Set Theory (MATH 6730)

HOMEWORK 2

Problems:

3. Mateo

(i) Prove in ZF that $\omega \times \omega$ and $n \times \omega$ ($n \in \omega \setminus \{0\}$) are equipotent with ω.

(ii) Prove in ZF that the union of a finite set of countable sets is countable. (You may use the statement proved in Problem 2.)

(iii) Modify your arguent for (ii) to prove in ZFC that the union of a countable set of countable sets is countable.

Proof. We start (i) by proving the equipotence of $\omega \times \omega$ and ω. We have for every $n \in \omega \setminus \{0\}$, $n = 2^{x_n}(2y_n + 1)$ for uniquely determined $x_n, y_n \in \omega$. Therefore we have a well-defined function $f : \omega \setminus \{0\} \to \omega \times \omega$ defined by $f(n) = (x_n, y_n)$ for all $n \in \omega \setminus \{0\}$. To show that f is injective, suppose $f(n) = f(m)$. Then $(x_n, y_n) = (x_m, y_m)$, hence $n = 2^{x_n}(2y_n + 1) = 2^{x_m}(2y_m + 1) = m$. f is also surjective, because for any $(x, y) \in \omega \times \omega$, we have that $2^x(2y + 1)$ is a nonzero natural number and $f(2^x(2y + 1)) = (x, y)$. Of course, the bijection we want is $g : \omega \to \omega \times \omega$ where $g(n) = f(n + 1)$. We know the successor function $n \mapsto n + 1$ is a bijection between ω and $\omega \setminus \{0\}$, so we have that g is a bijection between ω and $\omega \times \omega$.

Now we fix an $n \in \omega \setminus \{0\}$ and prove the equipotence of n and ω. For every $m \in \omega$, there exists unique $(r, q) \in n \times \omega$ such that $r + nq = m$, by the Divsion Algorithm. So the assignment $m \to (r, q)$ yields a function $h : \omega \to n \times \omega$. If $f(x) = (x_1, x_2) = (y_1, y_2) = f(y)$, then $x = x_1 + nx_2 = y_1 + ny_2 = y$, so f is injective. For every $(x, y) = n \times \omega$, we have $f(x + ny) = (x, y)$, hence f is surjective. So f is a bijection between ω and $n \times \omega$.

We prove (ii). Here we say a set S is countable if and only if there exists a surjection $g : \omega \to S$. Let A be a finite set of countable sets. Since A is finite, there exists $n \in \omega$ and a bijection $\varphi : n \to A$. For each $i \in n$, let $A_i := \varphi(i)$. Thus, $A = \{A_i : i \in n\}$ and each A_i is countable. Let $A' = \cup A$, the set whose existence is guaranteed by the axiom of union. Then we want to show there exists a surjection $g : \omega \to A'$. Since the composition of surjective functions is surjective, and we showed in (i) there exists a surjection $\omega \to n \times \omega$, it will suffice to show there exist a surjection $g : n \times \omega \to A'$. Define $\psi(x, y)$ to be the first-order formula that says y is the set of all functions such that $dom(f) = \omega$ and $ran(f) = x$, i.e.

$$\psi(x,y) \equiv \forall f(f \in y \leftrightarrow ((f \text{ is a function}) \wedge (dom(f) \text{ is the least inductive set}) \wedge (ran(f) = x))).$$

Here "f is a function", "$dom(f)$ is the least inductive set", and "$ran(f) = x$" are abbreviations . For example, "$dom(f)$ is the least inductive set" can be further expanded as follows.

$$\forall z(z \in dom(f) \leftrightarrow \forall v(v \text{ is inductive} \to z \in v)).$$

$\psi(x, y)$ defines a class function $\mathbb{J} : \mathbb{V} \to \mathbb{V}$ that maps each $x \in \mathbb{V}$ to the set of all surjections $\omega \to x$. Let $H_i = \mathbb{J}(A_i)$ for all $i \in n$. By the axioms of replacement and comprehension, we have that $\mathbb{J}[A] = \{H_i : i \in n\}$ is a set. To simplify notation, we denote this set by H. Each H_i is non-empty by assumption that each A_i is countable. A choice function for H is a function F with domain H such that $F(H_i) \in H_i$ for all $i \in n$. H is finite, so such a choice function exists by Problem 2. Let $h_i = F(H_i)$ for each $i \in n$. Then we define the function $g : n \times \omega \to A'$ via $g(x, y) = h_x(y)$ for all $x \in n$ and $y \in \omega$. The function g is surjective, for take any $a \in A'$. Then for some $x \in n$, $a \in A_x$ and $h_x : \omega \to A_x$ is a surjective function. There exists some $y \in \omega$ such that $g(x, y) = h_x(y) = a$ and we are done.

We prove (iii). Assume the axiom of choice, or more efficiently, assume the axiom of countable choice. Let A be a countable set of countable sets. We have shown the case where A is finite, so here we assume that there exists a bijection $\varphi : \omega \to A$. Let $A_i = \varphi(i)$ for every $i \in \omega$. Thus $A = \{A_i : i \in \omega\}$ and each A_i is countable. Let $A' = \bigcup A$, which is a set by the axiom of union. Then we want to show there exists a surjection $\omega \to A'$. It suffices to show there exists a surjection $\omega \times \omega \to A'$, by (i). Define $\psi(x, y)$ as before so that it defines the same class function $\mathbb{J}$. Let $H_i = \mathbb{J}(A_i)$ for all $i \in \omega$. By the axioms of replacement and comprehension, we have that $\mathbb{J}[A] = \{H_i : i \in \omega\}$ is a set. To simplify notation, we denote this set H. A choice function for H is a function F with domain H such that $F(H_i) \in H_i$ for all $i \in \omega$. Such a choice function exists by the axiom of countable choice. Let $h_i = F(H_i)$ for each $i \in \omega$. Then the function $g : \omega \times \omega \to A'$ defined by $g(x, y) = h_x(y)$ is a surjection for the same reasons as before.

$\square$
