SET THEORY HOMEWORK 1

CHASE MEADORS

Problem (6)**.** Prove that $\{\mathsf{Pair}^\sharp, \mathsf{Fnd}\} \vdash \forall x\ \neg x \in x$ by formalizing the informal proof given in class.

The statements in question are:

$$\mathsf{Pair}^\sharp := \forall x\ \forall y\ \exists p\ \forall u\ (u \in p \leftrightarrow (u = x \vee u = y))$$

$$\mathsf{Fnd} := \forall x\ ((\exists z\ z \in x) \to \exists y\ (y \in x \wedge \neg \exists z\ (z \in x \wedge z \in y)))$$

First we establish the following metatheorem about our proof system:

Lemma. If $\Gamma \vdash \exists x\ \phi$, then Γ is inconsistent if and only if $\Gamma' = \Gamma \cup \{\mathrm{Sub}^x_c \phi\}$ is inconsistent in the language with an additional constant symbol c.

Proof. The forward direction is clear. Conversely, suppose $\Gamma' \vdash \psi, \neg\psi$. Even if ψ mentions c, we may use ex falso to obtain contradictory formulas that don't:

(1)	ψ	(hyp)
(2)	$\neg\psi$	(hyp)
(3)	$\psi \to (\neg\psi \to \neg x = x)$	(Ax1 (tautology))
(4)	$\neg x = x$	((3), (1), (2), MP twice)
(5)	$x = x$	(Ax5)

Since (4) and (5) do not mention c, by Existential Instantiation they are proper theorems of $\Gamma \cup \{\exists x\ \phi\}$, and Γ is inconsistent. □

We will prove the claim by contradiction, using the following:

Claim. $\Gamma = \{\mathsf{Pair}^\sharp, \mathsf{Fnd}, \exists x\ x \in x\}$ is inconsistent.

Proof. By the lemma, it suffices to show $\Gamma' = \{\mathsf{Pair}^\sharp, \mathsf{Fnd}, c \in c\}$ is inconsistent in the language with an additional constant symbol c. First consider the following deduction from Γ':

(1)	$\forall x\ \forall y\ \exists p\ \forall u\ (u \in p \leftrightarrow (u = x \vee u = y))$	($\mathsf{Pair}^\sharp$)
(2)	$(1) \to \forall y\ \exists p\ \forall u\ (u \in p \leftrightarrow (u = c \vee u = y))$	(Ax2)

$$
\begin{array}{lll}
(3) & \forall y\, \exists p\, \forall u\, (u \in p \leftrightarrow (u = c \vee u = y)) & ((2), (1), \text{MP}) \\
(4) & (3) \rightarrow \exists p\, \forall u\, (u \in p \leftrightarrow (u = c \vee u = c)) & (\text{Ax2}) \\
(5) & \exists p\, \forall u\, (u \in p \leftrightarrow (u = c \vee u = c)) & ((4), (3), \text{MP})
\end{array}
$$

By the same argument, we may adjoin another constant symbol p and show

$$\Gamma'' = \left\{\mathsf{Pair}^\sharp, \mathsf{Fnd}, c \in c, \forall u\, (u \in p \leftrightarrow (u = c \vee u = c))\right\}$$

is inconsistent. We have the following deduction from Γ'':

$$
\begin{array}{lll}
(1) & \forall x\, ((\exists z\, z \in x) \rightarrow \exists y\, (y \in x \wedge \neg\exists z\, (z \in x \wedge z \in y))) & (\mathsf{Fnd}) \\
(2) & (1) \rightarrow ((\exists z\, z \in p) \rightarrow \exists y\, (y \in p \wedge \neg\exists z\, (z \in p \wedge z \in y))) & (\text{Ax2}) \\
(3) & (\exists z\, z \in p) \rightarrow \exists y\, (y \in p \wedge \neg\exists z\, (z \in p \wedge z \in y)) & ((2), (1), \text{MP}) \\
(4) & \forall u\, (u \in p \leftrightarrow (u = c \vee u = c)) & (\text{hyp}) \\
(5) & (4) \rightarrow (c \in p \leftrightarrow (c = c \vee c = c)) & (\text{Ax2}) \\
(6) & c \in p \leftrightarrow (c = c \vee c = c) & ((5), (4), \text{MP}) \\
(7) & (6) \rightarrow (c = c \rightarrow c \in p) & (\text{Ax1 (tautology)}) \\
(8) & c = c & (\text{Ax5}) \\
(9) & c \in p & ((7), (6), (8), \text{MP twice}) \\
(10) & (\forall z\, \neg z \in p) \rightarrow \neg(c \in p) & (\text{Ax2}) \\
(11) & (10) \rightarrow (c \in p \rightarrow \exists z\, z \in p) & (\text{Ax1 (tautology)}) \\
(12) & \exists z\, z \in p & ((11), (10), (9), \text{MP twice}) \\
(13) & \exists y\, (y \in p \wedge \neg\exists z\, (z \in p \wedge z \in y)) & ((3), (12), \text{MP})
\end{array}
$$

Finally, we apply the lemma a third time, to adjoin a constant symbol y and show that

$$\Gamma''' = \left\{\mathsf{Pair}^\sharp, \mathsf{Fnd}, c \in c, \forall u\, (u \in p \leftrightarrow (u = c \vee u = c)), y \in p \wedge \neg\exists z\, (z \in p \wedge z \in y)\right\}$$

is inconsistent. Indeed, we have the following derivation from Γ''':

$$
\begin{array}{lll}
(1) & y \in p \wedge \neg\exists z\, (z \in p \wedge z \in y) & (\text{hyp}) \\
(2) & (1) \rightarrow y \in p & (\text{Ax1 (tautology)}) \\
(3) & (1) \rightarrow \neg\exists z\, (z \in p \wedge z \in y) & (\text{Ax1 (tautology)}) \\
(4) & y \in p & ((2), (1), \text{MP}) \\
(5) & \neg\exists z\, (z \in p \wedge z \in y) & ((3), (1), \text{MP}) \\
(6) & (5) \rightarrow \forall z\, \neg(z \in p \wedge z \in y) & (\text{Ax1 (tautology)}) \\
(7) & \forall z\, \neg(z \in p \wedge z \in y) & ((6), (5), \text{MP}) \\
(8) & (7) \rightarrow \neg(c \in p \wedge c \in y) & (\text{Ax2})
\end{array}
$$

(9)	$\neg(c \in p \wedge c \in y)$	((8), (7), MP)
(10)	$\forall u\ (u \in p \leftrightarrow (u = c \vee u = c))$	(hyp)
(11)	$\forall u\ (u \in p \leftrightarrow (u = c \vee u = c)) \to y \in p \leftrightarrow (y = c \vee y = c)$	(Ax2)
(12)	$y \in p \leftrightarrow (y = c \vee y = c)$	((11), (10), MP)
(13)	$(12) \to (y \in p \to y = c)$	(Ax1 (tautology))
(14)	$y = c$	((13), (12), (4), MP twice)
(15)	$(14) \to ((4) \to c \in p)$	(Ax6)
(16)	$c \in p$	((15), (14), (4), MP twice)
(17)	$c \in c$	(hyp)
(18)	$y = y$	(Ax5)
(19)	$(14) \to ((18) \to c = y)$	(Ax6)
(20)	$c = y$	((19), (14), (18), MP twice)
(21)	$(20) \to ((17) \to c \in y)$	(Ax6)
(22)	$c \in y$	((21), (20), (17), MP twice)
(23)	$(16) \to ((22) \to (c \in p \wedge c \in y))$	(Ax1 (tautology))
(24)	$c \in p \wedge c \in y$	((23), (16), (22), MP twice)

where (9) is the negation of (24). □

Corollary. $\left\{\mathsf{Pair}^\sharp, \mathsf{Fnd}\right\} \vdash \forall x\ \neg x \in x$

Proof. We have $\exists x\ x \in x = \neg\forall x\ \neg x \in x$ by definition, so by Proof by Contradiction, this follows from the previous claim. □