Set Theory (MATH 6730)

Trees

From now on we will work in ZFC.

Definition 1. A *tree* is a partially ordered set $(T, <)$ with the property that for every $t \in T$ the set $\{s \in T : s < t\}$ is well-ordered by $<$. If $(T, <)$ is a tree,

- the *height* of an element $t \in T$ is the order type of the set $\{s \in T : s < t\}$ (a uniquely determined ordinal), and is denoted by $\mathrm{ht}(t, T)$ or $\mathrm{ht}(t)$;
- the *height* of T, denoted by $\mathrm{ht}(T)$, is the least ordinal greater than all ordinals $\mathrm{ht}(t)$, $t \in T$;
- an element of T of height 0 is called a *root* of T;
- for each ordinal α, *the α-th level* of T, denoted by $\mathrm{Lev}_\alpha(T)$, is the set of all elements of T of height α;
- a *chain* in T is a subset of T linearly ordered (hence well-ordered) by $<$;
- the *length of a chain* C in T is the order type of C (a uniquely determined ordinal);
- a *branch* of T is a maximal chain in T;
- an *antichain* in T is a subset X of T such that any two distinct elements of X are incomparable.

Notation 2. For any tree $(T, <)$ and $t \in T$ we will denote the set $\{u \in T : t \leq u\}$ by $\mathrm{Up}(t, T)$ or simply $\mathrm{Up}(t)$.

Notation 3. For any ordinal α and any set S, let ${}^{<\alpha}S$ denote the set of all functions $\beta \to S$ such that $\beta < \alpha$. Equivalently,

$${}^{<\alpha}S = \bigcup_{\beta<\alpha} {}^{\beta}S = \{\langle s_\gamma : \gamma < \beta\rangle : \beta < \alpha,\ s_\gamma \in S \text{ for all } \gamma < \beta\}.$$

Example 4.

(i) $(\alpha, <)$ is a tree for every ordinal; its height is α; it has a unique branch, namely α itself, and the length of the branch is also α.

(ii) $({}^{<\alpha}2, \subset)$ is a tree for every ordinal α;[1] its height is α, and every branch has length α.

(iii) $(\mathbb{Q}, <)$ is not a tree.

(iv) The level sets of a tree are antichains in the tree.

Theorem 5. (König's Tree Lemma) *Every tree of height ω in which all levels are finite has an infinite branch.*

[1]Throughout, $\subset$ denotes proper inclusion.

Definition 6. Let κ be an infinite cardinal. A tree is called

- a *κ-tree* if it has height κ and every level has size $< \kappa$;
- a *κ-Aronszajn tree* if it is a κ-tree and has no chain of size κ;
- a *κ-Suslin tree* if it has height κ and has no chains or antichains of size κ.

An *Aronszajn tree* is an ω_1-Aronszajn tree, while a *Suslin tree* is an ω_1-Suslin tree.

Facts 7.

(i) *König's Theorem is equivalent to saying that there is no ω-Aronszajn tree.*
(ii) *For any infinite cardinal κ and for any tree $(T, <)$,*

$$T \text{ is a } \kappa\text{-Suslin tree} \Rightarrow T \text{ is a } \kappa\text{-Aronszajn tree} \Rightarrow T \text{ is a } \kappa\text{-tree.}$$

Theorem 8. *If κ is a singular cardinal, then there exists a κ-Suslin tree.*

Proof. Since κ is singular, there exists a strictly increasing sequence $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ of cardinals such that $\bigcup_{\alpha < \mathrm{cf}(\kappa)} \lambda_\alpha = \kappa$. Consider the tree with a single root r which is the union of 'almost disjoint' branches B_α of lengths λ_α $(\alpha < \mathrm{cf}(\kappa))$, that is, $B_\alpha \cap B_\beta = \{r\}$ for all $\alpha < \beta < \mathrm{cf}(\kappa)$. □

There are no results in ZFC about the existence or nonexistence of κ-Suslin trees for uncountable regular cardinals κ. In particular, for $\kappa = \omega_1$ it is known that the existence of a Suslin tree is independent of ZFC.

However, for Aronszajn trees, we have the following theorem.

Theorem 9. *There exists an Aronszajn tree.*

Proof. The desired tree will be constructed as a subtree of $(T, \subset)$ where

$$T = \{s \in {}^{<\omega_1}\omega : s \text{ is one-to-one}\}.$$

We will define the system $\langle S_\alpha : \alpha < \omega_1 \rangle$ of levels of the desired tree by recursion so that the following conditions are satisfied for each $\beta < \omega_1$:

(1_β) $S_\beta \subseteq {}^\beta\omega \cap T$.
(2_β) $\omega \setminus \operatorname{rng}(s)$ is infinite for every $s \in S_\beta$.
(3_β) For all $s \in S_\gamma$ with $\gamma < \beta$ there exists $t \in S_\beta$ such that $s \subset t$.
(4_β) $|S_\beta| \leq \omega$.
(5_β) If $s \in S_\beta$ and $t \in {}^\beta\omega \cap T$ are such that $\{\gamma < \beta : s(\gamma) \neq t(\gamma)\}$ is finite, then $t \in S_\beta$.
(6_β) If $s \in S_\beta$ and $\gamma < \beta$, then $s \restriction \gamma \in S_\gamma$.

If this can be achieved, then the tree $\left(\bigcup_{\alpha<\omega_1} S_\alpha, \subset\right)$ is an Aronszajn tree. (Why?)

Let α be an ordinal $< \omega_1$. For $\alpha = 0$, let $S_0 = \{\emptyset\}$. Clearly, conditions (1_0)–(6_0) hold.

Now let $\alpha > 0$, and assume that the system $\langle S_\beta : \beta < \alpha \rangle$ of levels has been defined so that conditions (1_β)–(6_β) hold for all $\beta < \alpha$. Our goal is to define S_α so that conditions (1_α)–(6_α) hold.

First, let $\alpha\,(< \omega_1)$ be a successor ordinal, say $\alpha = \varepsilon + 1$. Define S_α to be the set of all $t \in {}^\alpha\omega \cap T$ that extend members of S_ε; that is,

$$S_\alpha = \big\{s \cup \{(\varepsilon, n)\} : s \in S_\varepsilon,\ n \in \omega \setminus \operatorname{rng}(s)\big\}.$$

It is straightforward to verify that conditions (1_α)–(6_α) are satisfied.

(1_β) $S_\beta \subseteq {}^\beta\omega \cap T$.
(2_β) $\omega \setminus \mathrm{rng}(s)$ is infinite for every $s \in S_\beta$.
(3_β) For all $s \in S_\gamma$ with $\gamma < \beta$ there exists $t \in S_\beta$ such that $s \subset t$.
(4_β) $|S_\beta| \leq \omega$.
(5_β) If $s \in S_\beta$ and $t \in {}^\beta\omega \cap T$ are such that $\{\gamma < \beta : s(\gamma) \neq t(\gamma)\}$ is finite, then $t \in S_\beta$.
(6_β) If $s \in S_\beta$ and $\gamma < \beta$, then $s{\restriction}\gamma \in S_\gamma$.

Now let $\alpha\,(< \omega_1)$ be a limit ordinal. Since α is countable, we have $\mathrm{cf}(\alpha) = \omega$. Hence, there exists a strictly increasing sequence $\langle \delta_n : n \in \omega \rangle$ of ordinals such that $\bigcup_{n\in\omega} \delta_n = \alpha$. Choose and fix such a sequence. Let $U = \bigcup_{\beta<\alpha} S_\beta$. Given any $s \in U$, say $\mathrm{dmn}(s) = \beta$, we want to define an extension $t_s \in {}^\alpha\omega \cap T$ of s such that $\omega \setminus \mathrm{rng}(t_s)$ is infinite. The steps are as follows:

- Let $n \in \omega$ be minimal with $\beta \leq \delta_n$. Use $(3)_{\delta_{n+i}}$ for $i \in \omega$ to define a sequence $\langle u_i : i \in \omega \rangle$ by recursion such that $s \subseteq u_0$, $u_i \in S_{\delta_{n+i}}$, and $u_i \subset u_{i+1}$ for all $i \in \omega$.
- $v = \bigcup_{i\in\omega} u_i$ satisfies $s \subseteq v \in {}^\alpha\omega \cap T$, but it may fail that $\omega \setminus \mathrm{rng}(v)$ is infinite.
- Modify v at all places δ_{n+i} ($i \in \omega$) to get t_s.

Define S_α by

$$S_\alpha = \bigcup_{s\in U} \big\{ w \in {}^\alpha\omega \cap T : \{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\} \text{ is finite} \big\},$$

and check that conditions (1_α)–(6_α) are satisfied. □

Remark 10. The Aronszajn tree $(S, \subset) = \big(\bigcup_{\alpha<\omega_1} S_\alpha, \subset\big)$ constructed in the proof of Theorem 9 is not a Suslin tree, because the sets

$$A_n = \bigcup_{\alpha<\omega_1} \{s \in S_{\alpha+1} : s(\alpha) = n\} \quad (n \in \omega)$$

are antichains in $(S, \subset)$, but since $\bigcup_{n\in\omega} A_n = \bigcup_{\alpha<\omega_1} S_{\alpha+1}$ and $\big|\bigcup_{\alpha<\omega_1} S_{\alpha+1}\big| = \omega_1$, we get that at least one of the antichains A_n ($n \in \omega$) has cardinality ω_1.

For the rest of this section we will focus on Suslin trees. Our main objectives are

(I) to establish a sufficient condition for a κ-tree (κ uncountable regular) to be a κ-Suslin tree, which we will use later on in the course to prove the existence of a Suslin tree under an extra assumption added to ZFC;

(II) to discuss the relationship between Suslin trees and Suslin lines, which motivates the study of Suslin trees.

We need some preparation.

Definition 11. Let $(T,<)$ be a tree, and let κ be an infinite cardinal.

- We say that T is *eventually branching* if for every $t \in T$ the set $\mathrm{Up}(t)$ is not a chain in T.
- A *normal subtree* of $(T,<)$ is a tree $(S,\prec)$ such that
  - $(S,\prec)$ is a *subtree* of $(T,<)$, that is, $S \subseteq T$ and $\prec\, = \,<\, \cap\, (S \times S)$; and
  - for any $t, t' \in T$, if $t < t'$ and $t' \in S$, then $t \in S$.
- T is called a *well-pruned κ-tree* if
  - T is a κ-tree with exactly one root, and
  - for all $\alpha < \beta < \mathrm{ht}(T)$ and for every $x \in \mathrm{Lev}_\alpha(T)$ there exists $y \in \mathrm{Lev}_\beta(T)$ such that $x < y$.

Example 12. The Aronszajn tree $(S,\subset) = \left(\bigcup_{\alpha<\omega_1} S_\alpha, \subset\right)$ constructed in the proof of Theorem 9 is

- a normal subtree of $(T,\subset)$ where $T = \{s \in {}^{<\omega_1}\omega : s \text{ is one-to-one}\}$; and is
- an eventually branching, well-pruned ω_1-tree.

Facts 13. *Let T be a tree, and let κ be an infinite cardinal.*

(i) *If S is a normal subtree of T, then* $\mathrm{ht}(s,T) = \mathrm{ht}(s,S)$ *for all $s \in S$.*

(ii) *A normal subtree of height κ of a κ-Aronszajn tree is a κ-Aronszajn tree; a normal subtree of height κ of a κ-Suslin tree is a κ-Suslin tree.*

(iii) *A well-pruned κ-Aronszajn tree is eventually branching.*

Theorem 14. *Let κ be a regular cardinal, and let T be an arbitrary κ-tree.*

(i) *T has a normal subtree T' which is a well-pruned κ-tree.*
(ii) *Moreover, if $x \in T$ is such that $|\mathrm{Up}(x)| = \kappa$, then T has a normal subtree T' containing x which is a well-pruned κ-tree.*

Idea of Proof. Argue that

- Under the assumptions of (i), T has a root r such that $|\mathrm{Up}(r)| = \kappa$.
  Under the assumptions of (ii), T has a root $r \leq x$ such that $|\mathrm{Up}(r)| = \kappa$.
- The (normal!) subtree T' of T defined by $T' = \{t \in T : r \leq t,\ |\mathrm{Up}(t)| = \kappa\}$ is a well-pruned κ-tree. □

Theorem 15. *Let κ be an uncountable regular cardinal. If T is an eventually branching κ-tree such that every antichain in T has size $< \kappa$, then T is a κ-Suslin tree.*

Idea of Proof. Arguing the contrapositive, we consider any eventually branching κ-tree T such that T has a chain C of length κ, and prove that T has an antichain of size κ.

- We may assume that C is a branch, i.e. contains elements from each level of T.
- There exists a function $f\colon C \to T$ such that $t < f(t) \notin C$ for all $t \in C$.
- Now define $\langle s_\alpha : \alpha < \kappa \rangle \in {}^\kappa C$ by recursion so that $\mathrm{ht}(s_\alpha) > \bigcup_{\beta<\alpha} \mathrm{ht}\big(f(s_\beta)\big)$ for all $\alpha < \kappa$.
- Then $\{f(s_\alpha) : \alpha < \kappa\}$ is an antichain of size κ. □

Suslin trees are closely related to Suslin lines, which occurred naturally in set theory (set theoretical topology) in connection with a possible weakening of a classical characterization of the real line $(\mathbb{R}, <)$. We need some definitions.

Definition 16. Let $(L, <)$ be a linear order.

- We say that $(L, <)$ is *densely ordered* (or briefly *dense*) if $|L| > 1$ and for any $a < b$ in L there exists $c \in L$ such that $a < c < b$. A subset X of L is *dense in L (in the order sense)* if for any $a < b$ in L there exists $c \in X$ such that $a < c < b$.
- The subsets of L of the form $(a, b) = \{x \in L : a < x < b\}$, $(-\infty, b) = \{x \in L : x < b\}$, $(a, \infty) = \{x \in L : a < x\}$ with $a, b \in L$ are called *open intervals*. A subset U of L is called *open* if $U = L$ or U is a union of open intervals.
- A subset D of L is *topologically dense* in L if $D \cap U \neq \emptyset$ for every nonempty open subset U of L.
- L is called *separable* if L has a countable subset which is topologically dense in L.
- An *antichain*[2] in L is a set of pairwise disjoint open subsets of L.
- L is said to satisfy the *countable chain condition* (*ccc*) if every antichain in L is countable.

Facts 17. *Let $(L, <)$ be a linear order.*

(i) *If L has a dense subset (in the order sense) and $|L| > 1$, then L is densely ordered.*
(ii) *Every dense subset of L (in the order sense) is topologically dense in L.*
(iii) *If L is densely ordered then, conversely, every topologically dense subset of L is dense in L (in the order sense).*
(iv) *If L is separable, then L satisfies ccc.*

Theorem 18. *The following conditions on a linear order $(L, \prec)$ are equivalent:*

(a) *$(L, \prec)$ is isomorphic to $(\mathbb{R}, <)$.*
(b) *$(L, \prec)$ has the following properties:*
  (†) *L is densely ordered and has no least or greatest elements; moreover, in L, every nonempty subset that is bounded above has a least upper bound.*
  (‡) *L is separable.*

Suslin asked (1920) whether the separability condition (‡) in this theorem could be replaced by the condition that L has ccc. The assumption that the answer to this question is 'yes' is referred to as the "Suslin Hypothesis".

Theorem 19. *The following statements are equivalent (in* ZFC*):*

(a) *There exists a linear order $(S, <)$ such that*
  - *$(S, <)$ satisfies ccc, and*
  - *$(S, <)$ is not separable.*
(b) *There exists a linear order $(L, \prec)$ that is a counterexample to the Suslin Hypothesis.*
(c) *There exists a linear order $(L, \prec)$ such that*
  - *$(L, \prec)$ satisfies (†),*
  - *$(L, \prec)$ satisfies ccc, and*
  - *no nonempty open subset of $(L, <)$ is separable.*

[2]This notion is different from antichains as defined in Definition 1.

Definition 20. We will call a linear order $(S,<)$ a *Suslin line*[3] if it satisfies the conditions in statement (a) of Theorem 19.

By Theorem 19, the Suslin Hypothesis can be restated as follows: *There is no Suslin line.*

Theorem 21. *The following statements are equivalent (in* ZFC*):*

(a) *There exists a Suslin line.*
(b) *There exists a Suslin tree.*

Idea of Proof. (b) $\Rightarrow$ (a): We use a general construction that creates a 'line' (a linear order) from a tree.

- Given any tree $(T,<)$ and any linear order $\prec$ on T (unrelated to $<$) we define a relation $\lessdot$ on the set $\mathcal{B}$ of all branches of T as follows:
  - Note that since each $B \in \mathcal{B}$ is a maximal chain in T, it contains a unique element b^B_α of height α for every $\alpha < \operatorname{len}(B)$ where $\operatorname{len}(B)$ is the length of B.
  - For any two distinct branches $B_1, B_2 \in \mathcal{B}$, define $B_1 \lessdot B_2$ to hold iff $b^{B_1}_\alpha \prec b^{B_2}_\alpha$ for the smallest ordinal $\alpha < \min\big(\operatorname{len}(B_1), \operatorname{len}(B_2)\big)$ such that $b^{B_1}_\alpha \neq b^{B_2}_\alpha$. (Such an α exists, because $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.)
- $(\mathcal{B},\lessdot)$ is a linear order.
- If $(T,<)$ is a well-pruned Suslin tree and $\prec$ is any linear order on T, then $(\mathcal{B},\lessdot)$ is a Suslin line.

So, if there exists a Suslin tree, then by Theorem 14(i) and Facts 13(ii), there also exists a well-pruned Suslin tree, and the construction above yields a Suslin line.

(a) $\Rightarrow$ (b): Assume there exists a Suslin line. By Theorem 19, there exists a Suslin line $(L,\prec)$ satisfying all conditions in part (c) of the theorem.[4] Let $\mathbb{I}$ denote the set of all intervals (a,b) with $a \prec b$ in L.

- One can define (by recursion) a system $\langle \mathbb{J}_\alpha : \alpha < \omega_1 \rangle$ of nonempty subsets of $\mathbb{I}$ with the following properties:
  - The elements of $\mathbb{J}_\alpha$ a pairwise disjoint for every $\alpha < \omega_1$.
  - The sets $\mathbb{J}_\alpha$ $(\alpha < \omega_1)$ are pairwise disjoint.
  - For $T = \bigcup_{\alpha<\omega_1} \mathbb{J}_\alpha$, the partially ordered set $(T, \supset)$ is a tree with $\operatorname{Lev}_\alpha(T) = \mathbb{J}_\alpha$ for every $\alpha < \omega_1$.
  - If $\gamma < \alpha < \omega_1$ and $I \in \mathbb{J}_\gamma$, then there are at least two $J \in \mathbb{J}_\alpha$ such that $I \supset J$.
  - If $\gamma < \alpha < \omega_1$ and $I \in \mathbb{J}_\gamma$, $J \in \mathbb{J}_\alpha$, then either $I \supset J$ or $I \cap J = \emptyset$.
- The last three items imply, respectively, the following:
  - T has height ω_1.
  - T is eventually branching.
  - Every antichain (in the sense of Definition 1) in T is an antichain (in the sense of Definition 16) in the Suslin line $(L,\prec)$. Since L satisfies ccc, every antichain in T is countable.

It follows from Theorem 15 that T is a Suslin tree.

□

[3]A Suslin line is often defined as a linear order that is a counterexample to the Suslin Hypothesis. This definition is not equivalent to our definition. Only the existence of the two kinds of Suslin lines is equivalent.

[4]Actually, the completeness property in the second line of (†) is not used in the construction.