Set Theory (MATH 6730)

Clubs and Stationary Sets

Definition 1. Let α be an ordinal, and let $C \subseteq \alpha$. We say that

- C is *unbounded in* α if for every $\beta < \alpha$ there exists $\gamma \in C$ such that $\beta \leq \gamma$;[1]
- C is *closed in* α if for every limit ordinal $\beta < \alpha$ such that $C \cap \beta$ is unbounded in β we have that $\beta \in C$;
- C is *club in* α if it is closed and unbounded in α.

Example 2. Let α be an ordinal.

(i) α is club in α; in particular, $\emptyset$ is club in 0.

(ii) If α is a successor ordinal, say $\alpha = \beta + 1$, then $\{\beta\}$ is club in α.

(iii) If α is a limit ordinal, then the set $[\beta, \alpha) = \{\gamma < \alpha : \gamma \geq \beta\}$ is club in α for every $\beta < \alpha$.

(iv) If α is a limit ordinal of countable cofinality, then for every strict order preserving function $f \colon \omega = \mathrm{cf}(\alpha) \to \alpha$ such that $C = \mathrm{rng}(\alpha)$ is unbounded in α we have that C is club in α.[2]

(v) If α is a limit ordinal of uncountable cofinality and $C \subseteq \alpha$ is club in α, then so are the following subsets of C:

$$D = \{\gamma \in C : \gamma \text{ is a limit ordinal}\},$$
$$E = \{\gamma < \alpha : \gamma \text{ is a limit ordinal and } C \cap \gamma \text{ is unbounded in } \gamma\} \ (\subseteq D).$$

(vi) A non-example: Under the assumptions of (v), $X = \{\gamma \in C : \gamma \text{ is a successor ordinal}\}$ is not club in α.

Clubs in an ordinal α are most interesting if α is a limit ordinal of uncountable cofinality.

---

[1] If α has no largest element (i.e., α is not a successor ordinal), then this definition coincides with our earlier definition; see Definition 4.8 in the handout "The Axiom of Choice. Cardinals and Cardinal Arithmetic".

[2] Cf. Theorem 4.11 in the same handout and Corollary 6 below.

Before stating our first result on clubs, the following observation on subsets of ordinals will be useful. Recall[3] that for every well-ordered set $(B, \prec)$ there exists a unique ordinal β such that $(\beta, <)$ is isomorphic to $(B, \prec)$. We will refer to this ordinal β as *the order type of* $(B, \prec)$.

Fact 3. *If α is an ordinal, $\Gamma \subseteq \alpha$, and Γ has order type β, then $\beta \leq \alpha$.*

Theorem 4. *Let α be a limit ordinal, and let $C \subseteq \alpha$. Then C is club in α if and only if*

(†) *C is unbounded in α, and*
there exist $\beta \in \mathbf{On}$ and a normal function[4] *$f\colon \beta \to \alpha$ such that $C = \operatorname{rng}(f)$.*

Idea of Proof.
$\Rightarrow$: Let C be club in α, let $\beta\,(\leq \alpha)$ be the order type of C and let f be an isomorphism $\beta \to (C, <)$, considered as a function $\beta \to \alpha$.

- Clearly, $C = \operatorname{rng}(f)$ is unbounded in α and f is strict order preserving.
- To prove that f is also continuous, let $\delta < \beta$ be a limit ordinal. Using that C is closed in α, show that $\bigcup_{\varepsilon<\delta} f(\varepsilon) \in C$, and conclude that $f(\delta) = \bigcup_{\varepsilon<\delta} f(\varepsilon)$.

$\Leftarrow$: Assume that $f\colon \beta \to \alpha$ is a normal function such that $C = \operatorname{rng}(f)$ is unbounded in α.

- To show that C is closed in α, let $\gamma < \alpha$ be a limit ordinal such that $C \cap \gamma$ is unbounded in γ. Verify that $\delta := \bigcup f^{-1}[C \cap \gamma]$ is a limit ordinal $< \beta$, and prove that
$$f(\delta) = \bigcup_{\varepsilon<\delta} f(\varepsilon) = \bigcup (C \cap \gamma) = \gamma.$$
Hence $\gamma \in \operatorname{rng}(f) = C$. □

[3] See Theorem 4.4 on the handout 'Ordinals. Transfinite Induction and recursion'.
[4] See Definition 5.1 on the same handout.

Corollary 5. *Let κ be a regular cardinal, and let $C \subseteq \kappa$. Then C is club in κ if and only if*

(‡) *there exists a normal function $f\colon \kappa \to \kappa$ such that $C = \mathrm{rng}(f)$.*

Proof. By Theorem 4, it suffices to show the following:

$$(\dagger) \text{ holds for } \alpha = \kappa \text{ if and only if } (\ddagger) \text{ holds.}$$

To prove this observe that

in $\Rightarrow$: since C is unbounded in κ, it must be that $\beta \geq |\beta| = |C| \geq \mathrm{cf}(\kappa) = \kappa$, so $\beta = \kappa$; and

in $\Leftarrow$: C is unbounded in κ, because (‡) forces $|C| = \kappa$. □

Corollary 6. *Every limit ordinal α has a club of order type* $\mathrm{cf}(\alpha)$*.*

Proof. We saw earlier[5] that there exists a strict order preserving function $f\colon \mathrm{cf}(\alpha) \to \alpha$ such that $\mathrm{rng}(f)$ is unbounded in α. Now we define a function $g\colon \mathrm{cf}(\alpha) \to \alpha$ by recursion as follows:

$$g(\delta) = \begin{cases} 0 & \text{if } \delta = 0, \\ \max\bigl(f(\delta), g(\varepsilon)+1\bigr) & \text{if } \delta = \varepsilon + 1 \text{ for some ordinal } \varepsilon, \\ \bigcup_{\varepsilon<\delta} g(\varepsilon) & \text{if } \delta \text{ is a limit ordinal} \end{cases} \qquad \bigl(\delta < \mathrm{cf}(\alpha)\bigr).$$

It follows that

- g is a normal function $\mathrm{cf}(\alpha) \to \alpha$;[6]

- $\mathrm{rng}(g)$ is unbounded in α, since $g(\delta) \geq f(\delta)$ for all $\delta < \mathrm{cf}(\alpha)$. □

Theorem 7. *If α is a limit ordinal of uncountable cofinality, then the intersection of fewer than* $\mathrm{cf}(\alpha)$ *clubs of α is a club of α.*

Example 8. If, in Theorem 7, we drop the assumption $\mathrm{cf}(\alpha) > \omega$ or the assumption that the number of clubs intersected is $< \mathrm{cf}(\alpha)$, then the conclusion of the theorem may fail.

(i) Let $\alpha = \omega$ (so $\mathrm{cf}(\alpha) = \omega$). Then $C_0 = \{n \in \omega : n \text{ even}\}$ and $C_1 = \{n \in \omega : n \text{ odd}\}$ are clubs in ω, $2 < \mathrm{cf}(\omega)$, but $C_0 \cap C_1 = \emptyset$.

(ii) Let $f\colon \mathrm{cf}(\alpha) \to \alpha$ (α a limit ordinal) be s.o.p. such that $\mathrm{rng}(f)$ is unbounded in α. Then each interval $C_\xi = [f(\xi), \alpha)$ ($\xi < \mathrm{cf}(\alpha)$) is club in α, but $\bigcap_{\xi<\mathrm{cf}(\alpha)} C_\xi = \emptyset$.

[5]See Theorem 4.11(i) on the handout "The Axiom of Choice. Cardinals and Cardinal Arithmetic".

[6]Use Theorem 5.2 on the handout "Ordinals. Transfinite Induction and Recursion".

Theorem 7. *If α is a limit ordinal of uncountable cofinality, then the intersection of fewer than* $\mathrm{cf}(\alpha)$ *clubs of α is a club of α.*

Idea of Proof of Theorem 7. Let $\langle C_\xi : \xi < \beta \rangle$ $(\beta < \mathrm{cf}(\alpha))$ be a system of clubs in α, and let $D = \bigcap_{\xi<\beta} C_\xi$.

- D is closed in α.
- D is unbounded in α: Let $\gamma < \alpha$. Show that
  - there exists a sequence $\langle \varepsilon_n : n \in \omega \rangle$ of ordinals $< \alpha$ such that $\varepsilon_0 = \gamma$ and for each $n \in \omega$ and $\xi < \beta$ we have that $\varepsilon_{n+1} \geq \theta_{n,\xi}$ for some $\theta_{n,\xi} \in C_\xi$ with $\varepsilon_n < \theta_{n,\xi}$.
  - Let $\delta = \bigcup_{n\in\omega} \varepsilon_n$. Then $\gamma < \delta < \alpha$ and $\delta \in C_\xi$ for all $\xi < \beta$, so $\delta \in D$. □

Definition 9. Let α be a limit ordinal. The *diagonal intersection* of a system $\langle C_\xi : \xi < \alpha \rangle$ of subsets of α is defined by

$$\Delta_{\xi<\alpha} C_\xi := \{\beta \in \alpha : \beta \in C_\xi \text{ for all } \xi < \beta\}.$$

Example 10. If, in Example 8(ii), α is a regular cardinal (hence, $\mathrm{cf}(\alpha) = \alpha$) and f is normal, then check that for a limit ordinal $\beta < \alpha$ we have $\beta \in \Delta_{\xi<\alpha} C_\xi = \Delta_{\xi<\alpha} [f(\xi), \alpha)$ iff $f(\beta) = \beta$.

Theorem 11. *Let α be a limit ordinal with $\mathrm{cf}(\alpha) > \omega$, and let $\langle C_\xi : \xi < \alpha \rangle$ be a system of clubs in α.*

(i) *If $\bigcap_{\xi<\beta} C_\xi$ is unbounded in α for all $\beta < \alpha$, then $\Delta_{\xi<\alpha} C_\xi$ is club in α.*
(ii) *If α is a regular cardinal, then $\Delta_{\xi<\alpha} C_\xi$ is club in α.*

Idea of Proof. (ii) follows from (i) by Theorem 7. To prove (i), let $D = \Delta_{\xi<\alpha} C_\xi$.

- D is closed: If $\beta < \alpha$ is a limit ordinal and $D \cap \beta$ is unbounded in β, then for each $\xi < \beta$, $C_\xi \cap \beta \left(\supseteq [\xi + 1, \beta) \cap (D \cap \beta)\right)$ is unbounded in β, so $\beta \in C_\xi$.
- D is unbounded in α: Let $\gamma < \alpha$. Show that
  - there exists a sequence $\langle \varepsilon_n : n \in \omega \rangle$ of ordinals $< \alpha$ such that $\varepsilon_0 = \gamma$ and for each $n \in \omega$, ε_{n+1} is an element of $\bigcap_{\xi<\varepsilon_n} C_\xi$ greater than ε_n.
  - As before, let $\delta := \bigcup_{n\in\omega} \varepsilon_n$, and show that $\gamma < \delta \in D$. $\square$

Definition 12. Let A be a set. A *finitary partial operation on A* is a function f with $\mathrm{dmn}(f) \subseteq {}^mA$ for some $m \in \omega$ and with $\mathrm{rng}(f) \subseteq A$. A subset B of A is closed under such an operation f if for every $b \in {}^mB \cap \mathrm{dmn}(f)$ we have that $f(b) \in B$.

Notation 13. For any set A and any cardinal κ, let

$$\begin{aligned} [A]^\kappa &= \{X \in \mathcal{P}(A) : |X| = \kappa\}, \\ [A]^{<\kappa} &= \{X \in \mathcal{P}(A) : |X| < \kappa\}, \\ [A]^{\leq\kappa} &= \{X \in \mathcal{P}(A) : |X| \leq \kappa\}. \end{aligned}$$

Theorem 14. *Let κ be an uncountable regular cardinal. If $X \in [\kappa]^{<\kappa}$ and $\mathcal{F}$ is a set of finitary partial operations on X with $|\mathcal{F}| < \kappa$, then the set*

$$C = \{\alpha < \kappa : X \subseteq \alpha \text{ and } \alpha \text{ is closed under each } f \in \mathcal{F}\}$$

is club in κ.

Definition 15. Let α be a limit ordinal. A subset S of α is said to be *stationary* if S has a nonempty intersection with every club of α.

Example 16. Let α be a limit ordinal with $\mathrm{cf}(\alpha) > \omega$. Then

- every club in α is stationary;
- every subset of α containing a club is stationary.

Theorem 17. *If α is a limit ordinal and κ is a regular cardinal such that $\kappa < \mathrm{cf}(\alpha)$, then*

$$S = \{\beta < \alpha : \mathrm{cf}(\beta) = \kappa\}$$

is a stationary subset of α.

Idea of Proof. Let C be a club in α. To show that $C \cap S \neq \emptyset$, argue that

- there exists a normal function $f\colon \mathrm{cf}(\alpha) \to \alpha$ such that $\mathrm{rng}(f)$ is club in α;
- there exists a normal function $g\colon \mathrm{cf}(\alpha) \to C$ such that $g(\beta + 1) > \max\{g(\beta), f(\beta)\}$ for all $\beta < \mathrm{cf}(\alpha)$; hence, $\mathrm{rng}(g)$ is club in α;
- $g(\kappa) \in C \cap S$. □

Lemma 18. *Let α be a limit ordinal with* $\mathrm{cf}(\alpha) > \omega$*. If* $\beta < \mathrm{cf}(\alpha)$*, then for any system* $\langle N_\xi : \xi < \beta \rangle$ *of nonstationary sets in α, the union* $\bigcup_{\xi<\beta} N_\xi$ *is also nonstationary in α.*

Definition 19. Let S be a set of ordinals. A function $f \in {}^S\mathbf{On}$ is called *regressive* if $f(\gamma) < \gamma$ for all $\gamma \in S \setminus \{0\}$.

Theorem 20. (Fodor's Lemma or "Pressing Down Lemma") *Let α be a limit ordinal with* $\mathrm{cf}(\alpha) > \omega$*, let S be a stationary subset of α, and let $f\colon S \to \alpha$ be a regressive function.*

(i) *Then there exists $\beta < \alpha$ such that $f^{-1}[\beta]$ is stationary in α.*

(ii) *Moreover, if α is a regular cardinal, then there exists $\gamma < \alpha$ such that $f^{-1}[\{\gamma\}]$ is stationary in α.*

Idea of Proof. (i) Assume there is no such β. Then there exists a system $\langle C_\beta : \beta < \alpha \rangle$ of clubs in α such that $C_\beta \cap f^{-1}[\beta] = \emptyset$ for all $\beta < \alpha$. Let D be a club in α of order type $\mathrm{cf}(\alpha)$ (cf. Corollary 6), and for each $\beta < \alpha$ let $\tau(\beta)$ denote the least member of D greater than β. For every $\beta < \alpha$ let

$$E_\beta = \bigcap_{\xi \in D \cap (\tau(\beta)+1)} C_\xi .$$

Use Theorems 7, 11, and Example 2(v) to show that

- for each $\beta < \alpha$, E_β is club in α and satisfies $E_\beta \cap f^{-1}[\beta] = \emptyset$;
- $F = \Delta_{\xi<\alpha} E_\xi$ is club in α;
- $G = \{\beta \in F : \beta \text{ is a limit ordinal}\}$ is club in α.

Now let $\delta \in G \cap S$, and argue that

- there exists $\xi < \delta$ such that $f(\delta) < \xi$;
- $\delta \in F$ and hence $\delta \in E_\xi$;
- $\delta \notin f^{-1}[\xi]$, which contradicts $f(\delta) < \xi$.

(ii) With the β from part (i) we have that $f^{-1}[\beta] = \bigcup_{\gamma<\beta} f^{-1}[\{\gamma\}]$ is stationary in α. By Lemma 18 at least one of the sets $f^{-1}[\{\gamma\}]$ $(\gamma < \beta)$ must be stationary in α. □

We will soon see an application of Fodor's Lemma.[7]

[7] See also Theorems 19.10-12 in *Lectures on Set Theory* by J. Donald Monk.

Next we introduce an important combinatorial principle, called $\diamondsuit$ (*diamond*), which can be proved to be consistent with ZFC (if ZFC is consistent). We will show that ZFC together with $\diamondsuit$ implies CH and the existence of a Suslin tree.

Definition 21. $\diamondsuit$ is the following statement:

There exists a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ of sets with the following properties:

- *$A_\alpha \subseteq \alpha$ for each $\alpha < \omega_1$, and*
- *For every subset A of ω_1, the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary in ω_1.*

A sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ with these properties is called a *$\diamondsuit$-sequence.*

A $\diamondsuit$-sequence may be thought of as an ω_1-sequence of subsets of ω_1 which — in a sense — captures all subsets of ω_1.

Theorem 22. $\mathsf{ZFC} \cup \{\diamondsuit\}$ *implies CH.*

Proof. Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. We prove $\big(|\mathcal{P}(\omega)| =\big)\ 2^{\aleph_0} \leq \aleph_1$ by showing that there exists an injective function $f\colon \mathcal{P}(\omega) \to \omega_1$.

- If $A \in \mathcal{P}(\omega)$, then — since $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary in ω_1 — there exists an infinite $\beta < \omega_1$ such that $A \cap \beta = A_\beta$. Since $A \subseteq \omega \subseteq \beta$, we get $A = A_\beta$.
- Therefore, the assignment
$$A \mapsto \text{the least } \beta < \omega_1 \text{ such that } A = A_\beta$$
defines a function $f\colon \mathcal{P}(\omega) \to \omega_1$, which is clearly injective. □

Our goal now is to prove that $\diamondsuit$ implies the existence of a Suslin tree. Since a Suslin tree has cardinality ω_1, we will construct a Suslin tree $T = (\omega_1, \prec)$ with ω_1 as its set of elements. We will use the following notation.

Notation 23. If $T = (\omega_1, \prec)$ is an ω_1-tree and $\alpha < \omega_1$, let
$$T{\restriction}\alpha = \{\beta < \omega_1 : \mathrm{ht}(\beta) < \alpha\}.$$
We will also use the notation $T{\restriction}\alpha$ for the (normal) subtree of T with underlying set $T{\restriction}\alpha$.

Lemma 24. *If $T = (\omega_1, \prec)$ is an ω_1-tree and A is a maximal antichain in T, then the set*

(1) $$C = \{\alpha < \omega_1 : T{\restriction}\alpha = \alpha \text{ and } A \cap \alpha \text{ is a maximal antichain in } T{\restriction}\alpha\}$$

is club in ω_1.

Lemma 24. *If $T = (\omega_1, \prec)$ is an ω_1-tree and A is a maximal antichain in T, then the set*

(1) $$C = \{\alpha < \omega_1 : T\restriction\alpha = \alpha \text{ and } A \cap \alpha \text{ is a maximal antichain in } T\restriction\alpha\}$$

is club in ω_1.

Proof. To prove that C is closed, let $\alpha < \omega_1$ be a limit ordinal such that $C \cap \alpha$ is unbounded in α. Our goal is to show that $\alpha \in C$. The following observation will be used repeatedly:

(†) For each $\gamma < \alpha$ there exists $\delta \in C \cap \alpha$ such that $\gamma < \delta$, so we have that
- $\gamma \in \delta = T\restriction\delta \subseteq T\restriction\alpha$, and
- $A \cap \delta$ is a maximal antichain in $T\restriction\delta$.

Now $\alpha \in C$ can be verified as follows.

- $T\restriction\alpha \subseteq \alpha$: If $\beta \in T\restriction\alpha$, then $\beta \in T\restriction\gamma$ for some $\gamma < \alpha$, so for any δ from (†), $\beta \in T\restriction\gamma \subseteq T\restriction\delta = \delta \subseteq \alpha$.
- $T\restriction\alpha \supseteq \alpha$: If $\gamma \in \alpha$, then for any δ from (†) we get that $\gamma \in \delta = T\restriction\delta \subseteq T\restriction\alpha$.
- $A\cap\alpha$ is a maximal antichain in $T\restriction\alpha$: Clearly $A\cap\alpha$ is an antichain in $T\restriction\alpha$, so we need to show only that every $\beta \in T\restriction\alpha$ is comparable (in T) to some element of $A \cap \alpha$.[8] Choose $\gamma < \alpha$ such that $\beta \in T\restriction\gamma$. For any δ from (†) we have that $A\cap\delta$ is a maximal antichain in $T\restriction\delta$, so β is comparable (in T) to an element of $A \cap \delta \subseteq A \cap \alpha$.

To prove that C is unbounded in ω_1, consider the following unary functions f, g, h on ω_1: for each $\beta < \omega_1$, let $f(\beta) = \mathrm{ht}(\beta)$, $g(\beta) = \bigcup \mathrm{Lev}_\beta(T)$, and let $h(\beta)$ be an element of A comparable (in T) to β. By Theorem 14,

$$D = \{\alpha < \omega_1 : \alpha \text{ is closed under } f, g, h\}$$

is club in ω_1. It suffices to show that $D \subseteq C$. Let $\alpha \in D$.

- $T\restriction\alpha \subseteq \alpha$: If $\beta \in T\restriction\alpha$, then $\gamma := \mathrm{ht}(\beta) \in \alpha$, so $\beta \in \mathrm{Lev}_\gamma(T)$ and $\beta \le g(\gamma) \in \alpha$.
- $T\restriction\alpha \supseteq \alpha$: If $\beta \in \alpha$, then $\mathrm{ht}(\beta) = f(\beta) \in \alpha$, so $\beta \in T\restriction\alpha$.
- $A \cap \alpha$ is a maximal antichain in $T\restriction\alpha$: If $\beta \in T\restriction\alpha$, then $h(\beta) \in A \cap \alpha$ is comparable (in T) to β. □

---

[8] We call two elements u, v of a tree $(T, \prec)$ — or, more generally, of a partially ordered set $(T, \prec)$ — *comparable* if $u \prec v$ or $u = v$ or $v \prec u$.

Lemma 25. (Assumes $\Diamond$) *Let $T = (\omega_1, \prec)$ be an eventually branching ω_1-tree, and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\Diamond$-sequence. Assume that*

(∗) *for every limit ordinal $\alpha < \omega_1$, if $T \restriction \alpha = \alpha$ and A_α is a maximal antichain in $T \restriction \alpha$, then for each $x \in \mathrm{Lev}_\alpha(T)$ there exists $y \in A_\alpha$ such that $y \prec x$.*

Then T is a Suslin tree.

Sketch of Proof. By our earlier sufficient condition[9] we have to show only that every maximal antichain A in T is countable.

- By Lemma 24, the set C in (1) is club in ω_1.
- There exists $\alpha \in C$ such that $A \cap \alpha = A_\alpha$; fix such an α.
- **Claim.** *For all β in T, if $\mathrm{ht}(\beta) \geq \alpha$, then $\beta \notin A$.*
- Therefore, if $\beta \in A$, then $\mathrm{ht}(\beta) < \alpha$, so $\beta \in T \restriction \alpha = \alpha$; this proves that $A \subseteq \alpha$, hence A is countable. □

[9] See Theorem 15 on the handout "Trees".

Theorem 26. $\mathsf{ZFC} \cup \{\Diamond\}$ *implies the existence of a Suslin tree.*

Sketch of Proof. Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\Diamond$-sequence. Using this sequence, we will construct a Suslin tree $T = (\omega_1, \prec)$ such that $\mathrm{Lev}_\beta(T) = \{\omega \cdot \beta + m : m \in \omega\}$ for each $\beta < \omega_1$. The construction proceeds by recursion, completely defining the normal subtree $T_\beta := (\omega \cdot \beta, \prec_\beta)$ of T (up to level β) for each $\beta < \omega_1$, all in such a way that the 'union' $T = (\omega_1, \prec)$ of these trees — i.e., the tree $T := (\omega_1, \prec)$ where the relation $\prec$ is defined on $\omega_1 = \omega \cdot \omega_1$ by $\prec := \bigcup_{\beta<\omega_1} \prec_\beta$ — satisfies the hypotheses of Lemma 25.

In more detail, we want to construct relations $\prec_\beta$ on $\omega \cdot \beta$ for all $\beta < \omega_1$, by recursion, so that the following conditions are satisfied:

(1_β) $T_\beta := (\omega \cdot \beta, \prec_\beta)$ is a tree.
(2_β) For each $\gamma < \beta$, T_γ is a subtree of T_β; that is, $\prec_\gamma = \prec_\beta \restriction (\omega \cdot \gamma)$.
(3_β) For each $\gamma < \beta$, $\mathrm{Lev}_\gamma(T_\beta) = \{\omega \cdot \gamma + m : m \in \omega\}$.
(4_β) For all $\gamma < \delta < \beta$ and $m \in \omega$ there exists $n \in \omega$ such that $\omega \cdot \gamma + m \prec_\beta \omega \cdot \delta + n$.
(5_β) Whenever $\delta < \beta$ is a limit ordinal satisfying $\omega \cdot \delta = \delta$, and A_δ is a maximal antichain in T_δ, we have that for each $x \in \mathrm{Lev}_\delta(T_\beta)$ there exists $y \in A_\delta$ such that $y \prec_\beta x$.

Conditions (1_β)–(3_β) ($\beta < \omega_1$) here just say that the tree $T = (\omega_1, \prec)$ $\left(\text{with } \prec := \bigcup_{\beta<\omega_1} \prec_\beta\right)$ has the form outlined at the beginning of the proof, conditions (4_β) ($\beta < \omega_1$) make sure that T is well-pruned from each root up (T will have infinitely many roots!), and conditions (5_β) ($\beta < \omega_1$) have the effect of forcing T to satisfy assumption ($*$) in Lemma 25.

Now we describe the construction of the relations $\prec_\alpha$ on $\omega \cdot \alpha$ ($\alpha < \omega_1$) by recursion.

- For $\alpha \le 1$, we define $\prec_\alpha := \emptyset$. Clearly, conditions ($1_\alpha$)–($5_\alpha$) hold.
  Notice that the set $\mathrm{Lev}_0(T_1)$ of roots of T_1 (and hence of T) is $\omega \cdot 1 = \omega$.

From now on let $\alpha > 1$, and assume that the relations $\prec_\beta$ on $\omega \cdot \beta$ have been constructed for all $\beta < \alpha$ so that all conditions (1_β)–(5_β) are met.

- If α is a limit ordinal, we define $\prec_\alpha$ on $\omega \cdot \alpha$ by $\prec_\alpha := \bigcup_{\beta<\alpha} \prec_\beta$.
  It is easy to see that conditions (1_α)–(5_α) are satisfied.

- If $\alpha = \varepsilon + 2$ for some ordinal ε, then we define $\prec_\alpha$ on $\omega \cdot \alpha$ by
$$\begin{aligned}\prec_\alpha := \prec_{\varepsilon+1} &\cup \{(\xi,\, \omega \cdot (\varepsilon+1) + 2m) : \xi \preceq_{\varepsilon+1} \omega \cdot \varepsilon + m,\, m \in \omega\} \\ &\cup \{(\xi,\, \omega \cdot (\varepsilon+1) + 2m + 1) : \xi \preceq_{\varepsilon+1} \omega \cdot \varepsilon + m,\, m \in \omega\}.\end{aligned}$$
Again, it is easy to check that conditions (1_α)–(5_α) hold.

- Finally, let $\alpha = \varepsilon + 1$ where ε is a limit ordinal. In this case, the definition of $\prec_\alpha$ requires several steps. The goal of the first four steps is to assign a(n appropriately chosen) branch of T_ε to every element of T_ε. So, for steps 1–4 below, let $x \in \omega \cdot \varepsilon$ be an arbitrary element of T_ε.
  1. First, we choose an element y_0^x of T_ε as follows:
     - If $\omega \cdot \varepsilon = \varepsilon$ and A_ε is a maximal antichain in T_ε, and hence there exists $z \in A_\varepsilon$ such that z is comparable to x, then fix such a z and let y_0^x be an element of T_ε such that $x, z \prec_\varepsilon y_0^x$.
     - Otherwise, let $y_0^x = x$.
  2. Let $\langle \xi_n : n \in \omega \rangle$ be a strictly increasing sequence of ordinals $< \varepsilon$ such that $\xi_0 := \mathrm{ht}(y_0^x, T_\varepsilon)$ and $\bigcup_{n \in \omega} \xi_n = \varepsilon$. (Such a sequence exists, because $\mathrm{cf}(\varepsilon) = \omega$.)
  3. Use the conditions (4_{ξ_n}) $(n < \omega)$ to extend y_0^x, by recursion on ω, to a sequence $\langle y_n^x : n < \omega \rangle$ such that $\mathrm{ht}(y_n^x, T_\varepsilon) = \xi_n$ for all $n < \omega$.
  4. Let $B(x)$ be the unique branch of T_ε containing all elements y_n^x $(n < \omega)$; that is, let
  $$B(x) := \{u \in \omega \cdot \varepsilon : u \le y_n^x \text{ for some } n < \omega\}.$$
  5. Now, choose and fix a bijection $\omega \to \omega \cdot \varepsilon$, $n \mapsto x_n$, and define $\prec_\alpha$ as follows:
  $$\prec_\alpha := \prec_\varepsilon \cup \{(u, \omega \cdot \varepsilon + n) : u \in B(x_n)\}.$$

  It is not hard to verify that conditions (1_α)–(5_α) hold.

This finishes the construction of the trees T_β $(\beta < \omega_1)$ so that all conditions (1_β)–(5_β) $(\beta < \omega_1)$ are satisfied. Hence, $T = (\omega_1, \prec)$ $\left(\text{with } \prec := \bigcup_{\beta<\omega_1} \prec_\beta\right)$ is an ω_1-tree. The construction at levels $\alpha = \varepsilon + 2$ shows that T is eventually branching, and conditions (5_β) $(\beta < \omega_1)$ ensure that T satisfies assumption $(*)$ of Lemma 25. Hence, T is a Suslin tree. $\square$