Set Theory (MATH 6730)

HOMEWORK 3

(First draft is due on April 12, 2021)

Definition. Let A be a set.

- A is called *Dedekind finite* if A is not equipotent with a proper subset of A (i.e., if there is no bijection between A and a proper subset of A).
- A is *Dedekind infinite* if it is not Dedekind finite.

Problems:

1. **Connor**
   Prove in ZF that a set A is Dedekind infinite if and only if there exists a one-to-one function $\omega \to A$.
2. **Nick**
   Use the result of Problem 1 to prove in ZF that if $\mathcal{A}$ is a Dedekind finite set of pairwise disjoint Dedekind finite sets, then $\bigcup \mathcal{A}$ is Dedekind finite. Explain why your proof would not work in ZF if the disjointness assumption was omitted.
3. **Mateo**
   (i) Show that if $\langle \kappa_\alpha : \alpha < \beta \rangle$ is a strictly increasing sequence of cardinals and $\kappa = \bigcup_{\alpha<\beta} \kappa_\alpha$, then $\mathrm{cf}(\kappa) = \mathrm{cf}(\beta)$.
   (ii) Let $\lambda < \lambda'$ be infinite cardinals. Use the statement in part (i) to construct a strictly increasing sequence $\langle \kappa_\alpha : \alpha < \beta \rangle$ of cardinals (for an appropriate choice of β) such that for $\kappa = \bigcup_{\alpha<\beta} \kappa_\alpha$ we have that
   $$\kappa^\lambda < \kappa^{\lambda'}.$$
4. **Dale**
   Prove that every Suslin tree has exactly 2^ω branches.
5. **Toby**
   Let $(S, \subset)$ be the Aronszajn tree constructed in the proof of Theorem 9 in the lecture notes "Trees". Show that there exists a function f mapping S into the set of real numbers such that for all $s, t \in S$ with $s < t$ we have $f(s) < f(t)$.
6. **Chase**
   Let κ be an uncountable regular cardinal, and view κ as a topological space where the open sets are the unions of intervals[1]. Prove that for every metric space X and continuous function $h\colon \kappa \to X$ there is a $\beta < \kappa$ such that $h(\alpha) = h(\beta)$ for all $\alpha \geq \beta$.
   Hint: Let S be the set of limit ordinals in κ. Find regressive functions $f_n\colon S \to \kappa$ ($n \in \omega$) such that all subsets of X of the form $\{h(\alpha) : f_n(\xi) \leq \alpha \leq \xi\}$ ($\xi < \kappa$) have diameter $< 1/n$.

[1] See Definition 16 in the lecture notes "Trees".

7. Raymond

Let $(\mathbb{R}, \leq)$ be the set of reals with the usual ordering. We will say that a subset B of $\mathbb{R}$ is well-ordered if the restriction of $<$ to B is a well-order on B. Let us fix an n-ary function $f\colon \mathbb{R}^n \to \mathbb{R}$ ($n \in \omega \setminus \{0\}$) which is (weak) order preserving; i.e., satisfies $f(x_0, \ldots, x_{n-1}) \leq f(y_0, \ldots, y_{n-1})$ whenever $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1} \in \mathbb{R}$ are such that $x_0 \leq y_0, \ldots, x_{n-1} \leq y_{n-1}$.

Prove that if $A_0, \ldots, A_{n-1}$ are well-ordered subsets of $\mathbb{R}$, then the subset

$$f[A_0, \ldots, A_{n-1}] = \{f(a_0, \ldots, a_{n-1}) : a_0 \in A_0, \ldots, a_{n-1} \in A_{n-1}\}$$

of $\mathbb{R}$ is also well-ordered.

Please write your solution in TeX, include your name at the top of the first page, and email your solution to me (`szendrei@colorado.edu`) as a pdf file so that the file name includes the string

hw3pr$k$

where k is the problem number.