Set Theory (MATH 6730)

The Axiom of Choice. Cardinals and Cardinal Arithmetic

1. The Axiom of Choice

We will discuss several statements that are equivalent (in ZF) to the Axiom of Choice.

Definition 1.1. Let A be a set of nonempty sets. A *choice function* for A is a function f with domain A such that $f(a) \in a$ for all $a \in A$.

Theorem 1.2. *In* ZF *the following statement are equivalent:*

AC (The Axiom of Choice)
For any set $\mathcal{A}$ of pairwise disjoint, nonempty sets there exists a set C which has exactly one element from every set in $\mathcal{A}$.

CFP (Choice Function Principle)
For any set A of nonempty sets there exists a choice function for A.

WOP (Well-Ordering Principle)
For every set B there exists a well-ordering $(B, \prec)$.

ZLm (Zorn's Lemma)
If $(D, <)$ is a partial order such that
$(*)$ *every subset of D that is linearly ordered by $<$ has an upper bound in D,*[1]
then $(D, <)$ has a maximal element.

Corollary 1.3. ZFC $\vdash$ CFP, WOP, ZLm.

Proof of Theorem 1.2. AC$\Rightarrow$CFP: Assume AC, and let A be a set of nonempty sets. By the Axiom of Replacement,

$$\mathcal{A} = \{\{a\} \times a : a \in A\}$$

is a set. Hence, by AC, there is a set C which has exactly one element from every set in $\mathcal{A}$.

- C is a choice function for A.

[1]Condition $(*)$ for the empty subset of D is equivalent to requiring that $D \neq \emptyset$.

CFP$\Rightarrow$ WOP: Assume CFP, and let B be a set. We want to prove that for some ordinal α, there exists a one-to-one function $f\colon \alpha \to B$ with range B. Then it will follow that for

$$\prec\, = \{\big(f(\beta), f(\gamma)\big) : \beta < \gamma < \alpha\},$$

$(B, \prec)$ is a well-order, and f is an isomorphism from $(\alpha, <)$ onto $(B, \prec)$. The proof of the existence of α and f will be very similar to the proof of the theorem that every well-order is isomorphic to an ordinal.[2]

We will use transfinite recursion. Let h be a choice function for $\mathcal{P}(B) \setminus \{\emptyset\}$, and let us define a class function $\mathbf{G}\colon \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ as follows: for any $\gamma \in \mathbf{On}$ and $x \in \mathbf{V}$, let

$$\mathbf{G}(\gamma, x) = \begin{cases} h\big(B \setminus \operatorname{rng}(x)\big) & \text{if } x \text{ is a function and } B \setminus \operatorname{rng}(x) \neq \emptyset, \\ B & \text{otherwise.} \end{cases}$$

Note that, since $B \notin B$, the equality $\mathbf{G}(\gamma, x) = B$ for a function x implies that $B \setminus \operatorname{rng}(x) = \emptyset$.

By the Transfinite Recursion Theorem there exists a class function $\mathbf{F}\colon \mathbf{On} \to \mathbf{V}$ such that

$$\mathbf{F}(\beta) = \mathbf{G}(\beta, \mathbf{F}{\restriction}\beta) \quad \text{for all } \beta \in \mathbf{On}.$$

The proof proceeds by showing that $\mathbf{F}$ has the following properties:

- For $\beta < \gamma$ in $\mathbf{On}$,
  - if $\mathbf{F}(\beta) = B$, then $\mathbf{F}(\gamma) = B$; and
  - if $\mathbf{F}(\gamma) \neq B$ — and hence $\mathbf{F}(\beta) \neq B$ —, then $\mathbf{F}(\beta) \neq \mathbf{F}(\gamma)$.
- There exists $\gamma \in \mathbf{On}$ such that $\mathbf{F}(\gamma) = B$.
- For the least ordinal α such that $\mathbf{F}(\alpha) = B$, the function $\mathbf{F}{\restriction}\alpha$ maps α onto B.

Thus, $f = \mathbf{F}{\restriction}\alpha$ is a one-to-one function $\alpha \to B$ with range B, as required.

WOP$\Rightarrow$ AC: Let $\mathcal{A}$ be a set of pairwise disjoint, nonempty sets, and let $U = \bigcup \mathcal{A}$. Then U is a set (by the union axiom), so by WOP, there exists a well-order $(U, \prec)$. Clearly,

$$C = \{x \in U : \exists A \in \mathcal{A}\, (x \text{ is the } \prec\text{-least element of } A)\}$$

is a set (by comprehension); moreover,

- C has exactly one element from every $A \in \mathcal{A}$,

[2] See Theorem 4.4 on the handout ‘Ordinals. Transfinite Induction and Recursion’.

WOP⇒ ZLm: Let $(D,<)$ be a partial order satisfying condition $(*)$. By WOP, we also have a well-order $(D,\prec)$. Now we define a class function $\mathbf{G}\colon D\times\mathbf{V}\to\{0,1\}$ by

$$\mathbf{G}(a,x)=\begin{cases}1 & \text{if } x \text{ is a function } \mathrm{pred}_{D,\prec}(a)\to\{0,1\} \text{ and}\\ & a>b \text{ for all } b\in\mathrm{dmn}(x) \text{ with } x(b)=1,\\ 0 & \text{otherwise.}\end{cases}$$

The Recursion Theorem yields the existence of a function $F\colon D\to\{0,1\}$ such that

$$F(a)=\mathbf{G}(a,F{\restriction}\mathrm{pred}_{D,\prec}(a))\quad\text{for all } a\in D.$$

Then for the subset $S=\{a\in D: F(a)=1\}$ of D we have the following:

- For any $a,b\in S$, $b\prec a$ implies that $b<a$; hence S is linearly ordered by $<$.
- Any upper bound for S in D (which exists by $(*)$) is a maximal element of $(D,<)$.

ZLm⇒ WOP: Assume ZLm, and let B be a set. Then

$$\mathcal{D}=\{(S,\prec)\in\mathcal{P}(B)\times\mathcal{P}(B\times B): S\subseteq B,\ (S,\prec) \text{ is a well-order}\}$$

is a set (by comprehension), and the elements of $\mathcal{D}$ are exactly the well-orders $(S,\prec)$ with $S\subseteq B$. Define a relation $<$ on $\mathcal{D}$ as follows:

$$(S,\prec)<(S',\prec')\quad\text{iff}\quad S\subsetneq S',\ \prec\,=\,\prec'\cap(S\times S)\ \text{ and }\ x\prec' y \text{ for all } x\in S,\, y\in S'\setminus S,$$

which expresses that "$(S,\prec)$ is a proper initial segment of $(S',\prec')$". It follows that

- $(\mathcal{D},<)$ is a partial order, and
- for any subset $\mathcal{C}$ of $\mathcal{D}$ that is linearly ordered by $<$, $(\bigcup_{(S,\prec)\in\mathcal{C}}S,\bigcup_{(S,\prec)\in\mathcal{C}}\prec)$ is in $\mathcal{D}$, and is an upper bound for $\mathcal{C}$.

By ZLm, $(\mathcal{D},<)$ has a maximal member: $(S_0,\prec_0)$. It must be that $S_0=B$, since otherwise for any $b\in B\setminus S_0$ we would get a well-order $(S_1,\prec_1)>(S_0,\prec_0)$ in $\mathcal{D}$ by setting $S_1=S_0\cup\{b\}$ and $\prec_1\,=\,\prec_0\cup(S_0\times\{b\})$. □

2. Cardinals

From now on we will work in ZFC; in particular, we will use that WOP and CFP are theorems of ZFC.

Definition 2.1. Let A, B be sets. A *bijection* $A \to B$ is a one-to-one function with range B. We say that A and B are *equipotent* (or *equinumerous*) if there exists a bijection $A \to B$.

Definition 2.2. A *cardinal* (or *cardinal number*) is an ordinal α that is not equipotent with any ordinal $\beta < \alpha$.

Cardinals will usually be denoted by the Greek letters $\kappa, \lambda, \mu, \ldots$. Since cardinals are special ordinals, the relation $<$ for ordinals will also be used for cardinals. Recall[3], that for any ordinals α, β we have

$$\alpha < \beta \quad \iff \quad \alpha \in \beta \quad \iff \quad \alpha \subsetneq \beta,$$

so the same is also true for cardinals.

Theorem 2.3. *For every set A there is a unique cardinal equipotent with A.*

Proof. A is equipotent with an ordinal by the WOP. The smallest such ordinal is a cardinal. The uniqueness of this cardinal is clear, since different cardinals are not equipotent. ☐

Definition 2.4. The unique cardinal equipotent with a set A is called *the cardinality* (or *the size* or *the magnitude*) *of* A, and is denoted by $|A|$.

Corollary 2.5. *For every ordinal α we have that $|\alpha| \leq \alpha$; moreover,*

$$|\alpha| = \alpha \quad \textit{iff} \quad \alpha \textit{ is a cardinal.}$$

Corollary 2.6. *For any sets A and B,*

$$|A| = |B| \quad \textit{iff} \quad A \textit{ and } B \textit{ are equipotent.}$$

Theorem 2.7. *For any sets A and B the following conditions are equivalent:*

(a) $|A| \leq |B|$;
(b) *there exists a one-to-one function $A \to B$;*
(c) $A = \emptyset$ *or there exists a function $B \to A$ with range A.*

Since $<$ is irreflexive, if κ, λ are cardinals such that $\kappa \leq \lambda$ and $\lambda \leq \kappa$, then $\kappa = \lambda$. Thus, we get the following corollary from the preceding theorem.

Corollary 2.8. (Cantor–Schröder–Bernstein Theorem)[4] *Let A and B be sets. If there exist one-to-one functions $A \to B$ and $B \to A$, then there also exists a bijection $A \to B$.*

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[3]See Definition 1.5 and Theorem 1.6(ii) on the handout "Ordinals. Transfinite Induction and Recursion".
[4]This theorem can be proved in ZF. Our proof relies on the Axiom of Choice, since we made use of WOP.

Theorem 2.9.

(i) *Every natural number is a cardinal; equivalently, there exist no bijections* $k \to n$ *if* $k, n \in \omega$ *and* $k < n$.
(ii) ω *is a cardinal; equivalently, there exist no bijections* $n \to \omega$ *if* $n \in \omega$.
(iii) *Every cardinal* κ *such that* $\kappa \geq \omega$ *is a limit ordinal.*
(iv) *If* Γ *is a set of cardinals, then* $\bigcup \Gamma$ *is a cardinal.*

Definition 2.10. A set A is called *finite* if $|A|$ is a natural number. Otherwise, A is said to be *infinite*.

Corollary 2.11. *A finite set* A *is not equipotent with any proper subset of* A.

Corollary 2.12. *If* A, B *are finite sets such that* $|A| = |B|$, *then for any function* $f \colon A \to B$,

$$f \text{ is one-to-one} \iff f \text{ is a bijection} \iff f \text{ maps onto } B.$$

Theorem 2.13. *For any set* A *the following conditions are equivalent:*

(a) A *is infinite;*
(b) $|A| \geq \omega$;
(c) *there exists a one-to-one function* $\omega \to A$;
(d) A *is equipotent with a proper subset of* A.

Theorems 2.9 and 2.13 imply that ω is the least infinite cardinal, and every infinite set has a subset of cardinality ω.

Definition 2.14. Let A be a set. If $|A| \leq \omega$, we say that A is *countable*. If A is not countable (i.e., if $|A| > \omega$), then we say that A is *uncountable*. We say that A is *denumerable* (or *countably infinite*) if $|A| = \omega$.

Since ω is the least infinite cardinal, a countable set is either finite or denumerable.

Cantor's Theorem below implies that there exist uncountable cardinals; in fact, there exist infinitely many of them.

Theorem 2.15. (Cantor's Theorem) *For every set A we have that $|A| < |\mathcal{P}(A)|$.*

Example 2.16. Cardinalities of some 'familiar' sets:[5]

- ω ($= \mathbb{N}$, the set of natural numbers), $\mathbb{Z}$, and $\mathbb{Q}$ are denumerable;
- $\mathbb{R}$ and the set $C(\mathbb{R})$ of all continuous functions $\mathbb{R} \to \mathbb{R}$ have cardinality $|\mathcal{P}(\omega)|$;
- The set of all functions $\mathbb{R} \to \mathbb{R}$ has cardinality larger than $|\mathcal{P}(\omega)|$.

Let κ be a cardinal. By Cantor's Theorem, there exists a cardinal larger than κ. Therefore,

$$\{\lambda \in \mathbf{On} : \lambda \text{ is cardinal, } \lambda > \kappa\}$$

is a nonempty subclass of **On**, so it has a least element (with respect to $<$).

Definition 2.17. For every cardinal κ, the least cardinal greater than κ is called *the successor*[6] *of* κ, and is denoted by κ^+. The cardinals of the form κ^+ are referred to as *successor cardinals*. The other infinite cardinals are called *limit cardinals*.

It follows from Cantor's Theorem that

(i) $|\mathcal{P}(\omega)| \geq \omega^+$; and in fact,
(ii) $|\mathcal{P}(\kappa)| \geq \kappa^+$ for every cardinal κ.

The statement that $=$ holds in (i) is the *continuum hypothesis* (CH), while the statement that $=$ holds in (ii) for every cardinal κ is the *generalized continuum hypothesis* (GCH).

[5]Some basic facts from cardinal arithmetic are needed to show that $|C(\mathbb{R})| = |\mathcal{P}(\omega)|$. The other statements have elementary proofs.

[6]Warning: The cardinal successor κ^+ of κ is not equal to the ordinal successor $\kappa + 1$ of κ, unless $\kappa \in \omega$; cf. Theorem 2.9(iii).

Theorem 2.18. (1) *There exists a unique ordinal class function* $\aleph\colon \mathbf{On} \to \mathbf{On}$, $\alpha \mapsto \aleph_\alpha$ *such that*

(o) $\aleph_\alpha$ *is a cardinal for every* $\alpha \in \mathbf{On}$*;*
(i) $\aleph_0 = \omega$*;*
(ii) $\aleph_\alpha = \aleph_\beta^+$ *if* $\alpha = \beta + 1$ *is a successor ordinal; and*
(iii) $\aleph_\alpha = \bigcup_{\beta<\alpha} \aleph_\beta$ *if* α *is a limit ordinal.*

(2) $\aleph$ *is normal (hence one-to-one), and maps onto the class of all infinite cardinals.*

Proof. (1) The existence of $\aleph$ is proved by transfinite recursion. We define a class function $\mathbf{G}\colon \mathbf{On} \times \mathbf{V} \to \mathbf{On}$ as follows: for any $\alpha \in \mathbf{On}$ and $x \in \mathbf{V}$ let

$$\mathbf{G}(\alpha, x) = \begin{cases} \omega & \text{if } \alpha = 0, \\ x(\beta)^+ & \text{if } \alpha = \beta + 1 \text{ and} \\ & x \text{ is a function with domain } \alpha \text{ and cardinal values,} \\ \bigcup_{\beta<\alpha} x(\beta) & \text{if } \alpha \text{ is a limit ordinal and} \\ & x \text{ is a function with domain } \alpha \text{ and cardinal values,} \\ \emptyset & \text{otherwise.} \end{cases}$$

By the Transfinite Recursion Theorem, there exists a class function $\mathbf{F}\colon \mathbf{On} \to \mathbf{On}$ such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F}{\restriction}\alpha)$ for all $\alpha \in \mathbf{On}$. It follows by transfinite induction on α that $\mathbf{F}(\alpha)$ is a cardinal for every $\alpha \in \mathbf{On}$. Thus,

- $\mathbf{F}(0) = \omega$;
- $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F}{\restriction}\alpha) = \mathbf{F}(\beta)^+$ if $\alpha = \beta + 1$ is a successor ordinal; and
- $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F}{\restriction}\alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta)$ if α is a limit ordinal.

Denoting $\mathbf{F}(\alpha)$ by $\aleph_\alpha$, we see that (o)–(iv) hold. The uniqueness of $\aleph$ follows easily by transfinite induction.

(2) The normality of $\aleph$ is a consequence of Theorem 5.2(ii) on the handout "Ordinals ..." and properties (ii)–(iii) above. Thus, by Theorem 5.2(i) on the same handout,

$$(\dagger) \qquad \alpha \le \aleph_\alpha \quad \text{for all } \alpha \in \mathbf{On}.$$

Since $\aleph$ is strict order preserving, it is one-to-one. Moreover, $\aleph_\alpha \ge \aleph_0 = \omega$ for all $\alpha \in \mathbf{On}$, so $\aleph$ maps into the class of all infinite cardinals.

To see that $\aleph$ maps onto the class of all infinite cardinals, let κ be an infinite cardinal. By (†) we have $\kappa \le \aleph_\kappa < \aleph_\kappa^+ = \aleph_{\kappa+1}$, so $\mathbf{X} = \{\gamma \in \mathbf{On} : \kappa < \aleph_\gamma\}$ is a nonempty class of ordinals. Let α be the least element of $\mathbf{X}$. Then

- $\alpha \neq 0$, and
- α is not a limit ordinal.

Hence, $\alpha = \beta + 1$ for some $\beta \in \mathbf{On}$. Since $\beta < \alpha$, we have $\aleph_\beta \le \kappa < \aleph_\alpha = \aleph_{\beta+1} = \aleph_\beta^+$. Thus, $\kappa = \aleph_\beta$. □

Corollary 2.19. *The class of all cardinals is not a set.*

3. Cardinal Operations and their Elementary Properties

Definition 3.1. We define the *sum of two cardinals* κ and λ by

$$\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|.$$

If I is a set and $\widehat{\kappa} = \langle \kappa_i : i \in I \rangle$ is a *system of cardinals* (i.e., $\widehat{\kappa}$ is a function with domain I and range contained in the class of cardinal), then we define the *sum of the cardinals* κ_i $(i \in I)$ by

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} (\kappa_i \times \{i\}) \right|.$$

Warning: If κ and λ are cardinals, their sum in the cardinal sense (defined above) is, in general, different from their sum in the ordinal sense (defined earlier); cf. Theorem 3.4(viii) below. However, as part (x) of the same theorem shows, if κ and λ are both natural numbers, then the two sums coincide.

Usually it will be clear from the context which $+$ is meant.

Theorem 3.2. *Let A, B, A', B', I be sets such that $A \cap B = \emptyset = A' \cap B'$, and let $\langle A_i : i \in I \rangle$ and $\langle A'_i : i \in I \rangle$ be systems of sets such that $A_i \cap A_j = \emptyset = A'_i \cap A'_j$ for all distinct $i, j \in I$.*

(i) *If A is equipotent with A' and B is equipotent with B', then $A \cup B$ is equipotent with $A' \cup B'$.*

(ii) *If A_i is equipotent with A'_i for all $i \in I$, then $\bigcup_{i \in I} A_i$ is equipotent with $\bigcup_{i \in I} A'_i$.*

Corollary 3.3. *Let A, B, I be sets, and let $\langle A_i : i \in I \rangle$ be a system of sets.*

(ii) *We have $|A \cup B| \le |A| + |B|$. If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.*

(iii) *We have $|\bigcup_{i \in I} A_i| \le \sum_{i \in I} |A_i|$. If $A_i \cap A_j = \emptyset$ for all distinct $i, j \in I$, then $|\bigcup_{i \in I} A_i| = \sum_{i \in I} |A_i|$.*

Theorem 3.4. *Let I, J be sets. The following hold for arbitrary cardinals κ, λ, μ and for arbitrary systems $\langle \kappa_i : i \in I \rangle$, $\langle \kappa'_i : i \in I \rangle$, $\langle \lambda_{ij} : (i,j) \in I \times J \rangle$, and $\langle \mu_i : i \in 2 \rangle$ of cardinals:*

(o) $\sum_{i \in 2} \mu_i = \mu_0 + \mu_1$.

(i) (General Commutative Law) $\sum_{i \in I} \kappa_i = \sum_{i \in I} \kappa_{f(i)}$ *for arbitrary bijection* $f \colon I \to I$; *in particular,* $\kappa + \lambda = \lambda + \kappa$.

(ii) $\sum_{i \in I} \kappa_i = 0$ *if* $I = \emptyset$.

(iii) $\sum_{i \in I} \kappa_i = \sum_{i \in I,\, \kappa_i \neq 0} \kappa_i$.

(iv) $\sum_{i \in I} 1 = |I|$.

(v) (General Associative Law) $\sum_{i \in I} \big(\sum_{j \in J} \lambda_{ij} \big) = \sum_{(i,j) \in I \times J} \lambda_{ij}$; *hence* $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$.

(vi) $\sum_{i \in I} \kappa_i \leq \sum_{i \in I} \kappa'_i$ *if* $\kappa_i \leq \kappa'_i$ *for all* $i \in I$.

(vii) $\bigcup_{i \in I} \kappa_i \leq \sum_{i \in I} \kappa_i$.

(viii) $\kappa + 1 = \kappa$ *if* κ *is infinite;* $\kappa + 1 = \kappa^+$ *if* $\kappa \in \omega$.

(ix) *If* I *is finite and* $\kappa_i \in \omega$ *for all* $i \in I$, *then* $\sum_{i \in I} \kappa_i \in \omega$.

(x) *If* $\kappa, \lambda \in \omega$, *then* $\kappa + \lambda$ *in the cardinal sense is the same as* $\kappa + \lambda$ *in the ordinal sense; equivalently,* $\kappa + \lambda$ *in the cardinal sense satisfies the following conditions:*
- $\kappa + 0 = \kappa$ *for all* $\kappa \in \omega$, *and*
- $\kappa + (\lambda + 1) = (\kappa + \lambda) + 1$ *for all* $\kappa, \lambda \in \omega$.

Definition 3.5. The *Cartesian product* of a system $\langle A_i : i \in I\rangle$ of sets, denoted by $\prod_{i\in I} A_i$, is the set

$$\Big\{f \in \mathcal{P}\Big(I \times \bigcup_{i\in I} A_i\Big) : f \text{ is a function with domain } I \text{ and } f(i) \in A_i \text{ for all } i \in I\Big\}.$$

Clearly, $\prod_{i\in I} A_i = \emptyset$ if there exists $i \in I$ such that $A_i = \emptyset$. Conversely, it is easy to see that in ZFC:

$(*)$ for every system $\widehat{A} = \langle A_i : i \in I\rangle$ of nonempty sets we have that $\prod_{i\in I} A_i \neq \emptyset$.

This follows by observing that if f is a choice function for $\operatorname{rng}(\widehat{A}) = \{A_i : i \in I\}$, then $f \circ \widehat{A}$ is a member of $\prod_{i\in I} A_i$.

In fact, it is not hard to prove in ZF that the statement $(*)$ is equivalent to AC.

Definition 3.6. We define the *product of two cardinals* κ and λ by

$$\kappa \cdot \lambda = |\kappa \times \lambda|.$$

If I is a set and $\langle \kappa_i : i \in I\rangle$ is a system of cardinals, then we define the *product of the cardinals* κ_i $(i \in I)$ by

$$\prod_{i\in I}^{\mathsf{c}} \kappa_i = \Big|\prod_{i\in I} \kappa_i\Big|.$$

Theorem 3.7. *Let A, B, A', B', I be arbitrary sets, and let $\langle A_i : i \in I\rangle$ and $\langle A'_i : i \in I\rangle$ be arbitrary systems of sets.*

(i) *If A is equipotent with A' and B is equipotent with B', then $A \times B$ is equipotent with $A' \times B'$.*

(ii) *If A_i is equipotent with A'_i for all $i \in I$, then $\prod_{i\in I} A_i$ is equipotent with $\prod_{i\in I} A'_i$.*

Corollary 3.8. *For arbitrary sets A, B, I and for any system $\langle A_i : i \in I\rangle$ of sets,*

(i) $|A \times B| = |A| \cdot |B|$, *and*

(ii) $|\prod_{i\in I} A_i| = \prod^{\mathsf{c}}_{i\in I} |A_i|$.

Theorem 3.9. *Let I, J be sets. The following hold for arbitrary cardinals κ, λ, μ and for arbitrary systems $\langle \kappa_i : i \in I \rangle$, $\langle \kappa'_i : i \in I \rangle$, $\langle \lambda_{ij} : (i,j) \in I \times J \rangle$, and $\langle \mu_i : i \in 2 \rangle$ of cardinals:*

(o) $\prod^{\mathsf{c}}_{i\in 2} \mu_i = \mu_0 \cdot \mu_1$.

(i) (General Commutative Law) $\prod^{\mathsf{c}}_{i\in I} \kappa_i = \prod^{\mathsf{c}}_{i\in I} \kappa_{f(i)}$ *for arbitrary bijection* $f \colon I \to I$; *in particular,* $\kappa \cdot \lambda = \lambda \cdot \kappa$.

(ii) $\prod^{\mathsf{c}}_{i\in I} \kappa_i = 0$ *if there exists* $i \in I$ *such that* $\kappa_i = 0$; *in particular,* $\kappa \cdot 0 = 0$.

(iii) $\prod^{\mathsf{c}}_{i\in I} \kappa_i = 1$ *if* $I = \emptyset$.

(iv) $\prod^{\mathsf{c}}_{i\in I} \kappa_i = \prod^{\mathsf{c}}_{i\in I,\, \kappa_i \neq 1} \kappa_i$; *in particular,* $\kappa \cdot 1 = \kappa$.

(v) (General Associative Law) $\prod^{\mathsf{c}}_{i\in I}\big(\prod^{\mathsf{c}}_{j\in J} \lambda_{ij}\big) = \prod^{\mathsf{c}}_{(i,j)\in I\times J} \lambda_{ij}$; *hence* $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$.

(vi) $\prod^{\mathsf{c}}_{i\in I} \kappa_i \leq \prod^{\mathsf{c}}_{i\in I} \kappa'_i$ *if* $\kappa_i \leq \kappa'_i$ *for all* $i \in I$.

(vii) *If* I *is finite and* $\kappa_i \in \omega$ *for all* $i \in I$, *then* $\prod^{\mathsf{c}}_{i\in I} \kappa_i \in \omega$.

(viii) *If* $\kappa, \lambda \in \omega$, *then* $\kappa \cdot \lambda$ *satisfies the following conditions:*
- $\kappa \cdot 0 = 0$ *for all* $\kappa \in \omega$, *and*
- $\kappa \cdot (\lambda + 1) = (\kappa \cdot \lambda) + \kappa$ *for all* $\kappa, \lambda \in \omega$.[7]

Theorem 3.10. *Let I be a set. The following hold for arbitrary cardinals κ, λ, μ and for arbitrary system $\langle \lambda_i : i \in I \rangle$ of cardinals:*

(i) (General Distributive Law) $\kappa \sum_{i\in I} \lambda_i = \sum_{i\in I}(\kappa \cdot \lambda_i)$; *in particular,* $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$.

(ii) $\sum_{i\in I} \kappa = |I| \cdot \kappa$.

(iii) $\sum_{i\in I} \lambda_i \leq |I| \cdot \bigcup_{i\in I} \lambda_i$.

[7] This shows that for $\kappa, \lambda \in \omega$ the product $\kappa \cdot \lambda$ in the cardinal sense (defined above) coincides with $\kappa \cdot \lambda$ in the ordinals sense, see Section 9 (p. 93) of *Lectures on Set Theory* by J. Donald Monk.

Notation 3.11. For arbitrary sets A, B the set of all functions $A \to B$ is denoted by ${}^{A}B$.

Definition 3.12. For any cardinals κ and λ we define κ^λ by

$$\kappa^\lambda = |{}^{\lambda}\kappa|.$$

Theorem 3.13. *Let A, B, A', B' be arbitrary sets. If A is equipotent with A' and B is equipotent with B', then ${}^{A}B$ is equipotent with ${}^{A'}B'$.*

Corollary 3.14. *For arbitrary sets A and B we have that $|{}^{A}B| = |B|^{|A|}$.*

Theorem 3.15. *Let I be a set. The following hold for arbitrary cardinals $\kappa, \lambda, \mu, \nu$ and for arbitrary systems $\langle \kappa_i : i \in I \rangle$ and $\langle \mu_i : i \in I \rangle$ of cardinals:*

(i) $\kappa^0 = 1$ *and* $\kappa^1 = \kappa$;

(ii) $0^\kappa = 0$ *if* $\kappa \neq 0$; $1^\kappa = 1$.

(iii) $\kappa^{\sum_{i\in I} \mu_i} = \prod^{\mathsf{c}}_{i\in I} \kappa^{\mu_i}$;
in particular, $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$ *and* $\kappa^{|I|} = \prod^{\mathsf{c}}_{i\in I} \kappa$.

(iv) $\left(\prod^{\mathsf{c}}_{i\in I} \kappa_i\right)^\mu = \prod^{\mathsf{c}}_{i\in I} \kappa_i^\mu$;
in particular, $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

(v) $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$.

(vi) $\kappa^\mu \leq \lambda^\nu$ *if* $\kappa \leq \lambda$, $\mu \leq \nu$, *and* $\lambda \neq 0$.

(vii) *If* $\kappa, \lambda \in \omega$, *then* $\kappa^\lambda \in \omega$.

(viii) *If* $\kappa, \mu \in \omega$, *then* κ^μ *satisfies the following conditions:*
- $\kappa^0 = 1$ *for all* $\kappa \in \omega$, *and*
- $\kappa^{\mu+1} = \kappa^\mu \cdot \kappa$ *for all* $\kappa, \mu \in \omega$.[8]

Theorem 3.16. *$|\mathcal{P}(A)| = 2^{|A|}$ holds for any set A.*

[8] This shows that for $\kappa, \mu \in \omega$, κ^μ in the cardinal sense (defined above) coincides with κ^μ in the ordinals sense, see Section 9 (pp. 96–97) of *Lectures on Set Theory* by J. Donald Monk.

4. Cardinal Arithmetic

Theorem 4.1. *$\kappa \cdot \kappa = \kappa$ holds for every infinite cardinal κ.*

Proof. Clearly, $\kappa = \kappa \cdot 1 \leq \kappa \cdot \kappa$. Assume $\kappa < \kappa \cdot \kappa$ for some infinite cardinal, and choose κ be the smallest such cardinal.

- The relation $\prec$ on $\kappa \times \kappa$ defined for $\delta, \varepsilon, \delta', \varepsilon' \in \kappa$ by

$$(\delta,\varepsilon) \prec (\delta',\varepsilon') \iff \begin{array}{l} \max(\delta,\varepsilon) < \max(\delta',\varepsilon'), \text{ or} \\ \max(\delta,\varepsilon) = \max(\delta',\varepsilon') \text{ and } \delta < \delta', \text{ or} \\ \max(\delta,\varepsilon) = \max(\delta',\varepsilon'), \ \delta = \delta' \text{ and } \varepsilon < \varepsilon'. \end{array} \tag{1}$$

  is a well-ordering on $\kappa \times \kappa$.

- For some ordinal α, there exists an isomorphism f from $(\kappa \times \kappa, \prec)$ onto $(\alpha, <)$.
- $\kappa < \kappa \cdot \kappa = |\alpha| \leq \alpha$.
- There exists $(\beta,\gamma) \in \kappa \times \kappa$ such that $f\big((\beta,\gamma)\big) = \kappa$.
- For $S = \{(\delta,\varepsilon) \in \kappa \times \kappa : (\delta,\varepsilon) \prec (\beta,\gamma)\}$ and $\bar\gamma = \max(\beta,\gamma) + 1$ we have:
  - $|S| = \kappa$, since $f{\restriction}S$ is a bijection $S \to \kappa$;
  - $|S| \leq |\bar\gamma \times \bar\gamma| = |\bar\gamma| \cdot |\bar\gamma| \overset{!}{=} |\bar\gamma| < \kappa$, where $\overset{!}{=}$ holds, because $|\bar\gamma| \leq \bar\gamma < \kappa$.

This contradiction proves the theorem. □

Corollary 4.2. *If κ and λ are nonzero cardinals such that not both are finite, then*

$$\kappa + \lambda = \max(\kappa,\lambda) = \kappa \cdot \lambda.$$

The following is a strengthening of Theorem 3.10(iii) in the case when infinite cardinals are involved.

Corollary 4.3. *Let I be a set and let $\langle \lambda_i : i \in I \rangle$ be a system of nonzero cardinals. If either I is infinite or at least one of the cardinals λ_i $(i \in I)$ is infinite, then*

$$\sum_{i\in I} \lambda_i = |I| \cdot \bigcup_{i\in I} \lambda_i.$$

Corollary 4.4. *The union of a countable set of countable sets is countable.*

Corollary 4.5. *If κ, λ are cardinals ≥ 2 such that not both are finite, then $\kappa^\lambda \leq \max(2^\kappa, 2^\lambda)$. In particular, if $\kappa \leq \lambda$ then $\kappa^\lambda = 2^\lambda$.*

Corollaries 4.2 and 4.3 show that the operation of addition for any system of cardinals and the binary operation of multiplication for cardinals are well understood. The operation of multiplication for arbitrary systems of cardinals is more complicated.

We start with the following generalization of Cantor's Theorem.

Theorem 4.6. (König's Theorem) *Let I be a set and let $\langle \kappa_i : i \in I\rangle$ and $\langle \lambda_i : i \in I\rangle$ be two systems of cardinals. If $\lambda_i < \kappa_i$ for all $i \in I$, then*

$$\sum_{i\in I} \lambda_i < \prod^{\mathsf{c}}_{i\in I} \kappa_i.$$

Proof. It suffices to show that if $\langle K_i : i \in I\rangle$ is a system of pairwise disjoint sets such that $|K_i| = \kappa_i$ for all $i \in I$, and L_i is a subset of K_i with $|L_i| = \lambda_i$ for all $i \in I$, then there is no one-to-one function $\prod_{i\in I} K_i \to \bigcup_{i\in I} L_i$.

Suppose there is a one-to-one function $F\colon \prod_{i\in I} K_i \to \bigcup_{i\in I} L_i$, and for each $i \in I$, let

$$K_i' = \Big\{h(i) \in K_i : h \in \prod_{i\in I} K_i,\ F(h) \in L_i\Big\}\ (\subseteq K_i).$$

Show that

- $|K_i'| \le \lambda_i < \kappa_i$ for every $i \in I$;
- there exists a function $g \in \prod_{i\in I} K_i$ such that $g(i) \notin K_i'$ for all $i \in I$;
- $F(g) \notin L_i$ holds for every $i \in I$,

a contradiction. □

Cantor's Theorem can be deduced from König's Theorem (and Theorem 3.16) by letting $\lambda_i = 1$ and $\kappa_i = 2$ for all $i \in I$:

$$|I| = \sum_{i\in I} 1 = \sum_{i\in I} \lambda_i < \prod^{\mathsf{c}}_{i\in I} \kappa_i = \prod^{\mathsf{c}}_{i\in I} 2 = 2^{|I|} = |\mathcal{P}(I)|.$$

The following analog of Corollary 4.3 is useful in evaluating infinite products.

Theorem 4.7. *If μ is an infinite cardinal and $\langle \kappa_\alpha : \alpha < \mu\rangle$ is a system of nonzero cardinals such that $\kappa_\alpha \le \kappa_\beta$ whenever $\alpha < \beta < \mu$, then*

$$\prod^{\mathsf{c}}_{\alpha<\mu} \kappa_\alpha = \Big(\bigcup_{\alpha<\mu} \kappa_\alpha\Big)^{\mu}. \tag{2}$$

It is easy to see that in (2), $\le$ holds for any system $\langle \kappa_\alpha : \alpha < \mu\rangle$ of nonzero cardinals. However, $=$ may fail if the monotonicity assumption is not satisfied. For example, for the system $\langle \kappa_n : n < \omega\rangle$ where $\kappa_0 = \aleph_\omega$ and $\kappa_n = \aleph_0$ for all $n \in \omega \setminus 1$, we have that

- $\prod^{\mathsf{c}}_{n<\omega} \kappa_n = \aleph_\omega \cdot \aleph_0^{\aleph_0} = \aleph_\omega \cdot 2^{\aleph_0} = \max(\aleph_\omega, 2^{\aleph_0})$, while
- $\big(\bigcup_{n<\omega} \kappa_n\big)^{\aleph_0} = \aleph_\omega^{\aleph_0}$.

It follows from Corollary 4.3 and from König's Theorem that $\aleph_\omega = \sum_{n\in\omega} \aleph_n < \prod^{\mathsf{c}}_{n\in\omega} \aleph_\omega = \aleph_\omega^{\aleph_0}$, and it is consistent with ZFC (e.g., true under CH) that $2^{\aleph_0} < \aleph_\omega$.

To discuss further properties of cardinal exponentiation, we need the concept of cofinality.

Definition 4.8. Let $(A, <)$ be a linear order with no largest element. A subset B of A is called *unbounded* if there is no $a \in A$ such that $b \leq a$ for all $b \in B$. The *cofinality of* $(A, <)$, denoted by $\mathrm{cf}(A)$, is the smallest cardinality of an unbounded subset of A.

Clearly, A is an unbounded subset of itself if $(A, <)$ has no largest element. Therefore, $\mathrm{cf}(A) \leq |A|$. In this section we will apply these concepts to infinite cardinals κ with their natural ordering $(\kappa, <)$. So, we have $\mathrm{cf}(\kappa) \leq \kappa$ for every cardinal κ.

Definition 4.9. An infinite cardinal κ is called *regular* if $\mathrm{cf}(\kappa) = \kappa$, and *singular* if $\mathrm{cf}(\kappa) < \kappa$.

Theorem 4.10. *Let κ be an infinite cardinal.*

(i) *The cardinal κ^+ is regular.*
(ii) *If κ is regular, then $\left|\bigcup \Gamma\right| \leq \sum_{\gamma \in \Gamma} |\gamma| < \kappa$ for every subset $\Gamma \subseteq \kappa$ with $|\Gamma| < \kappa$.*
(iii) *κ is regular if and only if for every system $\langle \lambda_i : i \in I \rangle$ of nonzero cardinals $< \kappa$ with $|I| < \kappa$ we have that $\sum_{i \in I} \lambda_i < \kappa$.*

Theorem 4.10(i) shows that infinite successor cardinals are regular. It is easy to see that $\aleph_0$ is a limit cardinal that is regular. Are there any uncountable limit cardinals that are regular? An uncountable regular limit cardinal is called *weakly inaccessible*. It is consistent with ZFC that there are no weakly inaccessible cardinals.

Theorem 4.11. *Let $(A,<)$ be a linear order with no largest element.*

(i) *There exists a strict order preserving function $f\colon \mathrm{cf}(A) \to A$ such that $\mathrm{rng}(f)$ is unbounded in A.*
(ii) $\mathrm{cf}\big(\mathrm{cf}(A)\big) = \mathrm{cf}(A)$*; that is, $\mathrm{cf}(A)$ is a regular cardinal.*
(iii) *If μ is a regular cardinal and $g\colon \mu \to A$ is a strict order preserving function such that $\mathrm{rng}(g)$ is unbounded in A, then $\mu = \mathrm{cf}(A)$.*

Sketch of Proof. The following claim is useful.

Claim 4.12. *If h is a function $\mathrm{cf}(A) \to A$ and Y is a subset of A such that both $X = \mathrm{rng}(h)$ and Y are unbounded in A, then there exists a function $f\colon \mathrm{cf}(A) \to Y$ such that*

- *f is strict order preserving, and*
- *$f(\alpha) > h(\alpha)$ for all $\alpha \in \mathrm{cf}(A)$, so $\mathrm{rng}(f)$ is unbounded in A.*

Proof of Claim 4.12. The function f can be constucted by recursion (using a choice function on $\mathcal{P}(A)\setminus\{\emptyset\}$): For any $\alpha \in \mathrm{cf}(A)$, if $f\restriction\alpha$ has been defined, let $f(\alpha)$ be an element in Y that is larger than an upper bound (in A) for $\mathrm{rng}(f\restriction\alpha)\cup\{h(\alpha)\}$. Such an element exists, because the set $\mathrm{rng}(f\restriction\alpha)\cup\{h(\alpha)\}$ is not unbounded in A (as $|\mathrm{rng}(f\restriction\alpha)\cup\{h(\alpha)\}| \leq |\alpha|+1 < \mathrm{cf}(A)$ for all $\alpha \in \mathrm{cf}(A)$), but Y is unbounded in A. ◇

Now, for the proof of the theorem:

(i) Let X be an unbounded subset of A with $|X| = \mathrm{cf}(A)$, and let h be any bijection $\mathrm{cf}(A) \to X$. Apply Claim 4.12 with $Y = X$.

(ii) By (i), we have strict order preserving (s.o.p.) functions $f\colon \mathrm{cf}(A) \to A$, $g\colon \mathrm{cf}\big(\mathrm{cf}(A)\big) \to \mathrm{cf}(A)$ with $\mathrm{rng}(f)$ unbounded in A and $\mathrm{rng}(g)$ unbounded in $\mathrm{cf}(A)$. For $\mathrm{cf}\big(\mathrm{cf}(A)\big) \geq \mathrm{cf}(A)$, show that $f\circ g\colon \mathrm{cf}\big(\mathrm{cf}(A)\big) \to A$ is s.o.p. with $\mathrm{rng}(f\circ g)$ unbounded in A.

(iii) Applying Claim 4.12 with $Y = \mathrm{rng}(g)$ we get a s.o.p. function $f\colon \mathrm{cf}(A) \to \mathrm{rng}(g)$ with $\mathrm{rng}(f)$ unbounded in A, hence in $\mathrm{rng}(g)$. Argue that $Z = \mathrm{rng}(g^{-1}\circ f)$ is unbounded in μ and $|Z| = \mathrm{cf}(A)$, so $\mathrm{cf}(A) \geq \mathrm{cf}(\mu) = \mu$. □

Corollary 4.13. *If κ is an infinite cardinal, then* $\mathrm{cf}(\kappa)$ *is a regular cardinal; moreover,* $\mathrm{cf}(\kappa)$ *is the unique regular cardinal μ with the property that there exists a strict order preserving function $f\colon \mu \to \kappa$ such that* $\mathrm{rng}(f)$ *is unbounded in κ.*

Theorem 4.14. *Let κ be an infinite cardinal. If κ is singular, then there exists a strictly increasing sequence $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ of infinite successor cardinals $< \kappa$ such that $\sum_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha = \kappa$.*

Proof. κ is a limit cardinal by Theorem 4.10(i), therefore the set Y of infinite successor cardinals in κ is unbounded. By Claim 4.12, there exists a s.o.p. function $\mathrm{cf}(\kappa) \to Y$, $\alpha \mapsto \lambda_\alpha$, such that $\bigcup_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha = \kappa$. This implies that $\sum_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha = \kappa$. □

Theorem 4.15. (König's Theorem on Cofinality) *For every infinite cardinal κ we have*

$$\kappa^{\mathrm{cf}(\kappa)} > \kappa.$$

Idea of proof. If κ is regular, then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^\kappa = 2^\kappa > \kappa$. If κ is singular, we get $\kappa < \kappa^{\mathrm{cf}(\kappa)}$ by applying König's Theorem to a system $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ of cardinals from Theorem 4.14. □

Corollary 4.16. *If λ is an infinite cardinal, then* $\mathrm{cf}(2^\lambda) > \lambda$.

The Main Theorem of Cardinal Arithmetic 4.17. *Let κ and λ be cardinals such that $\kappa \geq 2$ and λ is infinite.*

(i) *If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.*
(ii) *If there exists $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.*
(iii) *If $\kappa > 2$ and $\mu^\lambda < \kappa$ holds for all $\mu < \kappa$, then $\lambda < \kappa$ and we have*
 $(\text{iii})_1$ *$\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$ if $\mathrm{cf}(\kappa) \leq \lambda$ and*
 $(\text{iii})_2$ *$\kappa^\lambda = \kappa$ if $\mathrm{cf}(\kappa) > \lambda$.*

Before the proof we discuss some consequences.

It follows from the theorem that we can 'compute' κ^λ if we know the cardinal class functions $\lambda \mapsto 2^\lambda$ (the *continuum function*) and $\kappa \mapsto \kappa^{\mathrm{cf}(\kappa)}$, and the values of μ^λ for $\mu < \kappa$. Namely:

- If $\mu^\lambda \geq \kappa$ for some $\mu < \kappa$ — and hence, by (ii), $\mu^\lambda = \kappa^\lambda$ —, then let κ_0 be the least cardinal such that $\kappa_0^\lambda = \kappa^\lambda$. Clearly, $2 \leq \kappa_0 < \kappa$. Otherwise, let $\kappa_0 = \kappa$.
- In either case, $\mu^\lambda < \kappa_0$ holds for all $\mu < \kappa_0$.
- If $\kappa_0 = 2$ (which will be the case, by (i), if $\kappa \leq \lambda$), then $\kappa^\lambda = 2^\lambda$.
- If $\kappa_0 > 2$, then by (iii), $\kappa_0 > \lambda$ and we have one of the following cases:
  - $\kappa^\lambda = \kappa_0^\lambda = \kappa_0^{\mathrm{cf}(\kappa_0)}$ if $\mathrm{cf}(\kappa_0) \leq \lambda$;
  - $\kappa^\lambda = \kappa_0^\lambda = \kappa_0 = \kappa$ if $\mathrm{cf}(\kappa_0) > \lambda$.

Remark 4.18. For regular κ, the continuum function $\kappa \mapsto 2^\kappa$ and the function $\kappa \mapsto \kappa^{\mathrm{cf}(\kappa)}$ coincide, and by a theorem of Easton, nothing more can be proved in ZFC about them than what we already know. Precisely, the statement is as follows: Let **Cn** denote the class of all infinite cardinals, and let **RCn** denote its subclass consisting of all regular cardinals. For every class function $\mathbf{E}\colon \mathbf{RCn} \to \mathbf{Cn}$ such that

- $\mathbf{E}(\kappa) \leq \mathbf{E}(\lambda)$ for all $\kappa < \lambda$ in **RCn**, and
- $\mathrm{cf}\big(\mathbf{E}(\kappa)\big) > \kappa$ for all κ in **RCn**,

there exists a model of ZFC where $\mathbf{E}(\kappa) = 2^\kappa$ for all κ in **RCn**.

Shelah's PCF theory is used to obtain results about these functions for singular κ. For example, Shelah proved in ZFC that $\aleph_\omega^{\aleph_0} < \max\big(\aleph_{\omega_4}, (2^{\aleph_0})^+\big)$.

Theorem 4.19. *Let κ and λ be cardinals such that $\kappa \geq 2$ and λ is infinite. Assuming GCH, we have the following:*

- *If $\kappa \leq \lambda$, then $\kappa^\lambda = \lambda^+$.*
- *If $\mathrm{cf}(\kappa) \leq \lambda < \kappa$, then $\kappa^\lambda = \kappa^+$.*
- *If $\lambda < \mathrm{cf}(\kappa)$, then $\kappa^\lambda = \kappa$.*

The Main Theorem of Cardinal Arithmetic. *Let κ and λ be cardinals such that $\kappa \geq 2$ and λ is infinite.*

(i) *If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.*
(ii) *If there exists $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.*
(iii) *If $\kappa > 2$ and $\mu^\lambda < \kappa$ holds for all $\mu < \kappa$, then $\lambda < \kappa$ and we have*
  $(\text{iii})_1$ *$\kappa^\lambda = \kappa^{\text{cf}(\kappa)}$ if $\text{cf}(\kappa) \leq \lambda$ and*
  $(\text{iii})_2$ *$\kappa^\lambda = \kappa$ if $\text{cf}(\kappa) > \lambda$.*

Proof of the Main Theorem. (iii) $\lambda < \kappa$ follows from the assumption for $\mu = 2$: $\lambda < 2^\lambda < \kappa$.
If $\text{cf}(\kappa) \leq \lambda\,(< \kappa)$, then κ is singular, hence a limit cardinal. We will use

Lemma 4.20. *If κ is a limit cardinal and $\lambda \geq \text{cf}(\kappa)$, then*

$$\kappa^\lambda = \Bigg(\bigcup_{\substack{\mu<\kappa \\ \mu \text{ a cardinal}}} \mu^\lambda \Bigg)^{\text{cf}(\kappa)}.$$

Idea of proof of $\leq$ in the Lemma. Use the fact (see the proof of Theorem 4.14) that there exists a strictly increasing sequence $\langle \lambda_\alpha : \alpha < \text{cf}(\kappa) \rangle$ of infinite cardinals $< \kappa$ such that $\bigcup_{\alpha<\text{cf}(\kappa)} \lambda_\alpha = \kappa$ to construct a one-to-one function ${}^\lambda\kappa \to \prod_{\alpha<\text{cf}(\kappa)} {}^\lambda\lambda_\alpha$. ◇

So, under the assumptions of case $(\text{iii})_1$ of the Main Theorem, we get from the lemma that

$$\kappa^\lambda = \Bigg(\bigcup_{\substack{\mu<\kappa \\ \mu \text{ a cardinal}}} \mu^\lambda \Bigg)^{\text{cf}(\kappa)} \leq \kappa^{\text{cf}(\kappa)} \leq \kappa^\lambda.$$

Otherwise, under the assumptions of case $(\text{iii})_2$, we have $\text{cf}(\kappa) > \lambda$, therefore

$$\kappa^\lambda = |{}^\lambda\kappa| \overset{!}{=} \Big| \bigcup_{\alpha<\kappa} {}^\lambda\alpha \Big| \leq \sum_{\alpha<\kappa} |\alpha|^\lambda \overset{?}{\leq} \kappa \cdot \kappa = \kappa,$$

where $\overset{!}{=}$ holds because $\lambda < \text{cf}(\kappa)$, and $\overset{?}{=}$ holds because $|\alpha|^\lambda < \kappa$ for all $\alpha < \kappa$. □

Theorem 4.21. (Hausdorff's Theorem) *If κ, λ are infinite cardinals, then $(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$.*

5. Mathematics Without the Axiom of Choice

AC is independent of the axioms of ZF; that is, if ZF is consistent, then so are ZFC $=$ ZF $\cup$ {AC} and ZF $\cup$ {$\neg$AC}. Recall that the following definitions do not rely on AC:

- ω (the set of natual numbers) is the least inductive set;
- a set is *finite* if it is equipotent with a natural number;
- a set is countably infinite if it is equipotent with ω, and countable if it is finite or countably infinite.

Each statement listed below is the negation of a well-known and widely used theorem provable in ZFC. However, for each statement there exists a model of ZF in which the statement is true.

(1) There exists an infinite set A such that A is not the union of two disjoint infinite subsets of A. Such a set is called *amorphous*.
Consequently:
$(1)_1$ There exists an infinite set D (namely, $D = A$) such that
($*$) D is not equipotent with a proper subset of D;
a set D with property ($*$) is called *Dedekind finite*. It can be proved in ZF that ($*$) is equivalent to the condition that there is no one-to-one function $\omega \to D$.
$(1)_2$ There exist sets B, C (namely, $B = D$ and $C = \omega$) such that no function $B \to C$ or $C \to B$ is one-to-one.

(2) There exists a countable set of countable sets whose union is not countable.
$(2)_1$ In fact the set $\mathbb{R}$ of real numbers (or $\mathcal{P}(\omega)$) is the union of a countable set of countable subsets.

(3) There is a subset S of $\mathbb{R}$ and a point $x \in \mathbb{R}$ in the closure of S such that x is not the limit of any sequence $\langle x_n : n \in \omega \rangle$ of elements of S.

(4) There exists a vector space which has no basis.

(5) There exists a field which has no algebraic closure.

Some of these 'anomalies' can be avoided by assuming a weaker variant of the Axiom of Choice. Three weakenings that are often used in mathematics are the following:

◇ *Countable Axiom of Choice* (AC_ω): Every countable set of nonempty sets has a choice function.

◇ *The Principle of Dependent Choice* (DC): If A is a nonempty set and R is a relation on A such that for every $a \in A$ there exists $b \in R$ such that $(a, b) \in R$, then there exists a sequence $\langle x_n : n \in \omega \rangle$ of elements of A such that $(x_n, x_{n+1}) \in R$ for all $n \in \omega$.

◇ *Boolean Prime Ideal Theorem* (PIT): Every Boolean algebra has a prime ideal. (Equivalently: Every Boolean algebra has a maximal ideal.)

Theorem 5.1. *In* ZF*, we have that*

(i) $\mathsf{AC} \Rightarrow \mathsf{DC} \Rightarrow \mathsf{AC}_\omega$ *and* $\mathsf{AC} \Rightarrow \mathsf{PIT}$*;*

(ii) AC_ω *implies that every infinite set has a countably infinite subset (i.e., for every infinite set* S *there exists a one-to-one function* $\omega \to S$*);*

Remark 5.2. For each implication $\Rightarrow$ in (i) there is a model of ZF where the converse fails.

Proof of Theorem 5.1. (i) $\mathsf{AC} \Rightarrow \mathsf{PIT}$ follows by Zorn's Lemma.

To prove $\mathsf{AC} \Rightarrow \mathsf{DC}$ let A, R be as in DC, and let f be a choice function for $\mathcal{P}(A) \setminus \{\emptyset\}$. Define $g\colon \omega \to A$, $n \mapsto x_n$ by recursion: choose $x_0 \in A$ arbitrarily, and for any $n \in \omega$ let $x_{n+1} = f\big(\{y \in A : (x_n, y) \in R\}\big)$.

Finally, for $\mathsf{DC} \Rightarrow \mathsf{AC}_\omega$, let $\langle S_n : n \in \omega\rangle$ be a countable set of nonempty sets. Let A be the set of all finite sequences (= functions) $\widehat{x} = \langle x_0, x_1, \dots, x_k\rangle \in \prod_{i \in k+1} S_i$, and let R be the relation on A defined by $(\widehat{x}, \widehat{y}) \in R$ iff $\widehat{x} = \langle x_0, x_1, \dots, x_k\rangle$ and $\widehat{y} = \langle x_0, x_1, \dots, x_k, x_{k+1}\rangle$ for some $k \in \omega$ and $x_i \in S_i$ for all $i \in k+2$. Now, by DC, there is a sequence $\langle \widehat{x}_n : n \in \omega\rangle$ of elements of A such that $(\widehat{x}_n, \widehat{x}_{n+1}) \in R$ for all $n \in \omega$. Thus, $\bigcup_{n\in\omega} \widehat{x}_n \in \prod_{i\in\omega} S_i$.

(ii) Let S be an infinite set. For each $k \in \omega$ let

$$A_k = \{h \in {}^k S \colon h \text{ is one-to-one}\},$$

and let $\mathcal{A} = \{A_k : k \in \omega\}$. By AC_ω, there exists a choice function for $\mathcal{A}$; so, $f(A_k) \in A_k$ for every $k \in \omega$. It is easy to see that $\bigcup_{k\in\omega} \operatorname{rng}\big(f(A_k)\big)$ is a countably infinite subset of S. $\square$

By Theorem 5.1(ii), AC_ω implies that every infinite set is Dedekind infinite, or equivalently, every Dedekind finite set is finite. Hence, there are no amorphous sets. Axiom AC_ω also implies that the union of a countable set of countable sets is countable. It follows that if, in addition to ZF, we assume the Axiom of Countable Choice, then the 'desirable' statements $\neg(1)$, $\neg(1_1)$, $\neg\big((1)_2$ for sets B and $C = \omega\big)$, $\neg(2)$, $\neg(2)_1$, and $\neg(3)$ will all hold.

Statement $\neg(1)_2$ for arbitrary sets B and C can be proved to be equivalent, in ZF, to AC. What is more surprising is that the same holds for $\neg(4)$. In ZF, PIT implies the statement $\neg(5)$, but it seems to be open whether $\neg(5)$ implies PIT.