HOMEWORK 1

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Formal Proof That $\{Cmpr, Pair\} \vdash Pair^{\#}$

Statements Used:

$$Pair \equiv \forall x \forall y \exists z (x \in z \wedge y \in z)$$
$$Cmpr^{p} \equiv \forall x \forall w_1 \forall w_2 \exists z \forall t (t \in z \leftrightarrow ((t = w_1 \vee t = w_2) \wedge t \in x))$$

Premises Used:

$$\Gamma = \{Cmpr, Pair\}$$
$$\Gamma^{*} = \Gamma \cup \{a \in d \wedge b \in d\}$$
$$\Gamma^{**} = \Gamma^{*} \cup \{\forall t (t \in g \leftrightarrow ((t = a \vee t = b) \wedge t \in d))\}$$
$$\Gamma' = \Gamma^{**} \cup \{v \in g\}$$
$$\Gamma'' = \Gamma^{**} \cup \{v = a\}$$
$$\Gamma''' = \Gamma^{**} \cup \{v = a \vee v = b\}$$

Proof. Let $\Gamma = \{Cmpr, Pair\}$, where $Cpmr$ is all the axioms in the scheme's form. It follows that $Cmpr^{p} \in \{Cmpr, Pair\}$. To apply generalization of constants, define $\mathcal{L}'$ by adding the constant symbols a and b to the signature of $\mathcal{L}$. Now, in order to apply Existential Instantiation, define $\mathcal{L}''$ by adding a constant symbol d to the language $\mathcal{L}$. Define the set of premises $\Gamma^{*} = \Gamma \cup \{a \in d \wedge b \in d\}$. Taking on one more augmentation of the language and premises, let $\mathcal{L}''' = \mathcal{L}''_{\cup\{g\}}$, where g is a constant not in the signature of $\mathcal{L}''$. Let $\Gamma^{**} = \Gamma^{*} \cup \{\forall t (t \in g \leftrightarrow ((t = a \vee t = b) \wedge t \in d))\}$. Take $\mathcal{L}'''$ and Γ^{**} to be our language and our set of premises. We will see first what is provable from Γ^{**} in $\mathcal{L}'''$ and then look to apply Existential Instantiation. We will then repeat the process with Γ^{*} in the language $\mathcal{L}''$.

Let v be a variable. We are looking to show

$$\Gamma^{**} \vdash \forall v (v \in g \leftrightarrow (v = a \vee v = b))$$

We will do this with two successive applications of the deduction theorem to conclude that $\Gamma^{**} \vdash v \in g \rightarrow (v = a \vee v = b)$, as well as the converse of this statement. To begin the first deduction, take as premises $\Gamma' = \Gamma^{**} \cup \{v \in g\}$:

(1) $\Gamma' \vdash \forall t(t \in g \leftrightarrow ((t = a \vee t = b) \wedge t \in d)) \rightarrow (v \in g \leftrightarrow ((v = a \vee v = b) \wedge v \in d))$ Ax 2

(2) $\Gamma' \vdash \forall t(t \in g \leftrightarrow ((t = a \vee t = b) \wedge t \in d))$ Γ'

(3) $\Gamma' \vdash v \in g \leftrightarrow ((v = a \vee v = b) \wedge v \in d)$ MP 1,2

(4) $\Gamma' \vdash v \in g$ Γ'

(5) $\Gamma' \vdash (v = a \vee v = b) \wedge v \in d$ MP 3,4

(6) $\Gamma' \vdash ((v = a \vee v = b) \wedge v \in d) \rightarrow (v = a \vee v = b)$ Ax 1

(7) $\Gamma' \vdash v = a \vee v = b$ MP 5,6

An application of the deduction theorem allows us to conclude that

(8) $$\Gamma^{**} \vdash v \in g \rightarrow (v = a \vee v = b)$$ DT

Now, we look to show that Γ^{**} proves the converse of this statement. This will involve two subproofs using the deduction theorem and a few subsequent deductions. Take the premises $\Gamma'' = \Gamma^{**} \cup \{a = v\}$:

(9) $\Gamma'' \vdash a \in d \wedge b \in d$ Γ''

(10) $\Gamma'' \vdash (a \in d \wedge b \in d) \rightarrow a \in d$ Ax 1

(11) $\Gamma'' \vdash a \in d$ MP 9,10

(12) $\Gamma'' \vdash \forall z(z = v \rightarrow (z \in d \rightarrow v \in d))$ Ax 6

(13) $\Gamma'' \vdash [\forall z(z = v \rightarrow (z \in d \rightarrow v \in d))] \rightarrow [a = v \rightarrow (a \in d \rightarrow v \in d)]$ Ax 2

(14) $\Gamma'' \vdash a = v \rightarrow (a \in d \rightarrow v \in d)$ MP 12, 13

(15) $\Gamma'' \vdash a = v$ Γ''

(16) $\Gamma'' \vdash v \in d$ MP 15, 11, 14

Applying the deduction theorem, we obtain

(17) $$\Gamma^{**} \vdash v = a \rightarrow v \in d$$ DT 9-16

Repeating the steps 9-16 with the constant symbol b instead of a allows us to conclude that

(18) $$\Gamma^{**} \vdash v = b \rightarrow v \in d$$ DT 9*-16*

Now we have

(19) $\Gamma^{**} \vdash [v = a \rightarrow v \in d] \rightarrow [(v = b \rightarrow v \in d) \rightarrow ((v = a \vee v = b) \rightarrow v \in d)]$ Ax 1

(20) $\Gamma^{**} \vdash (v = a \vee v = b) \rightarrow v \in d$ MP 17, 18, 19

To finally prove the desired converse, take the set of premises $\Gamma''' = \Gamma^{**} \cup \{v = a \vee v = b\}$

(21) $\Gamma''' \vdash v = a \vee v = b$ Γ'''

(22) $\Gamma''' \vdash v \in d$ MP 21, 20

(23) $\Gamma''' \vdash (v = a \vee v = b) \to (v \in d \to ((v = a \vee v = b) \wedge v \in d))$ Ax 1

(24) $\Gamma''' \vdash (v = a \vee v = b) \wedge v \in d$ MP 21,22,23

(25) $\Gamma''' \vdash \forall t(t \in g \leftrightarrow ((t = a \vee t = b) \wedge t \in d))$ Γ'''

(26) $\Gamma''' \vdash \forall t(t \in g \leftrightarrow ((t = a \vee t = b) \wedge t \in d)) \to (v \in g \leftrightarrow ((v = a \vee v = b) \wedge v \in d))$ Ax 2

(27) $\Gamma'' \vdash v \in g \leftrightarrow ((v = a \vee v = b) \wedge v \in d)$ MP 25, 26

(28) $\Gamma'' \vdash [v \in g \leftrightarrow ((v = a \vee v = b) \wedge v \in d)] \to [((v = a \vee v = b) \wedge v \in d) \to v \in g]$ Ax 1

(29) $\Gamma'' \vdash ((v = a \vee v = b) \wedge v \in d) \to v \in g$ MP 27, 28

(30) $\Gamma'' \vdash v \in g$ MP 24, 29

Now, discharging our added premise and applying the deduction theorem, we have that

(31) $\Gamma^{**} \vdash (v = a \vee v = b) \to v \in g$ DT 21-30

(32) $\Gamma^{**} \vdash [(v = a \vee v = b) \to v \in g] \to [(v \in g \to (v = a \vee v = b))$

(33) $\to (v \in g \leftrightarrow (v = a \vee v = b))]$ Ax 1

(34) $\Gamma^{**} \vdash (v \in g \leftrightarrow (v = a \vee v = b))$ MP 31, 8, 32/33

Since v is a variable not free in any $\gamma \in \Gamma^{**}$, as every γ is a sentence, it follows from the generalization theorem that,

(35) $$\Gamma^{**} \vdash \forall v(v \in g \leftrightarrow (v = a \vee v = b))$$ GT

Now we look to establish the existential portion of our statement:

(36) $\Gamma^{**} \vdash [\forall z \neg(\forall v(v \in z \leftrightarrow (v = a \vee v = b)))] \to [\neg \forall v(v \in g \leftrightarrow (v = a \vee v = b))]$ Ax 2

(37) $\Gamma^{**} \vdash \big[[\forall z \neg \forall v(v \in z \leftrightarrow (v = a \vee v = b))] \to [\neg \forall v(v \in g \leftrightarrow (v = a \vee v = b))]\big]$

(38) $\to \big[[\neg\neg \forall v(v \in g \leftrightarrow (v = a \vee v = b))] \to [\neg \forall z \neg \forall v(v \in z \leftrightarrow (v = a \vee v = b))]\big]$ Ax 1

(39) $\Gamma^{**} \vdash [\neg\neg \forall v(v \in g \leftrightarrow (v = a \vee v = b))] \to [\neg \forall z \neg \forall v(v \in z \leftrightarrow (v = a \vee v = b))]$ MP 36, 37/38

(40) $\Gamma^{**} \vdash \forall v(v \in g \leftrightarrow (v = a \vee v = b)) \to \neg\neg \forall v(v \in g \leftrightarrow (v = a \vee v = b))$ Ax 1

(41) $\Gamma^{**} \vdash \neg\neg \forall v(v \in g \leftrightarrow (v = a \vee v = b))$ MP 35, 40

(42) $\Gamma^{**} \vdash \neg \forall z \neg(\forall v(v \in z \leftrightarrow (v = a \vee v = b)))$ MP 41, 39

But, an abbreviation for the above is just

(43) $$\Gamma^{**} \vdash \exists z \forall v(v \in z \leftrightarrow (v = a \vee v = b))$$

Now, we have just shown that

$$\Gamma^* \cup \{\forall t(t \in g \leftrightarrow ((t = a \vee t = b) \wedge t \in d))\} \vdash \exists z \forall v(v \in z \leftrightarrow (v = a \vee v = b))$$

in the language $\mathcal{L}'''$. Since d is not in the signature of $\mathcal{L}''$, we may apply the meta-theorem Existential Instantiation, allowing us to conclude that $\Gamma^* \cup \{\exists z \forall t(t \in z \leftrightarrow ((t = a \vee t = b) \wedge t \in d))\} \vdash \exists z \forall v(v \in z \leftrightarrow (v = a \vee v = b))$ in the language $\mathcal{L}''$. Thus, the deduction theorem implies that

(44) $\Gamma^* \vdash [\exists z \forall t(t \in z \leftrightarrow ((t = a \vee t = b) \wedge t \in d))] \to [\exists z \forall v(v \in z \leftrightarrow (v = a \vee v = b))]$

Now apply (Ax 2) to $Cmpr^p$ with the substitutions d for x, a for w_1, and b for w_2. This gives us

(45) $\Gamma^* \vdash \forall x \forall w_1 \forall w_2 \exists z \forall t(t \in z \leftrightarrow ((t = w_1 \vee t = w_2) \wedge t \in x))]$ Γ^*

(46) $\Gamma^* \vdash \forall x \forall w_1 \forall w_2 \exists z \forall t(t \in z \leftrightarrow ((t = w_1 \vee t = w_2) \wedge t \in x))]$

(47) $\to [\exists z \forall t(t \in z \leftrightarrow ((t = a \vee t = b) \wedge t \in d))]$ Ax 2

(48) $\Gamma^* \vdash \exists z \forall t(t \in z \leftrightarrow ((t = a \vee t = b) \wedge t \in d))$ MP 45, 46/47

Thus, we have shown that

$$\Gamma^* = \Gamma \cup \{a \in d \wedge b \in d\} \vdash \exists z \forall v(v \in z \leftrightarrow (v = a \vee v = b))$$

in the language $\mathcal{L}''$. But, since d is not in the signature of $\mathcal{L}'$, we may apply existential instantiation to obtain that $\Gamma \cup \{\exists z(a \in z \wedge b \in z)\} \vdash \exists z \forall v(v \in z \leftrightarrow (v = a \vee v = b))$ in the language $\mathcal{L}'$. We may apply the deduction theorem to obtain

(49) $\Gamma \vdash \exists z(a \in z \wedge b \in z) \to \exists z \forall v(v \in z \leftrightarrow (v = a \vee v = b))$ DT

(50) $\Gamma \vdash \forall x \forall y \exists z(x \in z \wedge y \in z)$ Γ

(51) $\Gamma \vdash [\forall x \forall y \exists z(x \in z \wedge y \in z)] \to [\exists z(a \in z \wedge b \in z)]$ Ax 2, twice

(52) $\Gamma \vdash \exists z(a \in z \wedge b \in z)$ MP 50, 51

(53) $\Gamma \vdash \exists z \forall v(v \in z \leftrightarrow (v = a \vee v = b))$ MP 49, 52

But, since a, b are not in the signature of $\mathcal{L}$, we may apply Generalization on Constants to conclude that $\Gamma \vdash \forall x \forall y \exists z \forall v(v \in z \leftrightarrow (v = x \vee v = y)) \equiv Pair^{\#}$ in the original language $\mathcal{L}$. Thus $\{Cmpr, Pair\} \vdash Pair^{\#}$. □