CHAPTER II
THE LIMIT OF A SEQUENCE OF NUMBERS
DEFINITION OF THE NUMBER e.

This chapter contains the beginnings of the most important, and probably the most subtle, notion in mathematical analysis, i.e., the concept of a limit. Though Newton and Leibniz discovered the calculus with its tangent lines described as limits of secant lines, and though the Greeks were already estimating areas of regions by a kind of limiting process, the precise notion of limit that we use today was not formulated until the 19th century by Cauchy and Weierstrass.

The main results of this chapter are the following:

(1) The definition of the **limit of a sequence**,
(2) The definition of the real number e (Theorem 2.3),
(3) The **Squeeze Theorem** (Theorem 2.5),
(4) the **Bolzano Weierstrass Theorem** (Theorems 2.8 and 2.10),
(5) The **Cauchy Criterion** (Theorem 2.9),
(6) the definition of an **infinite series**,
(7) the **Comparison Test** (Theorem 2.17), and
(8) the **Alternating Series Test** (Theorem 2.18).

These are powerful basic results about limits that will serve us well in later chapters.

SEQUENCES AND LIMITS

DEFINITION. A *sequence* of real or complex numbers is defined to be a function from the set $\mathbb{N}$ of natural numbers into the set$\mathbb{R}$ or $\mathbb{C}$. Instead of referring to such a function as an assignment $n \to f(n)$, we ordinarily use the notation $\{a_n\}$, $\{a_n\}_1^\infty$, or $\{a_1, a_2, a_3, \dots\}$. Here, of course, a_n denotes the number $f(n)$.

REMARK. We expand this definition slightly on occasion to make some of our notation more indicative. That is, we sometimes **index** the terms of a sequence beginning with an integer other than 1. For example, we write $\{a_n\}_0^\infty$, $\{a_0, a_1, \dots\}$, or even $\{a_n\}_{-3}^\infty$.

We give next what is the most significant definition in the whole of mathematical analysis, i.e., what it means for a sequence to converge or to have a limit.

DEFINITION. Let $\{a_n\}$ be a sequence of real numbers and let L be a real number. The sequence $\{a_n\}$ is said to *converge* to L, or that L is the *limit* of $\{a_n\}$, if the following condition is satisfied. For every positive number ϵ, there exists a natural number N such that if $n \geq N$, then $|a_n - L| < \epsilon$.

In symbols, we say $L = \lim a_n$ or

$$L = \lim_{n\to\infty} a_n.$$

We also may write $a_n \mapsto L$.

If a sequence $\{a_n\}$ of real or complex numbers converges to a number L, we say that the sequence $\{a_n\}$ is *convergent*.

We say that a sequence $\{a_n\}$ of real numbers *diverges* to $+\infty$ if for every positive number M, there exists a natural number N such that if $n \geq N$, then $a_n \geq M$. Note that we do **not** say that such a sequence is convergent.

Similarly, we say that a sequence $\{a_n\}$ of real numbers *diverges* to $-\infty$ if for every real number M, there exists a natural number N such that if $n \geq N$, then $a_n \leq M$.

The definition of convergence for a sequence $\{z_n\}$ of complex numbers is exactly the same as for a sequence of real numbers. Thus, let $\{z_n\}$ be a sequence of complex numbers and let L be a complex number. The sequence $\{z_n\}$ is said to *converge* to L, or that L is the *limit* of $\{z_n\}$, if the following condition is satisfied. For every positive number ϵ, there exists a natural number N such that if $n \geq N$, then $|z_n - L| < \epsilon$.

REMARKS. The natural number N of the preceding definition surely depends on the positive number ϵ. If ϵ' is a smaller positive number than ϵ, then the corresponding N' very likely will need to be larger than N. Sometimes we will indicate this dependence by writing $N(\epsilon)$ instead of simply N. It is always wise to remember that N depends on ϵ. On the other hand, the N or $N(\epsilon)$ in this definition is not unique. It should be clear that if a natural number N satisfies this definition, then any larger natural number M will also satisfy the definition. So, in fact, if there exists one natural number that works, then there exist infinitely many such natural numbers.

It is clear, too, from the definition that whether or not a sequence is convergent only depends on the "tail" of the sequence. Specifically, for any positive integer K, the numbers $a_1, a_2, \ldots, a_K$ can take on any value whatsoever without affecting the convergence of the entire sequence. We are only concerned with a_n's for $n \geq N$, and as soon as N is chosen to be greater than K, the first part of the sequence is irrelevant.

The definition of convergence is given as a fairly complicated sentence, and there are several other ways of saying the same thing. Here are two: For every $\epsilon > 0$, there exists a N such that, whenever $n \geq N$, $|a_n - L| < \epsilon$. And, given an $\epsilon > 0$, there exists a N such that $|a_n - L| < \epsilon$ for all n for which $n \geq N$. It's a good idea to think about these two sentences and convince yourself that they really do "mean" the same thing as the one defining convergence.

It is clear from this definition that we can't check whether a sequence converges or not unless we know the limit value L. The whole thrust of this definition has to do with estimating the quantity $|a_n - L|$. We will see later that there are ways to tell in advance that a sequence converges without knowing the value of the limit.

EXAMPLE 2.1. Let $a_n = 1/n$, and let us show that $\lim a_n = 0$. Given an $\epsilon > 0$, let us choose a N such that $1/N < \epsilon$. (How do we know we can find such a N?) Now, if $n \geq N$, then we have

$$|a_n - 0| = |\frac{1}{n}| = \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

which is exactly what we needed to show to conclude that $0 = \lim a_n$.

EXAMPLE 2.2. Let $a_n = (2n+1)/(1-3n)$, and let $L = -2/3$. Let us show that $L = \lim a_n$. Indeed, if $\epsilon > 0$ is given, we must find a N, such that if $n \geq N$ then $|a_n + (2/3)| < \epsilon$. Let us examine the quantity $|a_n + 2/3|$. Maybe we can make some estimates on it, in such a way that it becomes clear how to find the natural number N.

$$\begin{aligned} |a_n + (2/3)| &= |\frac{2n+1}{1-3n} + \frac{2}{3}| \\ &= |\frac{6n+3+2-6n}{3-9n}| \\ &= |\frac{5}{3-9n}| \\ &= \frac{5}{9n-3} \\ &= \frac{5}{6n+3n-3} \\ &\leq \frac{5}{6n} \\ &< \frac{1}{n}, \end{aligned}$$

for all $n \geq 1$. Therefore, if N is an integer for which $N > 1/\epsilon$, then

$$|a_n + 2/3| < 1/n \leq 1/N < \epsilon,$$

whenever $n \geq N$, as desired. (How do we know that there exists a N which is larger than the number $1/\epsilon$?)

EXAMPLE 2.3. Let $a_n = 1/\sqrt{n}$, and let us show that $\lim a_n = 0$. Given an $\epsilon > 0$, we must find an integer N that satisfies the requirements of the definition. It's a little trickier this time to choose this N. Consider the positive number ϵ^2. We know, from Exercise 1.16, that there exists a natural number N such that $1/N < \epsilon^2$. Now, if $n \geq N$, then

$$|a_n - 0| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} = \sqrt{\frac{1}{N}} < \sqrt{\epsilon^2} = \epsilon,$$

which shows that $0 = \lim 1/\sqrt{n}$.

REMARK. A good way to attack a limit problem is to immediately examine the quantity $|a_n - L|$, which is what we did in Example 2.2 above. This is the quantity we eventually wish to show is less than ϵ when $n \geq N$, and determining which N to use is always the hard part. Ordinarily, some algebraic manipulations can be performed on the expression $|a_n - L|$ that can help us figure out exactly how to choose N. Just know that this process takes some getting used to, so practice!

Exercise 2.1. (a) Using the basic definition, prove that $\lim 3/(2n+7) = 0$.

(b) Using the basic definition, prove that $\lim 1/n^2 = 0$.

(c) Using the basic definition, prove that $\lim (n^2+1)/(n^2+100n) = 1$.

HINT: Use the idea from the remark above; i.e., examine the quantity $|a_n - L|$.

(d) Again, using the basic definition, prove that

$$\lim \frac{n+n^2 i}{n-n^2 i} = -1.$$

Remember the definition of the absolute value of a complex number.

(e) Using the basic definition, prove that

$$\lim \frac{n^3 + n^2 i}{1 - n^3 i} = i.$$

(f) Let $a_n = (-1)^n$. Prove that 1 is **not** the limit of the sequence $\{a_n\}$.
HINT: Suppose the sequence $\{a_n\}$ does converge to 1. Use $\epsilon = 1$, let N be the corresponding integer that exists in the definition, satisfying $|a_n - 1| < 1$ for all $n \geq N$, and then examine the quantity $|a_n - 1|$ for various n's to get a contradiction.

Exercise 2.2. (a) Let $\{a_n\}$ be a sequence of (real or complex) numbers, and let L be a number. Prove that $L = \lim a_n$ if and only if for every positive integer k there exists an integer N, such that if $n \geq N$ then $|a_n - L| < 1/k$.

(b) Let $\{c_n\}$ be a sequence of complex numbers, and suppose that $c_n \mapsto L$. If $c_n = a_n + b_n i$ and $L = a + bi$, show that $a = \lim a_n$ and $b = \lim b_n$. Conversely, if $a = \lim a_n$ and $b = \lim b_n$, show that $a + bi = \lim(a_n + b_n i)$. That is, a sequence $\{c_n = a_n + b_n i\}$ of complex numbers converges if and only if the sequence $\{a_n\}$ of the real parts converges and the sequence $\{b_n\}$ of the imaginary parts converges.
HINT: You need to show that, given some hypotheses, certain quantities are less than ϵ. Part (c) of Exercise 1.25 should be of help.

Exercise 2.3. (a) Prove that a constant sequence ($a_n \equiv c$) converges to c.

(b) Prove that the sequence $\{\frac{2n^2+1}{1-3n}\}$ diverges to $-\infty$.

(c) Prove that the sequence $\{(-1)^n\}$ does not converge to any number L.
HINT: Argue by contradiction. Suppose it does converge to a number L. Use $\epsilon = 1/2$, let N be the corresponding integer that exists in the definition, and then examine $|a_n - a_{n+1}|$ for $n \geq N$. Use the following useful add and subtract trick:

$$|a_n - a_{n+1}| = |a_n - L + L - a_{n+1}| \leq |a_n - L| + |L - a_{n+1}|.$$

EXISTENCE OF CERTAIN FUNDAMENTAL LIMITS

We have, in the preceding exercises, seen that certain specific sequences converge. It's time to develop some general theory, something that will apply to lots of sequences, and something that will help us actually evaluate limits of certain sequences.

DEFINITION. A sequence $\{a_n\}$ of real numbers is called *nondecreasing* if $a_n \leq a_{n+1}$ for all n, and it is called *nonincreasing* if $a_n \geq a_{n+1}$ for all n. It is called *strictly increasing* if $a_n < a_{n+1}$ for all n, and *strictly decreasing* if $a_n > a_{n+1}$ for all n.

A sequence $\{a_n\}$ of real numbers is called *eventually nondecreasing* if there exists a natural number N such that $a_n \leq a_{n+1}$ for all $n \geq N$, and it is called *eventually nonincreasing* if there exists a natural number N such that $a_n \geq a_{n+1}$ for all $n \geq N$. We make analogous definitions of "eventually strictly increasing" and "eventually strictly decreasing."

It is ordinarily very difficult to tell whether a given sequence converges or not; and even if we know in theory that a sequence converges, it is still frequently difficult to tell what the limit is. The next theorem is therefore very useful. It is also very fundamental, for it makes explicit use of the existence of a least upper bound.

THEOREM 2.1. *Let* $\{a_n\}$ *be a nondecreasing sequence of real numbers. Suppose that the set* S *of elements of the sequence* $\{a_n\}$ *is bounded above. Then the sequence* $\{a_n\}$ *is convergent, and the limit* L *is given by*$L = \sup S = \sup a_n$.

Analogously, if $\{a_n\}$ *is a nonincreasing sequence that is bounded below, then* $\{a_n\}$ *converges to* $\inf a_n$.

PROOF. We prove the first statement. The second is done analogously, and we leave it to an exercise. Write L for the supremum $\sup a_n$. Let ϵ be a positive number. By Theorem 1.5, there exists an integer N such that $a_N > L - \epsilon$, which implies that $L - a_N < \epsilon$. Since $\{a_n\}$ is nondecreasing, we then have that $a_n \geq a_N > L - \epsilon$ for all $n \geq N$. Since L is an upper bound for the entire sequence, we know that $L \geq a_n$ for every n, and so we have that

$$|L - a_n| = L - a_n \leq L - a_N < \epsilon$$

for all $n \geq N$. This completes the proof of the first assertion.

Exercise 2.4. (a) Prove the second assertion of the preceding theorem.

(b) Show that Theorem 2.1 holds for sequences that are eventually nondecreasing or eventually nonincreasing. (Re-read the remark following the definition of the limit of a sequence.)

The next exercise again demonstrates the "denseness" of the rational and irrational numbers in the set $\mathbb{R}$ of all real numbers.

Exercise 2.5. (a) Let x be a real number. Prove that there exists a sequence $\{r_n\}$ of rational numbers such that $x = \lim r_n$. In fact, show that the sequence $\{r_n\}$ can be chosen to be nondecreasing.
HINT: For example, for each n, use Theorem 1.8 to choose a rational number r_n between $x - 1/n$ and x.

(b) Let x be a real number. Prove that there exists a sequence $\{r'{}_n\}$ of irrational numbers such that $x = \lim r'_n$.

(c) Let $z = x + iy$ be a complex number. Prove that there exists a sequence $\{\alpha_n\} = \{\beta_n + i\gamma_n\}$ of complex numbers that converges to z, such that each β_n and each γ_n is a rational number.

Exercise 2.6. Suppose $\{a_n\}$ and $\{b_n\}$ are two convergent sequences, and suppose that $\lim a_n = a$ and $\lim b_n = b$. Prove that the sequence $\{a_n + b_n\}$ is convergent and that

$$\lim(a_n + b_n) = a + b.$$

HINT: Use an $\epsilon/2$ argument. That is, choose a natural number N_1 so that $|a_n - a| < \epsilon/2$ for all $n \geq N_1$, and choose a natural number N_2 so that $|b_n - b| < \epsilon/2$ for all $n \geq N_2$. Then let N be the larger of the two numbers N_1 and N_2.

The next theorem establishes the existence of four nontrivial and important limits. This time, the proofs are more tricky. Some clever idea will have to be used before we can tell how to choose the N.

THEOREM 2.2.

(1) *Let* $z \in \mathbb{C}$ *satisfy* $|z| < 1$, *and define* $a_n = z^n$. *then the sequence* $\{a_n\}$ *converges to* 0. *We write* $\lim z^n = 0$.

(2) *Let* b *be a fixed positive number greater than 1, and define* $a_n = b^{1/n}$. *See Theorem 1.11. Then* $\lim a_n = 1$. *Again, we write* $\lim b^{1/n} = 1$.
(3) *Let* b *be a positive number less than 1. Then* $\lim b^{1/n} = 1$.
(4) *If* $a_n = n^{1/n}$, *then* $\lim a_n = \lim n^{1/n} = 1$.

PROOF. We prove parts (1) and (2) and leave the rest of the proof to the exercise that follows. If $z = 0$, claim (1) is obvious. Assume then that $z \neq 0$, and let $\epsilon > 0$ be given. Let $w = 1/|z|$, and observe that $w > 1$. So, we may write $w = 1 + h$ for some positive h. (That step is the clever idea for this argument.) Then, using the Binomial Theorem, $w^n > nh$, and so $1/w^n < 1/(nh)$. See part (a) of Exercise 1.20. But then

$$|z^n - 0| = |z^n| = |z|^n = (1/w)^n = 1/w^n < 1/(nh).$$

So, if N is any natural number larger than $1/(\epsilon h)$, then

$$|z^n - 0| = |z^n| = |z|^n < \frac{1}{nh} \leq \frac{1}{Nh} < \epsilon$$

for all $n \geq N$. This completes the proof of the first assertion of the theorem.

To see part (2), write $a_n = b^{1/n} = 1 + x_n$, i.e., $x_n = b^{1/n} - 1$, and observe first that $x_n > 0$. Indeed, since $b > 1$, it must be that the nth root $b^{1/n}$ is also > 1. (Why?) Therefore, $x_n = b^{1/n} - 1 > 0$. (Again, writing $b^{1/n}$ as $1 + x_n$ is the clever idea.) Now, $b = b^{1/n^n} = (1 + x_n)^n$, which, again by the Binomial Theorem, implies that $b > 1 + nx_n$. So, $x_n < (b-1)/n$, and therefore

$$|b^{1/n} - 1| = b^{1/n} - 1 = x_n < \frac{b-1}{n} < \epsilon$$

whenever $n > \epsilon/(b-1)$, and this proves part (2).

Exercise 2.7. (a) Prove part (3) of the preceding theorem.
HINT: For $b \leq 1$, use the following algebraic calculation:

$$|b^{1/n} - 1| = b^{1/n}|1 - (1/b)^{1/n}| \leq |1 - (1/b)^{1/n}|,$$

and then use part (2) as applied to the positive number $1/b$.

(b) Prove part (4) of the preceding theorem. Explain why it does not follow directly from part (2).
HINT: Write $n^{1/n} = 1 + h_n$. Observe that $h_n > 0$. Then use the third term of the binomial theorem in the expansion $n = (1 + h_n)^n$.

(c) Construct an alternate proof to part (2) of the preceding theorem as follows: Show that the sequence $\{b^{1/n}\}$ is nonincreasing and bounded below by 1. Deduce, from Theorem 2.1, that the sequence converges to a number L. Now prove that L must be 1.

DEFINITION OF e

Part (4) of Theorem 2.2 raises an interesting point. Suppose we have a sequence $\{a_n\}$, like $\{n\}$, that is diverging to infinity, and suppose we have another sequence $\{b_n\}$, like $\{1/n\}$, that is converging to 0. What can be said about the sequence $\{a_n^{b_n}\}$? The base a_n is blowing up, while the exponent b_n is going to 0. In other words, there are two competing processes going on. If a_n is blowing up, then its powers ought to be blowing up as well. On the other hand, anything to the 0 power should be 1, so that, as the exponents of the elements of a sequence converge to 0, the sequence ought to converge to 1. This competition between the convergence of the base to infinity and the convergence of the exponent to 0 makes it subtle, if not impossibly difficult, to tell what the combination does. For the special case of part (4) of Theorem 2.2, the answer was 1, indicating that, in that case at least, the exponents going to 0 seem to be more important than the base going to infinity. One can think up all kinds of such examples: $\{(2^n)^{1/n}\}$, $\{(n!)^{1/n}\}$, $\{(n!)^{1/n^2}\}$, and so on. We will see later that all sorts of things can happen.

Of course there is the reverse situation. Suppose $\{a_n\}$ is a sequence of numbers that decreases to 1, and suppose $\{b_n\}$ is a sequence of numbers that diverges to infinity. What can we say about the sequence $\{{a_n}^{b_n}\}$? The base is tending to 1, so that one might expect that the whole sequence also would be converging to 1. On the other hand the exponents are blowing up, so that one might think that the whole sequence should blow up as well. Again, there are lots of examples, and they don't all work the same way. Here is perhaps the most famous such example.

THEOREM 2.3. (Definition of e.) For $n \geq 1$, define $a_n = (1 + 1/n)^n$. Then the sequence $\{a_n\}$ is nondecreasing and bounded above, whence it is convergent. (We will denote the limit of this special sequence by the letter e.)

PROOF. To see that $\{a_n\}$ is nondecreasing, it will suffice to prove that $a_{n+1}/a_n \geq 1$ for all n. In the computation below, we will use the fact (part (c)of Exercise 1.20) that if $x > -1$ then $(1 + x)^n \geq 1 + nx$. So,

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} \\
&= \frac{(\frac{n+2}{n+1})^{n+1}}{(\frac{n+1}{n})^n} \\
&= \frac{n+1}{n}\frac{\frac{n+2}{n+1})^{n+1}}{(\frac{n+1}{n})^{n+1}} \\
&= \frac{n+1}{n}(\frac{n^2+2n}{n^2+2n+1})^{n+1} \\
&= \frac{n+1}{n}(1-\frac{1}{(n+1)^2})^{n+1} \\
&\geq \frac{n+1}{n}(1-(n+1)(\frac{1}{n+1})^2) \\
&= \frac{n+1}{n}(1-\frac{1}{n+1}) \\
&= \frac{n+1}{n}\frac{n}{n+1} \\
&= 1,
\end{aligned}$$

as desired.

We show next that $\{a_n\}$ is bounded above. This time, we use the binomial theorem, the geometric progression, and Exercise 1.19.

$$\begin{aligned}
a_n &= (1+\frac{1}{n})^n \\
&= \sum_{k=0}^{n} \binom{n}{k} (\frac{1}{n})^k \\
&< \sum_{k=0}^{n} 2\frac{n^k}{2^k} (\frac{1}{n})^k \\
&= 2\sum_{k=0}^{n} (\frac{1}{2})^k \\
&= 2\frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} \\
&< 4,
\end{aligned}$$

as desired.

That the sequence $\{a_n\}$ converges is now a consequence of Theorem 2.1.

REMARK. We have now defined the real number e. Its central role in mathematics is not at all evident yet; at this point we have no definition of exponential function, logarithm, or trigonometric functions. It does follow from the proof above that e is between 2 and 4, and with a little more careful estimates we can show that actually $e \leq 3$. For the moment, we will omit any further discussion of its precise value. Later, in Exercise 4.19, we will show that it is an irrational number.

PROPERTIES OF CONVERGENT SEQUENCES

Often, our goal is to show that a given sequence is convergent. However, as we study convergent sequences, we would like to establish various properties that they have in common. The first theorem of this section is just such a result.

THEOREM 2.4. *Suppose $\{a_n\}$ is a convergent sequence of real or complex numbers. Then the sequence $\{a_n\}$ forms a bounded set.*

PROOF. Write $L = \lim a_n$. Let ϵ be the positive number 1. Then, there exists a natural number N such that $|a_n - L| < 1$ for all $n \geq N$. By the backward triangle inequality, this implies that $||a_n| - |L|| < 1$ for all $n \geq N$, which implies that $|a_n| \leq |L| + 1$ for all $n \geq N$. This shows that at least the tail of the sequence is bounded by the constant $|L| + 1$.

Next, let K be a number larger than the finitely many numbers $|a_1|, \ldots, |a_{N-1}|$. Then, for any n, $|a_n|$ is either less than K or $|L| + 1$. Let M be the larger of the two numbers K and $|L| + 1$. Then $|a_n| < M$ for all n. Hence, the sequence $\{a_n\}$ is bounded.

Note that the preceding theorem is a partial converse to Theorem 2.1; i.e., a convergent sequence is necessarily bounded. Of course, not every convergent sequence must be either nondecreasing or nonincreasing, so that a full converse to theorem 2.1 is not true. For instance, take $z = -1/2$ in part (1) of Theorem 2.2. It converges to 0 all right, but it is neither nondecreasing nor nonincreasing.

Exercise 2.8. (a) Suppose $\{a_n\}$ is a sequence of real numbers that converges to a number a, and assume that $a_n \geq c$ for all n. Prove that $a \geq c$.
HINT: Suppose not, and let ϵ be the positive number $c - a$. Let N be a natural number corresponding to this choice of ϵ, and derive a contradiction.

(b) If $\{a_n\}$ is a sequence of real numbers for which $\lim a_n = a$, and if $a \neq 0$, then prove that $a_n \neq 0$ for all large enough n. Show in fact that there exists an N such that $|a_n| > |a|/2$ for all $n \geq N$.
HINT: Make use of the positive number $\epsilon = |a|/2$.

Exercise 2.9. (a) If $\{a_n\}$ is a sequence of positive real numbers for which $\lim a_n = a > 0$, prove that $\lim \sqrt{a_n} = \sqrt{a}$.
HINT: Multiply the expression $\sqrt{a_n} - \sqrt{a}$ above and below by $\sqrt{a_n} + \sqrt{a}$.

(b) If $\{a_n\}$ is a sequence of complex numbers, and $\lim a_n = a$, prove that $\lim |a_n| = |a|$.
HINT: Use the backward triangle inequality.

Exercise 2.10. Suppose $\{a_n\}$ is a sequence of real numbers and that $L = \lim a_n$. Let M_1 and M_2 be real numbers such that $M_1 \leq a_n \leq M_2$ for all n. Prove that $M_1 \leq L \leq M_2$.
HINT: Suppose, for instance, that $L > M_2$. Make use of the positive number $L - M_2$ to derive a contradiction.

We are often able to show that a sequence converges by comparing it to another sequence that we already know converges. The following exercise demonstrates some of these techniques.

Exercise 2.11. Let $\{a_n\}$ be a sequence of complex numbers.

(a) Suppose that, for each n, $|a_n| < 1/n$. Prove that $0 = \lim a_n$.

(b) Suppose $\{b_n\}$ is a sequence that converges to 0, and suppose that, for each n, $|a_n| < |b_n|$. Prove that $0 = \lim a_n$.

The next result is perhaps the most powerful technique we have for showing that a given sequence converges to a given number.

THEOREM 2.5. $\{$*Squeeze Theorem*) Suppose that $\{a_n\}$ is a sequence of real numbers and that $\{b_n\}$ and $\{c_n\}$ are two sequences of real numbers for which $b_n \leq a_n \leq c_n$ for all n. Suppose further that $\lim b_n = \lim c_n = L$. Then the sequence $\{a_n\}$ also converges to L.

PROOF. We examine the quantity $|a_n - L, |$ employ some add and subtract tricks, and make the following computations:

$$\begin{aligned}
|a_n - L| &\leq |a_n - b_n + b_n - L| \\
&\leq |a_n - b_n| + |b_n - L| \\
&= a_n - b_n + |b_n - L| \\
&\leq c_n - b_n + |b_n - L| \\
&= |c_n - b_n| + |b_n - L| \\
&\leq |c_n - L| + |L - b_n| + |b_n - L|.
\end{aligned}$$

So, we can make $|a_n - L| < \epsilon$ by making $|c_n - L| < \epsilon/3$ and $|b_n - L| < \epsilon/3$. So, let N_1 be a positive integer such that $|c_n - L| < \epsilon/3$ if $n \geq N_1$, and let N_2 be a positive integer so that $|b_n - L| < \epsilon/3$ if $n \geq N_2$. Then set $N = \max(N_1, N_2)$.

Clearly, if $n \geq N$, then both inequalities $|c_n - L| < \epsilon/3$ and $|b_n - L| < \epsilon/3$, and hence $|a_n - L| < \epsilon$. This finishes the proof.

The next result establishes what are frequently called the " limit theorems." Basically, these results show how convergence interacts with algebraic operations.

THEOREM 2.6. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers with $a = \lim a_n$ and $b = \lim b_n$. Then*

(1) *The sequence $\{a_n + b_n\}$ converges, and*

$$\lim(a_n + b_n) = \lim a_n + \lim b_n = a + b.$$

(2) *The sequence $\{a_n b_n\}$ is convergent, and*

$$\lim(a_n b_n) = \lim a_n \lim b_n = ab.$$

(3) *If all the b_n's as well as b are nonzero, then the sequence $\{a_n/b_n\}$ is convergent, and*

$$\lim(\frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} = \frac{a}{b}.$$

PROOF. Part (1) is exactly the same as Exercise 2.6. Let us prove part (2).

By Theorem 2.4, both sequences $\{a_n\}$ and $\{b_n\}$ are bounded. Therefore, let M be a number such that $|a_n| \leq M$ and $|b_n| \leq M$ for all n. Now, let $\epsilon > 0$ be given. There exists an N_1 such that $|a_n - a| < \epsilon/(2M)$ whenever $n \geq N_1$, and there exists an N_2 such that $|b_n - b| < \epsilon/(2M)$ whenever $n \geq N_2$. Let N be the maximum of N_1 and N_2. Here comes the add and subtract trick again.

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\
&\leq |a_n b_n - ab_n| + |ab_n - ab| \\
&= |a_n - a||b_n| + |a||b - b_n| \\
&\leq |a_n - a|M + M|b_n - b| \\
&< \epsilon
\end{aligned}$$

if $n \geq N$, which shows that $\lim(a_n b_n) = ab$.

To prove part (3), let M be as in the previous paragraph, and let $\epsilon > 0$ be given. There exists an N_1 such that $|a_n - a| < (\epsilon|b|^2)/(4M)$ whenever $n \geq N_1$; there also exists an N_2 such that $|b_n - b| < (\epsilon|b|^2)/(4M)$ whenever $n \geq N_2$; and there exists an N_3 such that $|b_n| > |b|/2$ whenever $n \geq N_3$. (See Exercise 2.8.) Let N be the maximum of the three numbers N_1, N_2 and N_3. Then:

$$\begin{aligned}
|\frac{a_n}{b_n} - \frac{a}{b}| &= |\frac{a_n b - b_n a}{b_n b}| \\
&= |a_n b - b_n a|\frac{1}{|b_n b|} \\
&< |a_n b - b_n a|\frac{1}{|b|^2/2} \\
&\leq (|a_n - a||b| + |a||b_n - b|)\frac{2}{|b|^2} \\
&< (M|a_n - a| + M|b_n - b|)\frac{2}{|b|^2} \\
&< \epsilon
\end{aligned}$$

if $n \geq N$. This completes the proof.

REMARK. The proof of part (3) of the preceding theorem may look mysterious. Where, for instance, does this number $\epsilon|b|^2/4M$ come from? The answer is that one begins such a proof by examining the quantity $|a_n/b_n - a/b|$ to see if by some algebraic manipulation one can discover how to control its size by using the quantities $|a_n - a|$ and $|b_n - b|$. The assumption that $a = \lim a_n$ and $b = \lim b_n$ mean exactly that the quantities $|a_n - a|$ and $|b_n - b|$ can be controlled by requiring n to be large enough. The algebraic computation in the proof above shows that

$$|\frac{a_n}{b_n} - \frac{a}{b}| \leq (M|a_n - a| + M|b_n - b|)\frac{2}{|b|^2},$$

and one can then see exactly how small to make $|a_n - a|$ and $|b_n - b|$ so that $|a_n/b_n - a/b| < \epsilon$. Indeed, this is the way most limit proofs work.

Exercise 2.12. If possible, determine the limits of the following sequences by using Theorems 2.2, 2.3, 2.6, and the squeeze theorem 2.5.

(a) $\{n^{1/n^2}\}$.
(b) $\{(n^2)^{1/n}\}$.
(c) $\{(1+n)^{1/n}\}$.
(d) $\{(1+n^2)^{1/n^3}\}$.
(e) $\{(1+1/n)^{2/n}\}$.
(f) $\{(1+1/n)^{2n}\}$.
(g) $\{(1+1/n)^{n^2}\}$.
(h) $\{(1-1/n)^n\}$.

HINT: Note that

$$1 - 1/n = \frac{n-1}{n} = \frac{1}{\frac{n}{n-1}} = \frac{1}{\frac{n-1+1}{n-1}} = \frac{1}{1+\frac{1}{n-1}}.$$

(i) $\{(1-1/(2n))^{3n}\}$.
(j) $\{(n!)^{1/n}\}$.

SUBSEQUENCES AND CLUSTER POINTS

DEFINITION. Let $\{a_n\}$ be a sequence of real or complex numbers. A *subsequence* of $\{a_n\}$ is a sequence $\{b_k\}$ that is determined by the sequence $\{a_n\}$ together with a strictly increasing sequence $\{n_k\}$ of natural numbers. The sequence $\{b_k\}$ is defined by $b_k = a_{n_k}$. That is, the kth term of the sequence $\{b_k\}$ is the n_kth term of the original sequence $\{a_n\}$.

Exercise 2.13. Prove that a subsequence of a subsequence of $\{a_n\}$ is itself a subsequence of $\{a_n\}$. Thus, let $\{a_n\}$ be a sequence of numbers, and let $\{b_k\} = \{a_{n_k}\}$ be a subsequence of $\{a_n\}$. Suppose $\{c_j\} = \{b_{k_j}\}$ is a subsequence of the sequence $\{b_k\}$. Prove that $\{c_j\}$ is a subsequence of $\{a_n\}$. What is the strictly increasing sequence $\{m_j\}$ of natural numbers for which $c_j = a_{m_j}$?

Here is an interesting generalization of the notion of the limit of a sequence.

DEFINITION. Let $\{a_n\}$ be a sequence of real or complex numbers. A number x is called a *cluster point* of the sequence $\{a_n\}$ if there exists a subsequence $\{b_k\}$ of $\{a_n\}$ such that $x = \lim b_k$. The set of all cluster points of a sequence $\{a_n\}$ is called the *cluster set* of the sequence.

Exercise 2.14. (a) Give an example of a sequence whose cluster set contains two points. Give an example of a sequence whose cluster set contains exactly n points. Can you think of a sequence whose cluster set is infinite?

(b) Let $\{a_n\}$ be a sequence with cluster set S. What is the cluster set for the sequence $\{-a_n\}$? What is the cluster set for the sequence $\{a_n^2\}$?

(c) If $\{b_n\}$ is a sequence for which $b = \lim b_n$, and $\{a_n\}$ is another sequence, what is the cluster set of the sequence $\{a_n b_n\}$?

(d) Give an example of a sequence whose cluster set is empty.

(e) Show that if the sequence $\{a_n\}$ is bounded above, then the cluster set S is bounded above. Show also that if $\{a_n\}$ is bounded below, then S is bounded below.

(f) Give an example of a sequence whose cluster set S is bounded above but not bounded below.

(g) Give an example of a sequence that is not bounded, and which has exactly one cluster point.

THEOREM 2.7. *Suppose $\{a_n\}$ is a sequence of real or complex numbers.*

(1) (Uniqueness of limits) Suppose $\lim a_n = L$, and $\lim a_n = M$. Then $L = M$. That is, if the limit of a sequence exists, it is unique.
(2) If $L = \lim a_n$, and if $\{b_k\}$ is a subsequence of $\{a_n\}$, then the sequence $\{b_k\}$ is convergent, and $\lim b_k = L$. That is, if a sequence has a limit, then every subsequence is convergent and converges to that same limit.

PROOF. Suppose $\lim a_n = L \text{and} \lim a_n = M$. Let ϵ be a positive number, and choose N_1 so that $|a_n - L| < \epsilon/2$ if $n \geq N_1$, and choose N_2 so that $|a_n - M| < \epsilon/2$ if $n \geq N_2$. Choose an n larger than both $N_1 \text{and} N_2$. Then

$$|L - M| = |L - a_n + a_n - M| \leq |L - a_n| + |a_n - M| < \epsilon.$$

Therefore, since $|L - M| < \epsilon$ for every positive ϵ, it follows that $L - M = 0$ or $L = M$. This proves part (1).

Next, suppose $\lim a_n = L$ and let $\{b_k\}$ be a subsequence of $\{a_n\}$. We wish to show that $\lim b_k = L$. Let $\epsilon > 0$ be given, and choose an N such that $|a_n - L| < \epsilon$ if $n \geq N$. Choose a K so that $n_K \geq N$. (How?) Then, if $k \geq K$, we have $n_k \geq n_K \geq N$, whence $|b_k - L| = |a_{n_k} - L| < \epsilon$, which shows that $\lim b_k = L$. This proves part (2).

REMARK. The preceding theorem has the following interpretation. It says that if a sequence converges to a number L, then the cluster set of the sequence contains only one number, and that number is L. Indeed, if x is a cluster point of the sequence, then there must be some subsequence that converges to x. But, by part (2), everysubsequence converges to L. Then, by part (1), $x = L$. Part (g) of Exercise 2.14 shows that the converse of this theorem is not valid. that is, the cluster set may contain only one point, and yet the sequence is not convergent.

We give next what is probably the most useful fundamental result about sequences, the Bolzano-Weierstrass Theorem. It is this theorem that will enable us to derive many of the important properties of continuity, differentiability, and integrability.

THEOREM 2.8. (Bolzano-Weierstrass) Every bounded sequence $\{a_n\}$ of real or complex numbers has a cluster point. In other words, every bounded sequence has a convergent subsequence.

The Bolzano-Weierstrass Theorem is, perhaps not surprisingly, a very difficult theorem to prove. We begin with a technical, but very helpful, lemma.

LEMMA. *Let $\{a_n\}$ be a bounded sequence of real numbers; i.e., assume that there exists an M such that $|a_n| \leq M$ for all n. For each $n \geq 1$, let S_n be the set whose elements are $\{a_n, a_{n+1}, a_{n+2}, \dots\}$. That is, S_n is just the elements of the tail of the sequence from n on. Define $x_n = \sup S_n = \sup_{k \geq n} a_k$. Then*

(1) *The sequence $\{x_n\}$ is bounded (above and below).*
(2) *The sequence $\{x_n\}$ is non-increasing.*
(3) *The sequence $\{x_n\}$ converges to a number x.*
(4) *The limit x of the sequence $\{x_n\}$ is a cluster point of the sequence $\{a_n\}$. That is, there exists a subsequence $\{b_k\}$ of the sequence $\{a_n\}$ that converges to x.*
(5) *If y is any cluster point of the sequence $\{a_n\}$, then $y \leq x$, where x is the cluster point of part (4). That is, x is the maximum of all cluster points of the sequence $\{a_n\}$.*

PROOF OF THE LEMMA. Since x_n is the supremum of the set S_n, and since each element of that set is bounded between $-M$ and M, part (1) is immediate.

Since $S_{n+1} \subseteq S_n$, it is clear that

$$x_{n+1} = \sup S_{n+1} \leq \sup S_n = x_n,$$

showing part (2).

The fact that the sequence $\{x_n\}$ converges to a number x is then a consequence of Theorem 2.1.

We have to show that the limit x of the sequence $\{x_n\}$ is a cluster point of $\{a_n\}$. Notice that $\{x_n\}$ may not itself be a subsequence of $\{a_n\}$, each x_n may or may not be one of the numbers a_k, so that there really is something to prove. In fact, this is the hard part of this lemma. To finish the proof of part (4), we must define an increasing sequence $\{n_k\}$ of natural numbers for which the corresponding subsequence $\{b_k\} = \{a_{n_k}\}$ of $\{a_n\}$ converges to x. We will choose these natural numbers $\{n_k\}$ so that $|x - a_{n_k}| < 1/k$. Once we have accomplished this, the fact that the corresponding subsequence $\{a_{n_k}\}$ converges to x will be clear. We choose the n_k's inductively. First, using the fact that $x = \lim x_n$, choose an n so that $|x_n - x| = x_n - x < 1/1$. Then, because $x_n = \sup S_n$, we may choose by Theorem 1.5 some $m \geq n$ such that $x_n \geq a_m > x_n - 1/1$. But then $|a_m - x| < 1/1$. (Why?) This m we call n_1. We have that $|a_{n_1} - x| < 1/1$.

Next, again using the fact that $x = \lim x_n$, choose another n so that $n > n_1$ and so that $|x_n - x| = x_n - x < 1/2$. Then, since this $x_n = \sup S_n$, we may choose another $m \geq n$ such that $x_n \geq a_m > x_n - 1/2$. This m we call n_2. Note that we have $|a_{n_2} - x| < 1/2$.

Arguing by induction, if we have found an increasing set $n_1 < n_2 < \dots < n_j$, for which $|a_{n_i} - x| < 1/i$ for $1 \leq i \leq j$, choose an n larger than n_j such that $|x_n - x| < 1/(j+1)$. Then, since $x_n = \sup S_n$, choose an $m \geq n$ so that $x_n \geq a_m > x_n - 1/(j+1)$. Then $|a_m - x| < 1/(j+1)$, and we let n_{j+1} be this m. It follows that $|a_{n_{j+1}} - x| < 1/(j+1)$.

So, by recursive definition, we have constructed a subsequence of $\{a_n\}$ that converges to x, and this completes the proof of part (4) of the lemma.

Finally, if y is any cluster point of $\{a_n\}$, and if $y = \lim a_{n_k}$, then $n_k \geq k$, and so $a_{n_k} \leq x_k$, implying that $x_k - a_{n_k} \geq 0$. Hence, taking limits on k, we see that $x - y \geq 0$, and this proves part (5).

Now, using the lemma, we can give the proof of the Bolzano-Weierstrass Theorem.

PROOF OF THEOREM 2.8. If $\{a_n\}$ is a sequence of real numbers, this theorem is an immediate consequence of part (4) of the preceding lemma.

If $a_n = b_n + c_n i$ is a sequence of complex numbers, and if $\{a_n\}$ is bounded, then $\{b_n\}$ and $\{c_n\}$ are both bounded sequences of real numbers. See Exercise 1.27. So, by the preceding paragraph, there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ that converges to a real number b. Now, the subsequence $\{c_{n_k}\}$ is itself a bounded sequence of real numbers, so there is a subsequence $\{c_{n_{k_j}}\}$ that converges to a real number c. By part (2) of Theorem 2.7, we also have that the subsequence $\{b_{n_{k_j}}\}$ converges to b. So the subsequence $\{a_{n_{k_j}}\} = \{b_{n_{k_j}} + c_{n_{k_j}} i\}$ of $\{a_n\}$ converges to the complex number $b + ci$; i.e., $\{a_n\}$ has a cluster point. This completes the proof.

There is an important result that is analogous to the Lemma above, and its proof is easily adapted from the proof of that lemma.

Exercise 2.15. Let $\{a_n\}$ be a bounded sequence of real numbers. Define a sequence $\{y_n\}$ by $y_n = inf_{k \geq n} a_k$. Prove that:

(a) $\{y_n\}$ is nondecreasing and bounded above.

(b) $y = \lim y_n$ is a cluster point of $\{a_n\}$.

(c) If z is any cluster point of $\{a_n\}$, then $y \leq z$. That is, y is the minimum of all the cluster points of the sequence $\{a_n\}$.

HINT: Let $\{\alpha_n\} = \{-a_n\}$, and apply the preceding lemma to $\{\alpha_n\}$. This exercise will then follow from that.

The Bolzano-Wierstrass Theorem shows that the cluster set of a bounded sequence $\{a_n\}$ is nonempty. It is also a bounded set itself.

The following definition is only for sequences of real numbers. However, like the Bolzano-Weierstrass Theorem, it is of very basic importance and will be used several times in the sequel.

DEFINITION. Let $\{a_n\}$ be a sequence of real numbers and let S denote its cluster set.

If S is nonempty and bounded above, we define $\limsup a_n$ to be the supremum $\sup S$ of S.

If S is nonempty and bounded below, we define $\liminf a_n$ to be the infimum $infS$ of S.

If the sequence $\{a_n\}$ of real numbers is not bounded above, we define $\limsup a_n$ to be ∞, and if $\{a_n\}$ is not bounded below, we define $\liminf a_n$ to be $-\infty$.

If $\{a_n\}$ diverges to ∞, then we define $\limsup a_n$ and $\liminf a_n$ both to be ∞. And, if $\{a_n\}$ diverges to $-\infty$, we define $\limsup a_n$ and $\liminf a_n$ both to be $-\infty$.

We call $\limsup a_n$ the *limit superior* of the sequence $\{a_n\}$, and $\liminf a_n$ the *limit inferior* of $\{a_n\}$.

Exercise 2.16. (a) Suppose $\{a_n\}$ is a bounded sequence of real numbers. Prove that the sequence $\{x_n\}$ of the lemma following Theorem 2.8 converges to $\limsup a_n$. Show also that the sequence $\{y_n\}$ of Exercise 2.15 converges to $\liminf a_n$.

(b) Let $\{a_n\}$ be a not necessarily bounded sequence of real numbers. Prove that

$$\limsup a_n = \inf_n \sup_{k\geq n} a_k = \lim_n \sup_{k\geq n} a_k.$$

and

$$\liminf a_n = \sup_n \inf_{k\geq n} a_k = \lim_n \inf k \geq n a_k.$$

HINT: Check all cases, and use the lemma following Theorem 2.8 and Exercise 2.15.

(c) Let $\{a_n\}$ be a sequence of real numbers. Prove that

$$\limsup a_n = -\liminf(-a_n).$$

(d) Give examples to show that all four of the following possibilities can happen.

(1) $\limsup a_n$ is finite, and $\liminf a_n = -\infty$.
(2) $\limsup a_n = \infty$ and $\liminf a_n$ is finite.
(3) $\limsup a_n = \infty$ and $\liminf a_n = -\infty$.
(4) both $\limsup a_n$ and $\liminf a_n$ are finite.

The notions of limsup and liminf are perhaps mysterious, and they are in fact difficult to grasp. The previous exercise describes them as the resultof a kind of two-level process, and there are occasions when this description is a great help. However, the limsup and liminf can also be characterized in other ways that are more reminiscent of the definition of a limit. These other ways are indicated in the next exercise.

Exercise 2.17. Let $\{a_n\}$ be a bounded sequence of real numbers with $\limsup a_n = L$ and $\liminf a_n = l$. Prove that L and l satisfy the following properties.

(a) For each $\epsilon > 0$, there exists an N such that $a_n < L + \epsilon$ for all $n \geq N$.

HINT: Use the fact that $\limsup a_n = L$ is the number x of the lemma following Theorem 2.8, and that x is the limit of a specific sequence $\{x_n\}$.

(b) For each $\epsilon > 0$, and any natural number k, there exists a natural number $j \geq k$ such that $a_j > L - \epsilon$. Same hint as for part (a).

(c) For each $\epsilon > 0$, there exists an N such that $a_n > l - \epsilon$ for all $n \geq N$.

(d) For each $\epsilon > 0$, and any natural number k, there exists a natural number $j > k$ such that $a_j < l + \epsilon$.

(e) Suppose L' is a number that satisfies parts (a) and (b). Prove that L' is the limsup of $\{a_n\}$.

HINT: Use part (a) to show that L' is greater than or equal to every cluster point of $\{a_n\}$. Then use part (b) to show that L' is less than or equal to some cluster point.

(f) If l' is any number that satisfies parts (c) and (d), show that l' is the liminf of the sequence $\{a_n\}$.

Exercise 2.18. (a) Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of real numbers, and write $L = \limsup a_n$ and $M = \limsup b_n$. Prove that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

HINT: Using part (a) of the preceding exercise, show that for every $\epsilon > 0$ there exists a N such that $a_n + b_n < L + M + \epsilon$ for all $n \geq N$, and conclude from this that every cluster point y of the sequence $\{a_n + b_n\}$ is less than or equal to $L + M$. This will finish the proof, since $\limsup(a_n + b_n)$ is a cluster point of that sequence.

(b) Again, let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of real numbers, and write $l = \liminf a_n$ and $m = \liminf b_n$. Prove that $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$.

HINT: Use part (c) of the previous exercise.

(c) Find examples of sequences $\{a_n\}$ and $\{b_n\}$ for which $\limsup a_n = \limsup b_n = 1$, but $\limsup(a_n + b_n) = 0$.

We introduce next another property that a sequence can possess. It looks very like the definition of a convergent sequence, but it differs in a crucial way, and that is that this definition only concerns the elements of the sequence $\{a_n\}$ and not the limit L.

DEFINITION. A sequence $\{a_n\}$ of real or complex numbers is a *Cauchy* sequence if for every $\epsilon > 0$, there exists a natural number N such that if $n \geq N$ and $m \geq N$ then $|a_n - a_m| < \epsilon$.

REMARK. No doubt, this definition has something to do with limits. Any time there is a positive ϵ and an N, we must be near some kind of limit notion. The point of the definition of a Cauchy sequence is that there is no explicit mention of what the limit is. It isn't that the terms of the sequence are getting closer and closer to some number L, it's that the terms of the sequence are getting closer and closer to each other. This subtle difference is worth some thought.

Exercise 2.19. Prove that a Cauchy sequence is bounded. (Try to adjust the proof of Theorem 2.4 to work for this situation.)

The next theorem, like the Bolzano-Weierstrass Theorem, seems to be quite abstract, but it also turns out to be a very useful tool for proving theorems about continity, differentiability, etc. In the proof, the completeness of the set of real numbers will be crucial. This theorem is not true in ordered fields that are not complete.

THEOREM 2.9. (Cauchy Criterion) A sequence $\{a_n\}$ of real or complex numbers is convergent if and only if it is a Cauchy sequence.

PROOF. If $lim a_n = a$ then given $\epsilon > 0$, choose N so that $|a_k - a| < \epsilon/2$ if $k \geq N$. From the triangle inequality, and by adding and subtracting a, we obtain that $|a_n - a_m| < \epsilon$ if $n \geq N$ and $m \geq N$. Hence, if $\{a_n\}$ is convergent, then $\{a_n\}$ is a Cauchy sequence.

Conversely, if $\{a_n\}$ is a cauchy sequence, then $\{a_n\}$ is bounded by the previous exercise. Now we use the fact that $\{a_n\}$ is a sequence of real or complex numbers. Let x be a cluster point of $\{a_n\}$. We know that one exists by the Bolzano-Weierstrass Theorem. Let us show that in fact this number x not only is a cluster point but that it is in fact the limit of the sequence $\{a_n\}$. Given $\epsilon > 0$, choose Nso that $|a_n - a_m| < \epsilon/2$ whenever both n and $m \geq N$. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$ that converges to x. Because $\{n_k\}$ is strictly increasing, we may choose a k so that $n_k > N$ and also so that $|a_{n_k} - x| < \epsilon/2$. Then, if $n \geq N$, then both n and this particular n_k are larger than or equal to N. Therefore, $|a_n - x| \leq |a_n - a_{n_k}| + |a_{n_k} - x| < \epsilon$. this completes the proof that $x = \lim a_n$.

A LITTLE TOPOLOGY

We now investigate some properties that subsets of $\mathbb{R}$ and $\mathbb{C}$ may possess. We will define "closed sets," "open sets," and "limit points" of sets. These notions are the rudimentary notions of what is called topology. As in earlier definitions, these topological ones will be enlightening when we come to continuity.

DEFINITION. Let S be a subset of $\mathbb{C}$. A complex number x is called a *limit point* of S if there exists a sequence $\{x_n\}$ of elements of S such that $x = \lim x_n$.

A set $S \subseteq \mathbb{C}$ is called *closed* if every limit point of S belongs to S.

Every limit point of a set of real numbers is a real number. Closed intervals $[a, b]$ are examples of closed sets in $\mathbb{R}$, while open intervals and half-open intervals may not be closed sets. Similarly, closed disks $\overline{B_r(c)}$ of radius r around a point c in $\mathbb{C}$, and closed neighborhoods $\overline{N}_r(S)$ of radius r around a set $S \subseteq \mathbb{C}$, are closed sets, while the open disks or open neighborhoods are not closed sets. As a first example of a limit point of a set, we give the following exercise.

Exercise 2.20. Let S be a nonempty bounded set of real numbers, and let $M = \sup S$. Prove that there exists a sequence $\{a_n\}$ of elements of S such that $M = \lim a_n$. That is, prove that the supremum of a bounded set of real numbers is a limit point of that set. State and prove an analogous result for infs.
HINT: Use Theorem 1.5, and let ϵ run through the numbers $1/n$.

Exercise 2.21. (a) Suppose S is a set of real numbers, and that $z = a + bi \in \mathbb{C}$ with $b \neq 0$. Show that z is not a limit point of S. That is, every limit point of a set of real numbers is a real number.
HINT: Suppose false; write $a + bi = \lim x_n$, and make use of the positive number $|b|$.

(b) Let c be a complex number, and let $S = \overline{B}_r(c)$ be the set of all $z \in \mathbb{C}$ for which $|z - c| \leq r$. Show that S is a closed subset of $\mathbb{C}$.
HINT: Use part (b) of Exercise 2.9.

(c) Show that the open disk $B_r(0)$ is not a closed set in $\mathbb{C}$ by finding a limit point of $B_r(0)$ that is not in $B_r(0)$.

(d) State and prove results analogous to parts b and c for intervals in $\mathbb{R}$.

(e) Show that every element x of a set S is a limit point of S.

(f) Let S be a subset of $\mathbb{C}$, and let x be a complex number. Show that x is not a limit point of S if and only if there exists a positive number ϵ such that if $|y - x| < \epsilon$, then y is not in S. That is, $S \cap B_\epsilon(x) = \emptyset$.
HINT: To prove the " only if" part, argue by contradiction, and use the sequence $\{1/n\}$ as ϵ's.

(g) Let $\{a_n\}$ be a sequence of complex numbers, and let S be the set of all the a_n's. What is the difference between a cluster point of the sequence $\{a_n\}$ and a limit point of the set S?

(h) Prove that the cluster set of a sequence is a closed set.
HINT: Use parts (e) and (f).

Exercise 2.22. (a) Show that the set $\mathbb{Q}$ of all rational numbers is not a closed set. Show also that the set of all irrational numbers is not a closed set.

(b) Show that if S is a closed subset of $\mathbb{R}$ that contains $\mathbb{Q}$, then S must equal all of $\mathbb{R}$.

Here is another version of the Bolzano-Weierstrass Theorem, this time stated in terms of closed sets rather than bounded sequences.

THEOREM 2.10. *Let S be a bounded and closed subset of $\mathbb{C}$. Then every sequence $\{x_n\}$ of elements of S has a subsequence that converges to an element of S.*

PROOF. Let $\{x_n\}$ be a sequence in S. Since S is bounded, we know by Theorem 2.8 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to some number x. Since each x_{n_k} belongs to S, it follows that x is a limit point of S. Finally, because S is a closed subset of $\mathbb{C}$, it then follows that $x \in S$.

We have defined the concept of a closed set. Now let's give the definition of an open set.

DEFINITION. Let S be a subset of $\mathbb{C}$. A point $x \in S$ is called an *interior point* of S if there exists an $\epsilon > 0$ such that the open disk $B_\epsilon(x)$ of radius ϵ around x is entirely contained in S. The set of all interior points of S is denoted by S^0 and we call S^0 the *interior* of S.

A subset S of $\mathbb{C}$ is called an *open* subset of $\mathbb{C}$ if every point of S is an interior point of S; i.e., if $S = S^0$.

Analogously, let S be a subset of $\mathbb{R}$. A point $x \in S$ is called an *interior point* of S if there exists an $\epsilon > 0$ such that the open interval $(x - \epsilon, x + \epsilon)$ is entirely contained in S. Again, we denote the set of all interior points of S by S^0 and call S^0 the *interior* of S.

A subset S of $\mathbb{R}$ is called an *open* subset of $\mathbb{R}$ if every point of S is an interior point of S; i.e., if $S = S^0$.

Exercise 2.23. (a) Prove that an open interval (a, b) in $\mathbb{R}$ is an open subset of $\mathbb{R}$; i.e., show that every point of (a, b) is an interior point of (a, b).

(b) Prove that any disk $B_r(c)$ is an open subset of $\mathbb{C}$. Show also that the *punctured disk* $B'_r(c)$ is an open set, where $B'_r(c) = \{z : 0 < |z - c| < r\}$, i.e., evrything in the disk $B_r(c)$ except the central point c.

(c) Prove that the neighborhood $N_r(S)$ of radius r around a set S is an open subset of $\mathbb{C}$.

(d) Prove that no nonempty subset of $\mathbb{R}$ is an open subset of $\mathbb{C}$.

(e) Prove that the set $\mathbb{Q}$ of all rational numbers is not an open subset of $\mathbb{R}$. We have seen in part (a) of Exercise 2.22 that $\mathbb{Q}$ is not a closed set. Consequently it is an example of a set that is neither open nor closed. Show that the set of all irrational numbers is neither open nor closed.

We give next a useful application of the Bolzano-Weierstrass Theorem, or more precisely an application of Theorem 2.10. This also provides some insight into the structure of open sets.

THEOREM 2.11. *Let S be a closed and bounded subset of $\mathbb{C}$, and suppose S is a subset of an open set U. Then there exists an $r > 0$ such that the neighborhood $N_r(S)$ is contained in U. That is, every open set containing a closed and bounded set S actually contains a neighborhood of S.*

PROOF. If S is just a singleton $\{x\}$, then this theorem is asserting nothing more than the fact that x is in the interior of U, which it is if U is an open set. However, when S is an infinite set, then the result is more subtle. We argue by contradiction. Thus, suppose there is no such $r > 0$ for which $N_r(S) \subseteq U$. then for each positive integer n there must be a point x_n that is not in U, and a corresponding point

$y_n \in S$, such that $|x_n - y_n| < 1/n$. Otherwise, the number $r = 1/n$ would satisfy the claim of the theorem. Now, because the y_n's all belong to S, we know from Theorem 2.10 that a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ must converge to a number $y \in S$. Next, we see that

$$|x_{n_k} - y| \leq |x_{n_k} - y_{n_k}| + |y_{n_k} - y|, < \frac{1}{n_k} + |y_{n_k} - y|,$$

and this quantity tends to 0. Hence, the subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ also converges to y.

Finally, because y belongs to S and hence to the open set U, we know that there must exist an $\epsilon > 0$ such that the entire disk $B_\epsilon(y) \subseteq U$. Then, since the subsequence $\{x_{n_k}\}$ converges to y, there must exist an_k such that $|x_{n_k} - y| < \epsilon$, implying that $x_{n_k} \in B_\epsilon(y)$, and hence belongs to U. But this is our contradiction, because all of the x_n's were not in U. So, the theorem is proved.

We give next a result that cliarifies to some extent the connection between open sets and closed sets. Always remember that there are sets that are neither open nor closed, and just because a set is not open **does not** mean that it is closed.

THEOREM 2.12. *A subset S of $\mathbb{C}$ ($\mathbb{R}$) is open if and only if its complement $\tilde{S} = \mathbb{C} \setminus S$ ($\mathbb{R} \setminus S$) is closed.*

PROOF. First, assume that S is open, and let us show that $\tilde{S}$ is closed. Suppose not. We will derive a contradiction. Suppose then that there is a sequence $\{x_n\}$ of elements of $\tilde{S}$ that converges to a number x that is not in $\tilde{S}$; i.e., x is an element of S. Since every element of S is an interior point of S, there must exist an $\epsilon > 0$ such that the entire disk $B_\epsilon(x)$ (or interval $(x - \epsilon, x + \epsilon)$) is a subset of S. Now, since $x = \lim x_n$, there must exist an$N$ such that $|x_n - x| < \epsilon$ for every $n \geq N$. In particular, $|x_N - x| < \epsilon$; i.e., x_N belongs to $B_\epsilon(x)$ (or $(x - \epsilon, x + \epsilon)$). This implies that $x_N \in S$. But $x_N \in \tilde{S}$, and this is a contradiction. Hence, if S is open, then $\tilde{S}$ is closed.

Conversely, assume that $\tilde{S}$ is closed, and let us show that S must be open. Again we argue by contradiction. Thus, assuming that S is not open, there must exist a point $x \in S$ that is not an interior point of S. Hence, for every $\epsilon > 0$ the disk $B_\epsilon(x)$ (or interval $(x - \epsilon, x + \epsilon)$) is not entirely contained in S. So, for each positive integer n, there must exist a point x_n such that $|x_n - x| < 1/n$ and $x_n \notin S$. It follows then that $x = \lim x_n$, and that each $x_n \in \tilde{S}$. Since $\tilde{S}$ is a closed set, we must have that $x \in \tilde{S}$. But $x \in S$, and we have arrived at the desired contradiction. Hence, if $\tilde{S}$ is closed, then S is open, and the theorem is proved.

The theorem below, the famous Heine-Borel Theorem, gives an equivalent and different description of closed and bounded sets. This description is in terms of open sets, whereas the original definitions were interms of limit points. Any time we can find two very different descriptions of the same phenomenon, we have found something useful.

DEFINITION. Let S be a subset of $\mathbb{C}$ (respectively $\mathbb{R}$). By an *open cover* of S we mean a sequence $\{U_n\}$ of open subsets of $\mathbb{C}$ (respectively $\mathbb{R}$) such that $S \subseteq \cup U_n$; i.e., for every $x \in S$there exists an n such that $x \in U_n$.

A subset S of $\mathbb{C}$ (respectively $\mathbb{R}$) is called *compact*, or is said to satisfy the *Heine-Borel property*, if every open cover of S has a finite subcover. That is, if

$\{U_n\}$ is an open cover of S, then there exists an integer N such that $S \subseteq \cup_{n=1}^N U_n$. In other words, only a finite number of the open sets are necessary to cover S.

REMARK. The definition we have given here for a set being compact is a little less general from the one found in books on topology. We have restricted the notion of an open cover to be a sequence of open sets, while in the general setting an open cover is just a collection of open sets. The distinction between a sequence of open sets and a collection of open sets is genuine in general topology, but it can be disregarded in the case of the topological spaces $\mathbb{R}$ and $\mathbb{C}$.

THEOREM 2.13. (Heine-Borel Theorem) A subset S of $\mathbb{C}$ (respectively $\mathbb{R}$) is compact if and only if it is a closed and bounded set.

PROOF. We prove this theorem for subsets S of $\mathbb{C}$, and leave the proof for subsets of $\mathbb{R}$ to the exercises.

Suppose first that $S \subseteq \mathbb{C}$ is compact, i.e., satisfies the Heine-Borel property. For each positive integer n, define U_n to be the open set $B_n(0)$. Then $S \subseteq \cup U_n$, because $\mathbb{C} = \cup U_n$. Hence, by the Heine-Borel property, there must exist an N such that $S \subseteq \cup_{n=1}^N U_n$. But then $S \subseteq B_N(0)$, implying that S is bounded. Indeed, $|x| \leq N$ for all $x \in S$.

Next, still assuming that S is compact, we will show that S is closed by showing that $\tilde{S}$ is open. Thus, let x be an element of $\tilde{S}$. For each positive integer n, define U_n to be the complement of the closed set $\overline{B_{1/n}(x)}$. Then each U_n is an open set by Theorem 2.12, and we claim that $\{U_n\}$ is an open cover of S. Indeed, if $y \in S$, then $y \neq x$, and $|y - x| > 0$. Choose an n so that $1/n < |y - x|$. Then $y \notin \overline{B_{1/n}(x)}$, implying that $y \in U_n$. This proves our claim that $\{U_n\}$ is an open cover of S. Now, by the Heine-Borel property, there exists an N such that $S \subseteq \cup_{n=1}^N U_n$. But this implies that for every $z \in S$ we must have $|z - x| \geq 1/N$, and this implies that the disk $B_{1/N}(x)$ is entirely contained in $\tilde{S}$. Therefore, every element x of $\tilde{S}$ is an interior point of $\tilde{S}$. So, $\tilde{S}$ is open, whence S is closed. This finishes the proof that compact sets are necessarily closed and bounded.

Conversely, assume that S is both closed and bounded. We must show that S satisfies the Heine-Borel property. Suppose not. Then, there exists an open cover $\{U_n\}$ that has no finite subcover. So, for each positive integer n there must exist an element $x_n \in S$ for which $x_n \notin \cup_{k=1}^n U_k$. Otherwise, there would be a finite subcover. By Theorem 2.10, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges to an element x of S. Now, because $\{U_n\}$ is an open cover of S, there must exist an N such that $x \in U_N$. Because U_N is open, there exists an $\epsilon > 0$ so that the entire disk $B_\epsilon(x)$ is contained in U_N. Since $x = \lim x_{n_j}$, there exists a J so that $|x_{n_j} - x| < \epsilon$ if $j \geq J$. Therefore, if $j \geq J$, then $x_{n_j} \in U_N$. But the sequence $\{n_j\}$ is strictly increasing, so that there exists a $j' \geq J$ such that $n_{j'} > N$, and by the choice of the point $x_{n_{j'}}$, we know that $x_{n_{j'}} \notin \cup_{k=1}^N U_k$. We have arrived at a contradiction, and so the second half of the theorem is proved.

Exercise 2.24. (a) Prove that the union $A \cup B$ of two open sets is open and the intersection $A \cap B$ is also open.

(b) Prove that the union $A \cup B$ of two closed sets is closed and the intersection $A \cap B$ is also closed.

HINT: Use Theorem 2.12 and the set equations $\widetilde{A \cup B} = \widetilde{A} \cap \widetilde{B}$, and $\widetilde{A \cap B} = \widetilde{A} \cup \widetilde{B}$. These set equations are known as Demorgan's Laws.

(c) Prove that the union $A \cup B$ of two bounded sets is bounded and the intersection $A \cap B$ is also bounded.

(d) Prove that the union $A \cup B$ of two compact sets is compact and the intersection $A \cap B$ is also compact.

(e) Prove that the intersection of a compact set and a closed set is compact.

(f) Suppose S is a compact set in $\mathbb{C}$ and r is a positive real number. Prove that the closed neighborhood $\overline{N}_r(S)$ of radius r around S is compact.
HINT: To see that this set is closed, show that its coplement is open.

INFINITE SERIES

Probably the most interesting and important examples of sequences are those that arise as the partial sums of an infinite series. In fact, it will be infinite series that allow us to explain such things as trigonometric and exponential functions.

DEFINITION. Let $\{a_n\}_0^\infty$ be a sequence of real or complex numbers. By the *infinite series* $\sum a_n$ we mean the sequence $\{S_N\}$ defined by

$$S_N = \sum_{n=0}^{N} a_n.$$

The sequence $\{S_N\}$ is called the *sequence of partial sums* of the infinite series $\sum a_n$, and the infinite series is said to be *summable* to a number S, or to be *convergent,* if the sequence $\{S_N\}$ of partial sums converges to S. **The sum of an infinite series is the limit of its partial sums.**

An infinite series $\sum a_n$ is called *absolutely summable* or *absolutely convergent* if the infinite series $\sum |a_n|$ is convergent.

If $\sum a_n$ is not convergent, it is called *divergent*. If it is convergent but not absolutely convergent, it is called *conditionally convergent*.

A few simple formulas relating the a_n's and the S_N's are useful:

$$S_N = a_0 + a_1 + a_2 + \ldots + a_N,$$

$$S_{N+1} = S_N + a_{N+1},$$

and

$$S_M - S_K = \sum_{n=K+1}^{M} a_n = a_{K+1} + a_{K+2} + \ldots + A_M,$$

for $M > K$.

REMARK. Determining whether or not a given infinite series converges is one of the most important and subtle parts of analysis. Even the first few elementary theorems depend in deep ways on our previous development, particularly the Cauchy criterion.

THEOREM 2.14. *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Then the infinite series $\sum a_n$ is summable if and only if the sequence $\{S_N\}$ of partial sums is bounded.*

PROOF. If $\sum a_n$ is summable, then $\{S_N\}$ is convergent, whence bounded according to Theorem 2.4. Conversely, we see from the hypothesis that each $a_n \geq 0$ that

$\{S_N\}$ is nondecreasing ($S_{N+1} = S_N + a_{N+1} \geq S_N$). So, if $\{S_N\}$ is bounded, then it automatically converges by Theorem 2.1, and hence the infinite series $\sum a_n$ is summable.

The next theorem is the first one most calculus students learn about infinite series. Unfortunately, it is often misinterpreted, so be careful! Both of the proofs to the next two theorems use Theorem 2.9, which again is a serious and fundamental result about the real numbers. Therefore, these two theorems must be deep results themselves.

THEOREM 2.15. *Let $\sum a_n$ be a convergent infinite series. Then the sequence $\{a_n\}$ is convergent, and* $\lim a_n = 0$.

PROOF. Because $\sum a_n$ is summable, the sequence $\{S_N\}$ is convergent and so is a Cauchy sequence. Therefore, given an $\epsilon > 0$, there exists an N_0 so that $|S_n - S_m| < \epsilon$ whenever both n and $m \geq N_0$. If $n > N_0$, let $m = n - 1$. We have then that $|a_n| = |S_n - S_m| < \epsilon$, which completes the proof.

REMARK. Note that this theorem is **not** an "if and only if" theorem. The harmonic series (part (b) of Exercise 2.26 below) is the standard counterexample. The theorem above is mainly used to show that an infinite series is **not** summable. If we can prove that the sequence $\{a_n\}$ does not converge to 0, then the infinite series $\sum a_n$ does not converge. The misinterpretation of this result referred to above is exactly in trying to apply the (false) converse of this theorem.

THEOREM 2.16. *If $\sum a_n$ is an absolutely convergent infinite series of complex numbers, then it is a convergent infinite series. (Absolute convergence implies convergence.)*

PROOF. If $\{S_N\}$ denotes the sequence of partial sums for $\sum a_n$, and if $\{T_N\}$ denotes the sequence of partial sums for $\sum |a_n|$, then

$$|S_M - S_N| = |\sum_{n=N+1}^{M} a_n| \leq \sum_{n=N+1}^{M} |a_n| = |T_M - T_N|$$

for all N and M. We are given that $\{T_N\}$ is convergent and hence it is a Cauchy sequence. So, by the inequality above, $\{S_N\}$ must also be a Cauchy sequence. (If $|T_N - T_M| < \epsilon$, then $|S_N - S_M| < \epsilon$ as well.) This implies that $\sum a_n$ is convergent.

Exercise 2.25. (The Infinite Geometric Series) Let z be a complex number, and define a sequence $\{a_n\}$ by $a_n = z^n$. Consider the infinite series $\sum a_n$. Show that $\sum_{n=0}^{\infty} a_n$ converges to a number S if and only if $|z| < 1$. Show in fact that $S = 1/(1-z)$, when $|z| < 1$.
HINT: Evaluate explicitly the partial sums S_N, and then take their limit. Show that $S_N = \frac{1-z^{N+1}}{1-z}$.

Exercise 2.26. (a) Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1, by computing explicit formulas for the partial sums.
HINT: Use a partial fraction decomposition for the a_n's.

(b) (The Harmonic Series.) Show that $\sum_{n=1}^{\infty} 1/n$ diverges by verifying that $S_{2^k} > k/2$.

HINT: Group the terms in the sum as follows,

$$1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+(\frac{1}{9}+\frac{1}{10}+\dots+\frac{1}{16})+\dots,$$

and then estimate the sum of each group. Remember this example as an infinite series that diverges, despite the fact that is terms tend to 0.

The next theorem is the most important one we have concerning infinite series of numbers.

THEOREM 2.17. (Comparison Test) Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative real numbers for which there exists a positive integer M and a constant C such that $b_n \leq Ca_n$ for all $n \geq M$. If the infinite series $\sum a_n$ converges, so must the infinite series $\sum b_n$.

PROOF. We will show that the sequence $\{T_N\}$ of partial sums of the infinite series $\sum b_n$ is a bounded sequence. Then, by Theorem 2.14, the infinite series $\sum b_n$ must be summable.

Write S_N for the Nth partial sum of the convergent infinite series $\sum a_n$. Because this series is summable, its sequence of partial sums is a bounded sequence. Let B be a number such that $S_N \leq B$ for all N. We have for all $N > M$ that

$$\begin{aligned} T_N &= \sum_{n=1}^{N} b_n \\ &= \sum_{n=1}^{M} b_n + \sum_{n=M+1}^{N} b_n \\ &\leq \sum_{n=1}^{M} b_n + \sum_{n=M+1}^{N} Ca_n \\ &= \sum_{n=1}^{M} b_n + C\sum_{n=M+1}^{N} a_n \\ &\leq \sum_{n=1}^{M} b_n + C\sum_{n=1}^{N} a_n \\ &\leq \sum_{n=1}^{M} b_n + CS_N \\ &\leq \sum_{n=1}^{M} b_n + CB, \end{aligned}$$

which completes the proof, since this final quantity is a fixed constant.

Exercise 2.27. (a) Let $\{a_n\}$ and $\{b_n\}$ be as in the preceding theorem. Show that if $\sum b_n$ diverges, then $\sum a_n$ also must diverge.

(b) Show by example that the hypothesis that the a_n's and b_n's of the Comparison Test are nonnegative can not be dropped.

Exercise 2.28. (The Ratio Test) Let $\{a_n\}$ be a sequence of positive numbers.

(a) If $\limsup a_{n+1}/a_n < 1$, show that $\sum a_n$ converges.
HINT: If $\limsup a_{n+1}/a_n = \alpha < 1$, let β be a number for which $\alpha < \beta < 1$. Using part (a) of Exercise 2.17, show that there exists an N such that for all $n > N$ we must have $a_{n+1}/a_n < \beta$, or equivalently $a_{n+1} < \beta a_n$, and therefore $a_{N+k} < \beta^k a_N$. Now use the comparison test with the geometric series $\sum \beta^k$.

(b) If $\liminf a_{n+1}/a_n > 1$, show that $\sum a_n$ diverges.

(c) As special cases of parts (a) and (b), show that $\{a_n\}$ converges if $\lim_n a_{n+1}/a_n < 1$, and diverges if $\lim_n a_{n+1}/a_n > 1$.

(d) Find two examples of infinite series' $\sum a_n$ of positive numbers, such that $\lim a_{n+1}/a_n = 1$ for both examples, and such that one infinite series converges and the other diverges.

Exercise 2.29. (a) Derive the Root Test: If $\{a_n\}$ is a sequence of positive numbers for which $\limsup a_n^{1/n} < 1$, then $\sum a_n$ converges. And, if $\liminf a_n^{1/n} > 1$, then $\sum a_n$ diverges.

(b) Let r be a positive integer. Show that $\sum 1/n^r$ converges if and only if $r \geq 2$.
HINT: Use Exercise 2.26 and the Comparison Test for $r = 2$.

(c) Show that the following infinite series are summable.

$$\sum 1/(n^2+1),\ \sum n/2^n,\ \sum a^n/n!,$$

for a any complex number.

Exercise 2.30. Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers, and let $\{S_N\}$ denote the sequence of partial sums of the infinite series $\sum a_n$. Derive the Abel Summation Formula:

$$\sum_{n=1}^{N} a_n b_n = S_N b_N + \sum_{n=1}^{N-1} S_n(b_n - b_{n+1}).$$

The Comparison Test is the most powerful theorem we have about infinite series of positive terms. Of course, most series do not consist entirely of positive terms, so that the Comparison Test is not enough. The next theorem is therefore of much importance.

THEOREM 2.18. (Alternating Series Test) Suppose $\{a_1, a_2, a_3, \dots\}$ is an alternating sequence of real numbers; i.e., their signs alternate. Assume further that the sequence $\{|a_n|\}$ is nonincreasing with $0 = \lim |a_n|$. Then the infinite series $\sum a_n$ converges.

PROOF. Assume, without loss of generality, that the odd terms a_{2n+1} of the sequence $\{a_n\}$ are positive and the even terms a_{2n} are negative. We collect some facts about the partial sums $S_N = a_1 + a_2 + \ldots + a_N$ of the infinite series $\sum a_n$.

1. Every even partial sum S_{2N} is less than the following odd partial sum $S_{2N+1} = S_{2N} + a_{2N+1}$, And every odd partial sum S_{2N+1} is greater than the following even partial sum $S_{2N+2} = S_{2N+1} + a_{2N+2}$.

2. Every even partial sum S_{2N} is less than or equal to the next even partial sum $S_{2N+2} = S_{2N} + a_{2N+1} + a_{2N+2}$, implying that the sequence of even partial sums $\{S_{2N}\}$ is nondecreasing.

3. Every odd partial sum S_{2N+1} is greater than or equal to the next odd partial sum $S_{2N+3} = S_{2N+1} + a_{2N+2} + a_{2N+3}$, implying that the sequence of odd partial sums $\{S_{2N+1}\}$ is nonincreasing.

4. Every odd partial sum S_{2N+1} is bounded below by S_2. For, $S_{2N+1} > S_{2N} \geq S_2$. And, every even partial sum S_{2N} is bounded above by S_1. For, $S_{2N} < S_{2N+1} \leq S_1$.

5. Therefore, the sequence $\{S_{2N}\}$ of even partial sums is nondecreasing and bounded above. That sequence must then have a limit, which we denote by S_e. Similarly, the sequence $\{S_{2N+1}\}$ of odd partial sums is nonincreasing and bounded below. This sequence of partial sums also must have a limit, which we denote by S_o.

Now

$$S_o - S_e = \lim S_{2N+1} - \lim S_{2N} = \lim(S_{2N+1} - S_{2N}) = \lim a_{2N+1} = 0,$$

showing that $S_e = S_o$, and we denote this common limit by S. Finally, given an $\epsilon > 0$, there exists an N_1 so that $|S_{2N} - S| < \epsilon$ if $2N \geq N_1$, and there exists an N_2 so that $|S_{2N+1} - S| < \epsilon$ if $2N + 1 \geq N_2$. Therefore, if $N \geq \max(N_1, N_2)$, then $|S_N - S| < \epsilon$, and this proves that the infinite series converges.

Exercise 2.31. (a) (The Alternating Harmonic Series) Show that $\sum_{n=1}^{\infty}(-1)^n/n$ converges, but that it is not absolutely convergent.

(b) Let $\{a_n\}$ be an alternating series, as in the preceding theorem. Show that the sum $S = \sum a_n$ is trapped between S_N and S_{N+1}, and that $|S - S_N| \leq |a_N|$.

(c) State and prove a theorem about "eventually alternating infinite series."

(d) Show that $\sum z^n/n$ converges if and only if $|z| \leq 1$, and $z \neq 1$.

HINT: Use the Abel Summation Formula to evaluate the partial sums.

Exercise 2.32. Let $s = p/q$ be a positive rational number.

(a) For each $x > 0$, show that there exists a unique $y > 0$ such that $y^s = x$; i.e., $y^p = x^q$.

(b) Prove that $\sum 1/n^s$ converges if $s > 1$ and diverges if $s \leq 1$.

HINT: Group the terms as in part (b) of Exercise 2.26.

THEOREM 2.19. (Test for Irrationality) Let x be a real number, and suppose that $\{p_N/q_N\}$ is a sequence of rational numbers for which $x = \lim p_N/q_N$ and $x \neq p_N/q_N$ for any N. If $\lim q_N|x - p_N/q_N| = 0$, then x is irrational.

PROOF. We prove the contrapositive statement; i.e., if $x = p/q$ is a rational number, then $\lim q_N|x - p_N/q_N| \neq 0$. We have

$$x - p_N/q_N = p/q - p_N/q_N = \frac{pq_N - qp_N}{qq_N}.$$

Now the numerator $pq_N - qp_N$ is not 0 for any N. For, if it were, then $x = p/q = p_N/q_N$, which we have assumed not to be the case. Therefore, since $pq_N - qp_N$ is an integer, we have that

$$|x - p_N/q_N| = |\frac{pq_N - qp_N}{qq_N}| \geq \frac{1}{|qq_N|}.$$

So,

$$q_N|x - p_N/q_N| \geq \frac{1}{|q|},$$

and this clearly does not converge to 0.

Exercise 2.33. (a) Let $x = \sum_{n=0}^{\infty} (-1)^n/2^n$. Prove that x is a rational number.

(b) Let $y = \sum_{n=0}^{\infty} (-1)^n/2^{n^2}$. Prove that y is an irrational number.

HINT: The partial sums of this series are rational numbers. Now use the preceding theorem and part (b) of Exercise 2.31.