Set Theory (MATH 6730)

HOMEWORK 3

(First draft is due on April 12, 2021)

Problems:

3. (i) Show that if $\langle \kappa_\alpha : \alpha < \beta \rangle$ is a strictly increasing sequence of cardinals and $\kappa = \bigcup_{\alpha<\beta} \kappa_\alpha$, then $\mathrm{cf}(\kappa) = \mathrm{cf}(\beta)$.

(ii) Let $\lambda < \lambda'$ be infinite cardinals. Use the statement in part (i) to construct a strictly increasing sequence $\langle \kappa_\alpha : \alpha < \beta \rangle$ of cardinals (for an appropriate choice of β) such that for $\kappa = \bigcup_{\alpha<\beta} \kappa_\alpha$ we have that

$$\kappa^\lambda < \kappa^{\lambda'}.$$

Proof. We prove (i). Let $A = \{\kappa_\alpha : \alpha < \beta\}$. Since $\langle \kappa_\alpha : \alpha < \beta \rangle$ is strictly increasing, A is order isomorphic to β. Then $\mathrm{cf}(A) = \mathrm{cf}(\beta)$. Now note that $\kappa = \sup(A)$. So any unbounded set in A is also unbounded in κ. Hence $\mathrm{cf}(\kappa) \leq \mathrm{cf}(A)$.
Let $\lambda = \mathrm{cf}(\kappa)$. Let $f : \lambda \to \kappa$ be a strict order preserving function such that $f[\lambda]$ is unbounded in κ. The existence of such a function is asserted in Theorem 4.11(i). We define $\mathbf{G} : \lambda \times \mathbf{V} \to A \cup \emptyset$ as follows.

$$\mathbf{G}(\alpha, x) = \begin{cases} \min\{a \in A : a > x(\gamma),\ \forall \gamma < \alpha \text{ and } a > f(\alpha)\} & \text{if } \alpha < \lambda \text{ and} \\ & x \text{ is a function with domain } \alpha \text{ and} \\ & \{a \in A : a > x(\gamma),\ \forall \gamma < \alpha \\ & \text{ and } a > f(\alpha)\} \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

By the Transfinite Recursion Theorem, there exists a class function $\mathbf{F} : \lambda \to A \cup \emptyset$ such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$ for $\alpha < \lambda$. We have that $\mathbf{F}(\alpha) \in A$ for all $\alpha < \lambda$. We prove this by transfinite induction. Assume $\alpha < \lambda$ and $\mathbf{F}(\gamma) \in A$ for all $\gamma < \alpha$. To prove $\mathbf{F}(\alpha) \in A$, it suffices to show the set $\{a \in A : a > \mathbf{F}(\gamma),\ \forall \gamma < \alpha \text{ and } a > f(\alpha)\}$ is nonempty. The set $\mathbf{F}[\alpha]$ has cardinality at most $|\alpha| < \lambda \leq \mathrm{cf}(A)$. $\mathbf{F}[\alpha]$ cannot be unbounded in A, so $\mathbf{F}[\alpha]$ has an upper bound in A, say κ_τ, where $\tau < \beta$. Also, $f(\alpha) \in \kappa$, meaning $f(\alpha) \in \kappa_\sigma$ for some $\sigma < \beta$. Then $\max\{\kappa_{\tau+1}, \kappa_{\sigma+1}\} \in \{a \in A : a > \mathbf{F}(\gamma),\ \forall \gamma < \alpha \text{ and } a > f(\alpha)\}$ and $\mathbf{F}(\alpha) \neq \emptyset$. We have that $\mathbf{F}$ is strictly increasing. Take any $\alpha < \alpha'$. Then $\mathbf{F}(\alpha') = \min\{a \in A : a > \mathbf{F}(\gamma), \forall \gamma < \alpha' \text{ and } a > f(\alpha')\} > \mathbf{F}(\alpha)$.

Now we want to show that $\mathbf{F}[\lambda]$ is unbounded in A. Suppose there exists some $a \in A$ such that $a > \mathbf{F}(\alpha)$ for all $\alpha < \lambda$. We have that

$$\mathbf{F}(\alpha) = \min\{a \in A : a > x(\gamma),\ \forall \gamma < \alpha \text{ and } a > f(\alpha)\} > f(\alpha)$$

for all $\alpha < \lambda$. So $a > \mathbf{F}(\alpha) > f(\alpha)$ for all $\alpha \in \lambda$. Then $a < \kappa$ is an upper bound of $f[\lambda]$, a contradiction. Hence there can be no such bound $a \in A$. So we have that $\mathbf{F}[\lambda]$ is an unbounded subset of A with cardinality at most $\lambda = \mathrm{cf}(\kappa)$. Then $\mathrm{cf}(A) \leq \mathrm{cf}(\kappa)$.

We now construct the example in (ii). Let $\lambda < \lambda'$ be a pair of fixed infinite cardinals. We define $\mathbf{G} : \lambda^+ \times \mathbf{V} \to \mathbf{On}$ as follows

$$\mathbf{G}(\alpha, x) = \begin{cases} \lambda' & \text{if } \alpha = 0 \\ (x(\gamma)^\lambda)^+ & \text{if } \alpha = \gamma + 1 < \lambda^+ \\ & \quad \text{and } x \text{ is a function with domain } \alpha \text{ and cardinal values} \\ \bigcup_{\gamma<\alpha} x(\gamma) & \text{if } \alpha < \lambda^+ \text{ is a limit ordinal} \\ & \quad \text{and } x \text{ is a function with domain } \alpha \text{ and cardinal values} \\ \emptyset & \text{otherwise} \end{cases}$$

By the Transfinite Recursion Theorem, there exists a class function $\mathbf{F} : \lambda^+ \to \mathbf{Card}$ such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$ for all $\alpha < \lambda^+$. All values of $\mathbf{G}$ were cardinals, so $\mathbf{F}$ has cardinal values. Note that $\mathbf{F}$ is continuous since for limit ordinals α we have that $\mathbf{F}(\alpha) = \bigcup_{\gamma<\alpha} \mathbf{F}(\gamma)$. Also, for every successor ordinal $\alpha + 1$, we have $\mathbf{F}(\alpha + 1) = (\mathbf{F}(\alpha)^\lambda)^+ > \mathbf{F}(\alpha)$. Then by Theorem 5.2(ii) of the ordinal lecture notes, we have that $\mathbf{F}$ is strictly increasing.

Denote $\mathbf{F}(\alpha)$ by κ_α for $\alpha < \lambda^+$. Let $\kappa = \bigcup_{\alpha<\lambda^+} \kappa_\alpha$. Clearly, $\kappa > 2$ and $\kappa > \lambda' > \lambda$. Let $\mu < \kappa$ be a cardinal. Then there must exist some $\alpha < \lambda^+$ with $\kappa_\alpha > \mu$. Then $\mu^\lambda \leq \kappa_\alpha^\lambda < \kappa_{\alpha+1} < \kappa$. By part (i), we have that $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda^+) = \lambda^+ > \lambda$. By the Main Theorem of Cardinal Arithmetic(iii)$_2$ we have that $\kappa^\lambda = \kappa$. Since $\mathrm{cf}(\kappa) = \lambda^+ \leq \lambda'$, by the Main Theorem of Cardinal Arithmetic(iii)$_1$ we have $\kappa^{\lambda'} = \kappa^{\mathrm{cf}(\kappa)} = \kappa^{\lambda^+}$. So using König's Theorem on cofinality, we have $\kappa^\lambda = \kappa < \kappa^{\mathrm{cf}(\kappa)} = \kappa^{\lambda^+} \leq \kappa^{\lambda'}$, as desired. $\square$