Set Theory HW 2

Toby Aldape

March 2021

Let μ be an infinite cardinal, and let $\prec$ be the well-ordering of $\mu \times \mu$ we used earlier[1], which is defined as follows: for any $\delta, \epsilon, \delta', \epsilon' \in \mu$,

$$(\delta, \epsilon) \prec (\delta', \epsilon') \iff \begin{array}{l} \max(\delta, \epsilon) < \max(\delta', \epsilon'), \text{ or} \\ \max(\delta, \epsilon) = \max(\delta', \epsilon') \text{ and } \delta < \delta', \text{ or} \\ \max(\delta, \epsilon) = \max(\delta', \epsilon'), \ \delta = \delta' \text{ and } \epsilon < \epsilon'. \end{array}$$

(i) Show that $(\mu \times \mu, \prec)$ and $(\mu, <)$ are isomorphic well-ordered sets.

Proof. Since $(\mu \times \mu, \prec)$ is a well-ordered set of the same cardinality as $(\mu, <)$, it suffices to show that $(\mu \times \mu, \prec)$ has the order isomorphism type of a cardinal. To do this, it suffices to show that for any $(\alpha, \beta) \in \mu \times \mu$, the set $\text{pred}_{\mu \times \mu, \prec}(\alpha, \beta)$ has strictly smaller cardinality than μ. But we have

$$\text{pred}_{\mu \times \mu, \prec}(\alpha, \beta) \subseteq (\max(\alpha, \beta) + 1) \times (\max(\alpha, \beta) + 1),$$

so

$$|\text{pred}_{\mu \times \mu, \prec}(\alpha, \beta)| \leq |(\max(\alpha, \beta) + 1) \times (\max(\alpha, \beta) + 1)| < \mu \cdot \mu = \mu.$$

□

(ii) Use the statement in part (i) and the General Associative Law for cardinal multiplication to prove Theorem 4.7 on the handout "The Axiom of Choice. Cardinals and Cardinal Arithmetic".

Proof. Suppose that μ is an infinite cardinal and $\langle \kappa_\alpha : \alpha < \mu \rangle$ is a system of nonzero cardinals such that $\kappa_\alpha \leq \kappa_\beta$ whenever $\alpha < \beta < \mu$. We have to show that

$$\prod_{\alpha < \mu}^{c} \kappa_\alpha = (\bigcup_{\alpha < \mu} \kappa_\alpha)^\mu.$$

[1]See the proof of Theorem 4.1 on the handout "The Axiom of Choice. Cardinals and Cardinal Arithmetic".

It is easy to see that

$$\prod_{\alpha<\mu}^{c} \kappa_\alpha \leq (\bigcup_{\alpha<\mu} \kappa_\alpha)^\mu.$$

Let $\phi : (\mu\times\mu, \prec) \to (\mu, <)$ be an order isomorphism (existence guaranteed by part (i)). For any $\beta < \mu$, the sequence $(\beta, \gamma)_{\gamma<\mu}$ is unbounded in $\mu \times \mu$. Therefore $(\phi(\beta, \gamma))_{\gamma<\mu}$ is an unbounded sequence in μ. Then by monotonicity,

$$\prod_{\gamma<\mu}^{c} \kappa_{\phi(\beta,\gamma)} \geq \bigcup_{\gamma<\mu} \kappa_{\phi(\beta,\gamma)} = \bigcup_{\alpha<\mu} \kappa_\alpha.$$

Then by the General Associative Law,

$$\prod_{\alpha<\mu}^{c} \kappa_\alpha = \prod_{(\beta,\gamma)\in\mu\times\mu}^{c} \kappa_{\phi(\beta,\gamma)} = \prod_{\beta<\mu}^{c} \prod_{\gamma<\mu}^{c} \kappa_{\phi(\beta,\gamma)} \geq \prod_{\beta<\mu}^{c} \bigcup_{\alpha<\mu} \kappa_\alpha = (\bigcup_{\alpha<\mu} \kappa_\alpha)^\mu.$$

□