HOMEWORK 1

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Prove the Following Statements to be Equivalent in ZFC\{Fnd}

Fnd

$$\phi \equiv \neg\exists f(f \text{ is a function} \wedge \text{ dmn}(f) = \omega \wedge \forall n \in \omega(f(n+1) \in f(n)))$$

Proof. To prove that these to statments are equivalent under ZFC\{Fnd}, we will see that ZFC\{Fnd}∪{Fnd}=ZFC⊢ ϕ and ZFC\{Fnd}∪{ϕ} ⊢Fnd. To prove the former, we will utilize a proof by contradiction and show that ZFC∪{¬ϕ} is inconsistent.

Take the set of premises ZFC∪{¬ϕ}. From ¬ϕ it follows that $\exists f(f \text{ is a function} \wedge \text{ dmn}(f) = \omega \wedge \forall n(n \in \omega \to f(n+1) \in f(n)))$. But rng($f$) is also a set. Since, the domain is ω, we know that rng(f) is non-empty. So, applying the axiom of foundations to f, we see that there is some $x \in \text{rng}(f)$ such that $x \cap \text{rng}(f) = \varnothing$. But, we have that $x = f(n)$ for some $n \in \omega$. By the definition of f, $f(n+1) \in x$. But $f(n+1) \in \text{rng}(f)$ implying $f(n+1) \in x \cap \text{rng}(f) \neq \varnothing$, a contradiction. Since ZFC∪{¬ϕ} is inconsistent, we may conclude that ZFC⊢ ¬¬ϕ, i.e. ZFC⊢ ϕ.

Now, to prove that ZFC\{Fnd} ∪ {ϕ} ⊢ Fnd, we will prove the contrapositive, i.e. demonstrating that ZFC\{Fnd} ∪ {¬Fnd} ⊢ ¬ϕ. So, take the premises ZFC\{Fnd} ∪ {¬Fnd}. From ¬Fnd, it follows that $\exists x(x \neq \varnothing \wedge \forall y \in x(y \cap x \neq \varnothing))$. Denote such a set with the symbol A. Using the General Recursion Theorem, we will construct a function f with $\text{rng}(f) \subseteq A$ satisfying ¬ϕ. In order to construct this function, we will invoke the axiom of choice. We will use the Choice Function Principle, shown to be equivalent to the axiom of choice. Since A is non-empty, $\mathscr{P}(A)\backslash\{\varnothing\}$ is a family of non-empty subsets of A. Let C be a choice function on $\mathscr{P}(A)\backslash\{\varnothing\}$. Define $G : \omega \times V \to V$ by

$$G(n,v) = \begin{cases} A & \text{if } n = 0 \\ C(v(m) \cap A) & \text{if } n = m+1,\ v \text{ is a function with dmn} = n, \text{ and } v(m) \cap A \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases}$$

By the recursion theorem, there exists an $f : \omega \to V$ such that $f(n) = G(n, f\restriction_n)$, for all $n \in \omega$. Clearly f is a function and $\text{dmn}(f) = \omega$. We will prove by induction that f satisfies $\forall n \in \omega(f(n+1) \in f(n))$. Consider the set $S = \{n \in \omega : f(n+1) \in f(n)\}$. We have that $f(0) = A$. Additionally, $f(1) = G(1, f\restriction_1)$. Clearly $1 = 0+1$, $f\restriction_1$ is a function with domain 1 and $f(0) \cap A = A$ is non-empty, so $f(1) = C(f\restriction_1(0) \cap A) = C(A) \in A$. Thus $f(1) \in f(0)$ and $0 \in S$. Take $n \neq 0$ and assume that $n \in S$. Since $f(n+1) \in f(n) \neq \varnothing$, we may conclude that $f(n) = G(n, f\restriction_n) = C(f\restriction_n(m) \cap A)$, where $n = m + 1$. Thus $f(n) \in A$. By the selection of A, $f(n) \cap A \neq \varnothing$. It follows that $f(n+1) = G(n+1, f\restriction_{(n+1)}) = C(f\restriction_{(n+1)}(n) \cap A)$. So $f(n+1) \in A$. It follows that $f(n+1) \cap A \neq \varnothing$. Now consider $f(n+2) = G(n+2, f\restriction_{(n+2)})$. Clearly $n+2 = (n+1)+1$, $f\restriction_{(n+2)}$ is a function with dmn= $n+2$, and $f\restriction_{(n+2)}(n+1) \cap A \neq \varnothing$. So, $f(n+2) = C(f\restriction_{(n+2)}(n+1) \cap A)$. It follows that $f(n+2) \in f(n+1)$. Thus, $n+1 \in S$. By the inductive principle for ω, $S = \omega$. So, $\forall n \in \omega (f(n+1) \in f(n))$. Therefore, $\exists f (f$ is a function $\wedge$ $\text{dmn}(f) = \omega \wedge \forall n \in \omega (f(n+1) \in f(n)))$, which implies $\neg\neg\exists f (f$ is a function $\wedge$ $\text{dmn}(f) = \omega \wedge \forall n \in \omega (f(n+1) \in f(n)))$. Thus, $\text{ZFC} \setminus \{\text{Fnd}\} \cup \{\neg\text{Fnd}\} \vdash \neg\phi$. So we may conclude that $\text{ZFC} \setminus \{\text{Fnd}\} \cup \{\phi\} \vdash$ Fnd.

Consequently, Fnd and ϕ are equivalent under $\text{ZFC} \setminus \{\text{Fnd}\}$. $\square$