varphi \to \psi$ MP(1)(2)
(4) φ hypothesis
(5) ψ MP(3)(4)

A similar proof, using the tautology $((\varphi \to \psi) \wedge (\psi \to \varphi)) \to (\psi \to \varphi)$, gives us $\Gamma \cup \{\psi\} \vdash \varphi$.

We now prove the statements in (ii). By (i), we have that it suffices to show that $\forall x\forall y\varphi \vdash \forall y\forall x\varphi$ and $\forall y\forall x\varphi \vdash \forall x\forall y\varphi$. We will show $\forall x\forall y\varphi \vdash \forall y\forall x\varphi$, as the arguments are symmetric. We first want to explain that $\text{Subf}^{x}_{x}(\theta) \equiv \theta$ for all formulas θ and variables x. This is because any free occurrence of x in θ is being substituted with x, and so the formulas read the same. Therefore, we will write $\forall x\theta \to \theta$ for any axiom of the form $\forall x\theta \to \text{Subf}^{x}_{x}(\theta)$ in axiom group (Ax2). Note also that substituting x for x will always satisfy the restriction on (Ax2), since x is a variable such that no quantifier $\forall x$ in θ can have a free occurrence of x in its scope. We now show that $\forall x\forall y\varphi \vdash \varphi$

(1) $\forall x\forall y\varphi \to \forall y\varphi$ (Ax2)
(2) $\forall x\forall y\varphi$ hypothesis
(3) $\forall y\varphi$ MP(1)(2)
(4) $\forall y\varphi \to \varphi$ (Ax2)
(5) φ MP(4)(5)

We have that x has no free occurrence in $\forall x\forall y\varphi$, so by Metatheorem(iv) we have $\forall x\forall y\varphi \vdash \forall x\varphi$. Applying the metatheorem again gives the desired result $\forall x\forall y\varphi \vdash \forall y\forall x\varphi$.

We know that $\exists x\varphi$ is an abbreviation for $\neg\forall x\neg\varphi$. We then have $\exists x\exists y\varphi \equiv \neg\forall x(\neg\exists y\varphi) \equiv \neg\forall x(\neg\neg\forall y\neg\varphi)$ and $\exists y\exists x\varphi \equiv \neg\forall y(\neg\neg\forall x\neg\varphi)$. We then want to show

(*) $$\neg\forall x(\neg\neg\forall y\neg\varphi) \vdash \neg\forall y(\neg\neg\forall x\neg\varphi).$$

By Metatheorem (v), it suffices to show

(**) $$\forall y(\neg\neg\forall x\neg\varphi) \vdash \neg\neg\forall x(\neg\neg\forall y\neg\varphi).$$

We first show that $\forall y(\neg\neg\forall x\neg\varphi) \vdash \neg\varphi$.

(1) $\forall y(\neg\neg\forall x\neg\varphi)$ hypothesis
(2) $(\forall y(\neg\neg\forall x\neg\varphi)) \to \neg\neg\forall x\neg\varphi$ (Ax2)
(3) $\neg\neg\forall x\neg\varphi$ MP(1)(2)
(4) $\neg\neg\forall x\neg\varphi \to \forall x\neg\varphi$ (Ax1)
(5) $\forall x\neg\varphi$ MP(3)(4)
(6) $\forall x\neg\varphi \to \neg\varphi$ (Ax2)
(7) $\neg\varphi$ MP(5)(6)

There is no free occurrence of y in $\forall y(\neg\neg\forall x\neg\varphi)$, so by Metathoerem(iv), we have $\forall y(\neg\neg\forall x\neg\varphi) \vdash \forall y\neg\varphi$.

We need a small result here that for any formula α we have $\alpha \vdash \neg\neg\alpha$.

(1) α hypothesis
(2) $\alpha \to \neg\neg\alpha$ (Ax1)
(3) $\neg\neg\alpha$ MP(1)(2)

This shows that $\forall y\neg\varphi \vdash \neg\neg\forall y\neg\varphi$. By Metatheorem(i) we have $\{\forall y\neg\varphi\} \cup \{\forall y(\neg\neg\forall x\neg\varphi)\} \vdash \neg\neg\forall y\neg\varphi$. We have also that $\forall y(\neg\neg\forall x\neg\varphi) \vdash \forall y\neg\varphi$ and so by Metatheorem(ii) we have $\forall y(\neg\neg\forall x\neg\varphi) \vdash \neg\neg\forall y\neg\varphi$. There is no free occurrence of x in $\forall y(\neg\neg\forall x\neg\varphi)$, so by Metatheorem(iv), we have that $\forall y \neg\neg\forall x \neg\varphi \vdash \forall x \neg\neg\forall y \neg\varphi$. We can argue similarly to show (**). Since (*) follows from (**), and what we wanted to prove was an abbreviation for (*), we have shown that $\exists x \exists y \varphi \vdash \exists y \exists x \varphi$. The proof of $\exists y \exists x \vdash \exists x \exists y \varphi$ follows similarly.

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