SET THEORY HOMEWORK 3

CHASE MEADORS

Problem 6. Let κ be an uncountable regular cardinal, and view κ as a topological space where the open sets are the unions of intervals. Prove that for every metric space (X,ρ) and continuous function $h:\kappa \to X$ there is a $\beta < \kappa$ such that $h(\alpha) = h(\beta)$ for all $\alpha \geq \beta$.

For any $\alpha \neq 0$ in κ and $n \in \omega \setminus \{0\}$, consider the set $O_{n,\alpha} = h^{-1}(B_{\alpha,n})$, where $B_{n,\alpha} = B\left(h(\alpha), \frac{1}{2n}\right)$ is an open ball around the image of α (note that the diameter of $B_{n,\alpha}$ is at most $1/n$). As h is continuous, $O_{n,\alpha}$ is open and must contain some open interval around α. Then we may define

$$\varphi_n(\alpha) = \min\{\delta : \delta < \alpha \text{ and } (\delta,\gamma) \subseteq O_{n,\alpha} \text{ for some } \gamma > \alpha\}$$

With this, we obtain a countable family of regressive functions $\kappa \to \kappa$ (define $\varphi_n(0) = 0$). Note that $\alpha \leq \beta$ implies $\varphi_n(\alpha) \leq \varphi_n(\beta)$ (if $\varphi_n(\beta) < \varphi_n(\alpha)$, the interval $(\varphi_n(\beta), \gamma)$ containing β also contains α, contradicting the minimality of $\varphi_n(\alpha)$).

Since the φ_n are regressive and κ is uncountable and regular, Fodor's lemma now gives a stationary set $S_n \subseteq \kappa$ on which φ_n is constant, which is necessarily unbounded in κ. In fact, letting $\alpha_n := \min S_n$, it must be the case that φ_n is constant on $[\alpha_n, \kappa)$. Indeed, if $\beta \geq \alpha_n$, there is a $\beta' > \beta$ in S_n with $\varphi_n(\beta') = \varphi_n(\alpha_n)$; then we have $\alpha_n \leq \beta \leq \beta'$ and thus $\varphi_n(\beta) = \varphi_n(\alpha_n)$ from the order preserving property noted earlier. This means that for any two ordinals $\beta \leq \beta'$ in $[\alpha_n, \kappa)$, there is an open interval $(\varphi_n(\alpha_n), \gamma)$ containing β' and thus β that is mapped by h into $B_{n,\beta'}$; in particular, $\rho(h(\beta), h(\beta')) < 1/n$.

Now consider $\lambda = \sup_{n<\omega} \alpha_n < \kappa$ (since κ is uncountable and regular, $\{\alpha_n\}$ cannot be unbounded in κ). For any $\beta \leq \beta'$ in $[\lambda, \kappa)$, we have $\beta, \beta' \geq \alpha_n$ for each $n \in \omega$; thus $\rho(h(\beta), h(\beta')) < 1/n$ for each n. That is, we must have $h(\beta) = h(\beta')$. In particular, $h(\beta) = h(\lambda)$ for all $\beta \geq \lambda$.