CHAPTER VI
INTEGRATION OVER SMOOTH CURVES IN THE PLANE
$C = 2\pi r$

In this chapter we will define what we mean by a smooth curve in the plane and what is meant by its arc length. These definitions are a good bit more tricky than one might imagine. Indeed, it is the subtlety of the definition of arc length that prevented us from defining the trigonometric functions in terms of wrapping the real line around the circle, a definition frequently used in high school trigonometry courses. Having made a proper definition of arc length, we will then be able to establish the formula $C = 2\pi r$ for the circumference of a circle of radius r.

By the "plane," we will mean $\mathbb{R}^2 \equiv \mathbb{C}$, and we will on occasion want to carefully distinguish between these two notions of the plane, i.e., two real variables x and y as opposed to one complex variable $z = x + iy$. In various instances, for clarity, we will use notations like $x + iy$ and (x, y), remembering that both of these represent the same point in the plane. As $x + iy$, it is a single complex number, while as (x, y) we may think of it as a vector in $\mathbb{R}^2$ having a magnitude and, if nonzero, a direction.

We also will define in this chapter three different kinds of integrals over such curves. The first kind, called "integration with respect to arc length," will be completely analogous to the integral defined in Chapter V for functions on a closed and bounded interval, and it will only deal with functions whose domain is the set consisting of the points on the curve. The second kind of integral, called a "contour integral," is similar to the first one, but it emphasizes in a critical way that we are integrating a complex-valued function over a curve in the complex plane $\mathbb{C}$ and not simply over a subset of $\mathbb{R}^2$. The applications of contour integrals is usually to functions whose domains are open subsets of the plane that contain the curve as a proper subset, i.e., whose domains are larger than just the curve. The third kind of integral over a curve, called a "line integral," is conceptually very different from the first two. In fact, we won't be integrating functions at all but rather a new notion that we call "differential forms." This is actually the beginnings of the subject called differential geometry, whose intricacies and power are much more evident in higher dimensions than 2.

The main points of this chapter include:

(1) The definition of a **smooth curve**, and the definition of its **arc length**,
(2) the derivation of the formula $C = 2\pi r$ for the circumference of a circle of radius r (Theorem 6.5),
(3) the definition of the **integral with respect to arc length**,
(4) the definition of a **contour integral**,
(5) the definition of a **line integral**, and
(6) **Green's Theorem** (Theorem 6.14).

SMOOTH CURVES IN THE PLANE

Our first project is to make a satisfactory definition of a smooth curve in the plane, for there is a good bit of subtlety to such a definition. In fact, the material in this chapter is all surprisingly tricky, and the proofs are good solid analytical arguments, with lots of ϵ's and references to earlier theorems.

Whatever definition we adopt for a curve, we certainly want straight lines, circles, and other natural geometric objects to be covered by our definition. Our intuition is that a curve in the plane should be a "1-dimensional" subset, whatever that may mean. At this point, we have no definition of the dimension of a general set, so this is probably not the way to think about curves. On the other hand, from the point of view of a physicist, we might well define a curve as the trajectory followed by a particle moving in the plane, whatever that may be. As it happens, we do have some notion of how to describe mathematically the trajectory of a moving particle. We suppose that a particle moving in the plane proceeds in a continuous manner relative to time. That is, the position of the particle at time t is given by a continuous function $f(t) = x(t) + iy(t) \equiv (x(t), y(t))$, as t ranges from time a to time b. A good first guess at a definition of a curve joining two points z_1 and z_2 might well be that it is the range C of a continuous function f that is defined on some closed bounded interval $[a, b]$. This would be a curve that joins the two points $z_1 = f(a)$ and $z_2 = f(b)$ in the plane. Unfortunately, this is also not a satisfactory definition of a curve, because of the following surprising and bizarre mathematical example, first discovered by Guiseppe Peano in 1890.

THE PEANO CURVE. *The so-called "Peano curve" is a continuous function f defined on the interval $[0, 1]$, whose range is the entire unit square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$.*

Be careful to realize that we're talking about the "range" of f and not its graph. The graph of a real-valued function could never be the entire square. This Peano function is a complex-valued function of a real variable. Anyway, whatever definition we settle on for a curve, we do not want the entire unit square to be a curve, so this first attempt at a definition is obviously not going to work.

Let's go back to the particle tracing out a trajectory. The physicist would probably agree that the particle should have a continuously varying velocity at all times, or at nearly all times, i.e., the function f should be continuously differentiable. Recall that the velocity of the particle is defined to be the rate of change of the position of the particle, and that's just the derivative f' of f. We might also assume that the particle is never at rest as it traces out the curve, i.e., the derivative $f'(t)$ is never 0. As a final simplification, we could suppose that the curve never crosses itself, i.e., the particle is never at the same position more than once during the time interval from $t = a$ to $t = b$. In fact, these considerations inspire the formal definition of a curve that we will adopt below.

Recall that a function f that is continuous on a closed interval $[a, b]$ and continuously differentiable on the open interval (a, b) is called a smooth function on $[a, b]$. And, if there exists a partition $\{t_0 < t_1 < \ldots < t_n\}$ of $[a, b]$ such that f is smooth on each subinterval $[t_{i-1}, t_i]$, then f is called piecewise smooth on $[a, b]$. Although the derivative of a smooth function is only defined and continuous on the open interval (a, b), and hence possibly is unbounded, it follows from part (d) of Exercise 5.22 that this derivative is improperly-integrable on that open interval. We recall also that just because a function is improperly-integrable on an open interval, its absolute value may not be improperly-integrable. Before giving the formal definition of a smooth curve, which apparently will be related to smooth or piecewise smooth functions, it is prudent to present an approximation theorem about smooth functions. Theorem 3.20 asserts that every continuous function on a closed bounded interval is the uniform limit of a sequence of step functions. We give next a similar, but stronger, result about smooth functions. It asserts that a smooth function can be approximated "almost uniformly" by piecewise linear functions.

THEOREM 6.1. *Let f be a smooth function on a closed and bounded interval $[a,b]$, and assume that $|f'|$ is improperly-integrable on the open interval (a,b). Given an $\epsilon > 0$, there exists a piecewise linear function p for which*

(1) $|f(x) - p(x)| < \epsilon$ *for all* $x \in [a,b]$.

(2) $\int_a^b |f'(x) - p'(x)|\,dx < \epsilon$.

That is, the functions f and p are close everywhere, and their derivatives are close on average in the sense that the integral of the absolute value of the difference of the derivatives is small.

PROOF. Because f is continuous on the compact set $[a,b]$, it is uniformly continuous. Hence, let $\delta > 0$ be such that if $x, y \in [a,b]$, and $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon/2$.

Because $|f'|$ is improperly-integrable on the open interval (a,b), we may use part (b) of Exercise 5.22 to find a $\delta' > 0$, which may also be chosen to be $< \delta$, such that $\int_a^{a+\delta'} |f'| + \int_{b-\delta'}^b |f'| < \epsilon/2$, and we fix such a δ'.

Now, because f' is uniformly continuous on the compact set $[a+\delta', b-\delta']$, there exists an $\alpha > 0$ such that $|f'(x) - f'(y)| < \epsilon/4(b-a)$ if x and y belong to $[a+\delta', b-\delta']$ and $|x-y| < \alpha$. Choose a partition $\{x_0 < x_1 < \ldots < x_n\}$ of $[a,b]$ such that $x_0 = a, x_1 = a+\delta', x_{n-1} = b-\delta', x_n = b$, and $x_i - x_{i-1} < \min(\delta, \alpha)$ for $2 \leq i \leq n-1$. Define p to be the piecewise linear function on $[a,b]$ whose graph is the polygonal line joining the $n+1$ points $(a, f(x_1))$, $\{(x_i, f(x_i))\}$ for $1 \leq i \leq n-1$, and $(b, f(x_{n-1}))$. That is, p is constant on the outer subintervals $[a, x_1]$ and $[x_{n-1}, b]$ determined by the partition, and its graph between x_1 and x_{n-1} is the polygonal line joining the points $\{(x_1, f(x_1)), \ldots, (x_{n-1}, f(x_{n-1}))\}$. For example, for $2 \leq i \leq n-1$, the function p has the form

$$p(x) = f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x - x_{i-1})$$

on the interval $[x_{i-1}, x_i]$. So, $p(x)$ lies between the numbers $f(x_{i-1})$ and $f(x_i)$ for all i. Therefore,

$$|f(x) - p(x)| \leq |f(x) - f(x_i)| + |f(x_i) - l(x)| \leq |f(x) - f(x_i)| + |f(x_i) - f(x_{i-1})| < \epsilon.$$

Since this inequality holds for all i, part (1) is proved.

Next, for $2 \leq i \leq n-1$, and for each $x \in (x_{i-1}, x_i)$, we have $p'(x) = (f(x_i) - f(x_{i-1}))/(x_i - x_{i-1})$, which, by the Mean Value Theorem, is equal to $f'(y_i)$ for some $y_i \in (x_{i-1}, x_i)$. So, for each such $x \in (x_{i-1}, x_i)$, we have $|f'(x) - p'(x)| = |f'(x) - f'(y_i)|$, and this is less than $\epsilon/4(b-a)$, because $|x - y_i| < \alpha$. On the two outer intervals, $p(x)$ is a constant, so that $p'(x) = 0$. Hence,

$$\begin{aligned}
\int_a^b |f' - p'| &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f' - p'| \\
&= \int_a^{x_1} |f'| + \sum_{i=2}^{n-1} |f' - p'| + \int_{x_{n-1}}^b |f'| \\
&\leq \int_a^{a+\delta'} |f'| + \int_{b-\delta'}^b |f'| + \frac{\epsilon}{4(b-a)} \int_{x_1}^{x_{n-1}} 1 \\
&< \epsilon.
\end{aligned}$$

The proof is now complete.

REMARK. It should be evident that the preceding theorem can easily be generalized to a piecewise smooth function f, i.e., a function that is continuous on $[a,b]$, continuously differentiable on each subinterval (t_{i-1},t_i) of a partition $\{t_0 < t_1 < \ldots < t_n\}$, and whose derivative f' is absolutely integrable on (a,b). Indeed, just apply the theorem to each of the subintervals (t_{i-1},t_i), and then carefully piece together the piecewise linear functions on those subintervals.

Now we are ready to define what a smooth curve is.

DEFINITION. By a *smooth curve* from a point z_1 to a different point z_2 in the plane, we mean a set $C \subseteq \mathbb{C}$ that is the range of a 1-1, smooth, function $\phi : [a,b] \to \mathbb{C}$, where $[a,b]$ is a bounded closed interval in $\mathbb{R}$, where $z_1 = \phi(a)$ and $z_2 = \phi(b)$, and satisfying $\phi'(t) \neq 0$ for all $t \in (a,b)$.

More generally, if $\phi : [a,b] \to \mathbb{R}^2$ is 1-1 and piecewise smooth on $[a,b]$, and if $\{t_0 < t_1 < \ldots < t_n\}$ is a partition of $[a,b]$ such that $\phi'(t) \neq 0$ for all $t \in (t_{i-1},t_i)$, then the range C of ϕ is called a *piecewise smooth curve* from $z_1 = \phi(a)$ to $z_2 = \phi(b)$.

In either of these cases, ϕ is called a *parameterization* of the curve C.

Note that we do not assume that $|\phi'|$ is improperly-integrable, though the preceding theorem might have made you think we would.

REMARK. Throughout this chapter we will be continually faced with the fact that a given curve can have many different parameterizations. Indeed, if $\phi_1 : [a,b] \to C$ is a parameterization, and if $g : [c,d] \to [a,b]$ is a smooth function having a nonzero derivative, then $\phi_2(s) = \phi_1(g(s))$ is another parameterization of C. Since our definitions and proofs about curves often involve a parametrization, we will frequently need to prove that the results we obtain are independent of the parameterization. The next theorem will help; it shows that any two parameterizations of C are connected exactly as above, i.e., there always is such a function g relating ϕ_1 and ϕ_2.

THEOREM 6.2. *Let $\phi_1 : [a,b] \to C$ and $\phi_2 : [c,d] \to C$ be two parameterizations of a piecewise smooth curve C joining z_1 to z_2. Then there exists a piecewise smooth function $g : [c,d] \to [a,b]$ such that $\phi_2(s) = \phi_1(g(s))$ for all $s \in [c,d]$. Moreover, the derivative g' of g is nonzero for all but a finite number of points in $[c,d]$.*

PROOF. Because both ϕ_1 and ϕ_2 are continuous and 1-1, it follows from Theorem 3.10 that the function $g = \phi_1^{-1} \circ \phi_2$ is continuous and 1-1 from $[c,d]$ onto $[a,b]$. Moreover, from Theorem 3.11, it must also be that g is strictly increasing or strictly decreasing. Write $\phi_1(t) = u_1(t) + iv_1(t) \equiv (u_1(t), v_1(t))$, and $\phi_2(s) = u_2(s) + iv_2(s) \equiv (u_2(s), v_2(s))$. Let $\{x_0 < x_1 < \ldots < x_p\}$ be a partition of $[a,b]$ for which ϕ_1' is continuous and nonzero on the subintervals (x_{j-1},x_j), and let $\{y_0 < y_1 < \ldots < y_q\}$ be a partition of $[c,d]$ for which ϕ_2' is continuous and nonzero on the subintervals (y_{k-1},y_k). Then let $\{s_0 < s_1 < \ldots < s_n\}$ be the partition of $[c,d]$ determined by the finitely many points $\{y_k\} \cup \{g^{-1}(x_j)\}$. We will show that g is continuously differentiable at each point s in the subintervals (s_{i-1},s_i).

Fix an s in one of the intervals (s_{i-1},s_i), and let $t = \phi_1^{-1}(\phi_2(s)) = g(s)$. Of course this means that $\phi_1(t) = \phi_2(s)$, or $u_1(t) = u_2(s)$ and $v_1(t) = v_2(s)$. Then t is in some one of the intervals (x_{j-1},x_j), so that we know that $\phi_1'(t) \neq 0$. Therefore, we must have that at least one of $u_1'(t)$ or $v_1'(t)$ is nonzero. Suppose it is $v_1'(t)$ that is nonzero. The argument, in case it is $u_1'(t)$ that is nonzero, is completely analogous. Now, because v_1' is continuous at t and $v_1'(t) \neq 0$, it follows that v_1 is strictly monotonic in some neighborhood $(t-\delta, t+\delta)$ of t and therefore is 1-1 on that interval. Then v_1^{-1} is continuous by Theorem 3.10, and is differentiable at the point $v_1(t)$ by the Inverse Function Theorem. We will show that on this small interval $g = v_1^{-1} \circ v_2$, and this will prove that g is continuously differentiable at s.

Note first that if $\phi_2(\sigma) = x+iy$ is a point on the curve C, then $v_2(\phi_2^{-1}(x+iy)) = y$. Then, for any $\tau \in [a,b]$, we have

$$\begin{aligned} v_1^{-1}(v_2(g^{-1}(\tau))) &= v_1^{-1}(v_2(\phi_2^{-1}(\phi_1(\tau)))) \\ &= v_1^{-1}(v_2(\phi_2^{-1}(u_1(\tau)+iv_1(\tau)))) \\ &= v_1^{-1}(v_1(\tau)) \\ &= \tau, \end{aligned}$$

showing that $v_1^{-1} \circ v_2 = g^{-1^{-1}} = g$. Hence g is continuously differentiable at every point s in the subintervals (s_{i-1}, s_i). Indeed $g'(\sigma) = v_1^{-1'}(v_2(\sigma))v_2'(\sigma)$ for all σ near s, and hence g is piecewise smooth.

Obviously, $\phi_2(s) = \phi_1(g(s))$ for all s, implying that $\phi_2'(s) = \phi_1'(g(s))g'(s)$. Since $\phi_2'(s) \neq 0$ for all but a finite number of points s, it follows that $g'(s) \neq 0$ for all but a finite number of points, and the theorem is proved.

COROLLARY. *Let ϕ_1 and ϕ_2 be as in the theorem. Then, for all but a finite number of points $z = \phi_1(t) = \phi_2(s)$ on the curve C, we have*

$$\frac{\phi_1'(t)}{|\phi_1'(t)|} = \frac{\phi_2'(s)}{|\phi_2'(s)|}.$$

PROOF OF THE COROLLARY. From the theorem we have that

$$\phi_2'(s) = \phi_1'(g(s))g'(s) = \phi_1'(t)g'(s)$$

for all but a finite number of points $s \in (c,d)$. Also, g is strictly increasing, so that $g'(s) \geq 0$ for all points s where g is differentiable. And in fact, $g'(s) \neq 0$ for all but a finite number of s's, because $g'(s)$ is either $(v_1^{-1} \circ v_2)'(s)$ or $(u_1^{-1} \circ u_2)'(s)$, and these are nonzero except for a finite number of points. Now the corollary follows by direct substitution.

REMARK. If we think of $\phi'(t) = (x'(t), y'(t))$ as a vector in the plane $\mathbb{R}^2$, then the corollary asserts that the direction of this vector is independent of the parameterization, at least at all but a finite number of points. This direction vector will come up again as the unit tangent of the curve.

The adjective "smooth" is meant to suggest that the curve is bending in some reasonable way, and specifically it should mean that the curve has a tangent, or tangential direction, at each point. We give the definition of tangential direction below, but we note that in the context of a moving particle, the tangential direction is that direction in which the particle would continue to move if the force that is keeping it on the curve were totally removed. If the derivative $\phi'(t) \neq 0$, then this vector is the velocity vector, and its direction is exactly what we should mean by the tangential direction.

The adjective "piecewise" will allow us to consider curves that have a finite number of points where there is no tangential direction, e.g., where there are "corners."

We are carefully orienting our curves at the moment. A curve C from z_1 to z_2 is being distinguished from the same curve from z_2 to z_1, even though the set C is the same in both instances. Which way we traverse a curve will be of great importance at the end of this chapter, when we come to Green's Theorem.

DEFINITION. Let C, the range of $\phi : [a,b] \to C$, be a piecewise smooth curve, and let $z = (x,y) = \phi(c)$ be a point on the curve. We say that the curve C has a tangential direction at z, relative to the parameterization ϕ, if the following limit exists:

$$\lim_{t\to c} \frac{\phi(t) - z}{|\phi(t) - z|} = \lim_{t\to c} \frac{\phi(t) - \phi(c)}{|\phi(t) - \phi(c)|}.$$

If this limit exists, it is a vector of length 1 in $\mathbb{R}^2$, and this unit vector is called the unit tangent (relative to the parameterization ϕ) to C at z.

The curve C has a *unit tangent* at the point z if there exists a parameterization ϕ for which the unit tangent at z relative to ϕ exists.

Exercise 6.1. (a) Restate the definition of tangential direction and unit tangent using the $\mathbb{R}^2$ version of the plane instead of the $\mathbb{C}$ version. That is, restate the definition in terms of pairs (x,y) of real numbers instead of a complex number z.

(b) Suppose $\phi : [a,b] \to C$ is a parameterization of a piecewise smooth curve C, and that $t \in (a,b)$ is a point where ϕ is differentiable with $\phi'(t) \neq 0$. Show that the unit tangent (relative to the parameterization ϕ) to C at $z = \phi(t)$ exists and equals $\phi'(t)/|\phi'(t)|$. Conclude that, except possibly for a finite number of points, the unit tangent to C at z is independent of the parameterization.

(c) Let C be the graph of the function $f(t) = |t|$ for $t \in [-1,1]$. Is C a smooth curve? Is it a piecewise smooth curve? Does C have a unit tangent at every point?

(d) Let C be the graph of the function $f(t) = t^{2/3} = (t^{1/3})^2$ for $t \in [-1,1]$. Is C a smooth curve? Is it a piecewise smooth curve? Does C have a unit tangent at every point?

(e) Consider the set C that is the right half of the unit circle in the plane. Let $\phi_1 : [-1,1] \to C$ be defined by

$$\phi_1(t) = (\cos(t\frac{\pi}{2}), \sin(t\frac{\pi}{2})),$$

and let $\phi_2 : [-1,1] \to C$ be defined by

$$\phi_2(t) = (\cos(t^3\frac{\pi}{2}), \sin(t^3\frac{\pi}{2})).$$

Prove that ϕ_1 and ϕ_2 are both parameterizations of C. Discuss the existence of a unit tangent at the point $(1,0) = \phi_1(0) = \phi_2(0)$ relative to these two parameterizations.

(f) Suppose $\phi : [a,b] \to C$ is a parameterization of a curve C from z_1 to z_2. Define ψ on $[a,b]$ by $\psi(t) = \phi(a+b-t)$. Show that ψ is a parameterization of a curve from z_2 to z_1.

Exercise 6.2. (a) Suppose f is a smooth, real-valued function defined on the closed interval $[a,b]$, and let $C \subseteq \mathbb{R}^2$ be the graph of f. Show that C is a smooth curve, and find a "natural" parameterization $\phi : [a,b] \to C$ of C. What is the unit tangent to C at the point $(t, f(t))$?

(b) Let z_1 and z_2 be two distinct points in $\mathbb{C}$, and define $\phi : [0,1] \rightarrow$ by $\phi(t) = (1-t)z_1 + tz_2$. Show that ϕ is a parameterization of the straight line from the point z_1 to the point z_2. Consequently, a straight line is a smooth curve. (Indeed, what is the definition of a straight line?)

(c) Define a function $\phi : [-r, r] \rightarrow \mathbb{R}^2$ by $\phi(t) = (t, \sqrt{r^2 - t^2})$. Show that the range C of ϕ is a smooth curve, and that ϕ is a parameterization of C.

(d) Define ϕ on $[0, \pi/2)$ by $\phi(t) = e^{it}$. For what curve is ϕ a parametrization?

(e) Let $z_1, z_2, \ldots, z_n$ be n distinct points in the plane, and suppose that the polygonal line joing these points in order never crosses itself. Construct a parameterization of that polygonal line.

(f) Let S be a piecewise smooth geometric set determined by the interval $[a, b]$ and the two piecewise smooth bounding functions u and l. Suppose z_1 and z_2 are two points in the interior S^0 of S. Show that there exists a piecewise smooth curve C joining z_1 to z_2, i.e., a piecewise smooth function $\phi : [\widehat{a}, \widehat{b}] \rightarrow C$ with $\phi(\widehat{a}) = z_1$ and $\phi(\widehat{b}) = z_2$, that lies entirely in S^0.

(g) Let C be a piecewise smooth curve, and suppose $\phi : [a, b] \rightarrow \mathbb{C}$ is a parameterization of C. Let $[c, d]$ be a subinterval of $[a, b]$. Show that the range of the restriction of ϕ to $[c, d]$ is a smooth curve.

Exercise 6.3. Suppose C is a smooth curve, parameterized by $\phi = u + iv : [a, b] \rightarrow C$.

(a) Suppose that $u'(t) \neq 0$ for all $t \in (a, b)$. Prove that there exists a smooth, real-valued function f on some closed interval $[a', b']$ such that C coincides with the graph of f.
HINT: f should be something like $v \circ u^{-1}$.

(b) What if $v'(t) \neq 0$ for all $t \in (a, b)$?

Exercise 6.4. Let C be the curve that is the range of the function $\phi : [-1, 1] \rightarrow \mathbb{C}$, where $\phi(t) = t^3 + t^6 i)$.

(a) Is C a piecewise smooth curve? Is it a smooth curve? What points z_1 and z_2 does it join?

(b) Is ϕ a parameterization of C?

(c) Find a parameterization for C by a function $\psi : [3, 4] \rightarrow \mathbb{C}$.

(d) Find the unit tangent to C and the point $0 + 0i$.

Exercise 6.5. Let C be the curve parameterized by $\phi : [-\pi, \pi - \epsilon] \rightarrow C$ defined by $\phi(t) = e^{it} = \cos(t) + i\sin(t)$.

(a) What curve does ϕ parameterize?

(b) Find another parameterization of this curve, but base on the interval $[0, 1-\epsilon]$.

ARC LENGTH

Suppose C is a piecewise smooth curve, parameterized by a function ϕ. Continuing to think like a physicist, we might guess that the length of this curve could be computed as follows. The particle is moving with velocity $\phi'(t)$. This velocity is thought of as a vector in $\mathbb{R}^2$, and as such it has a direction and a magnitude or speed. The speed is just the absolute value $|\phi'(t)|$ of the velocity vector $\phi'(t)$. Now distance is speed multiplied by time, and so a good guess for the formula for the length L of the curve C would be

$$L = \int_a^b |\phi'(t)| \, dt. \tag{6.1}$$

Two questions immediately present themselves. First, and of primary interest, is whether the function $|\phi'|$ is improperly-integrable on (a,b)? We know by Exercise 5.22 that ϕ' itself is improperly-integrable, but we also know from Exercise 5.23 that a function can be improperly-integrable on an open interval and yet its absolute value is not. In fact, the answer to this first question is no (See Exercise 6.6.). We know only that $|\phi'|$ exists and is continuous on the open subintervals of a partition of $[a,b]$.

The second question is more subtle. What if we parameterize a curve in two different ways, i.e., with two different functions ϕ_1 and ϕ_2? How do we know that the two integral formulas for the length have to agree? Of course, maybe most important of all to us, we also must justify the physicist's intuition. That is, we must give a rigorous mathematical definition of the length of a smooth curve and show that Formula (6.1) above does in fact give the length of the curve. First we deal with the independence of parameterization question.

THEOREM 6.3. *Let C be a smooth curve joining (distinct) points z_1 to z_2 in $\mathbb{C}$, and let $\phi_1 : [a,b] \to C$ and $\phi_2 : [c,d] \to C$ be two parameterizations of C. Suppose $|\phi_2'|$ is improperly-integrable on (c,d). Then $|\phi_1'|$is improperly-integrable on (a,b), and*

$$\int_a^b \|\phi_1'(t)\|\,dt = \int_c^d \|\phi_2'(s)\|\,ds.$$

PROOF. We will use Theorem 6.2. Thus, let $g = \phi_1^{-1} \circ \phi_2$, and recall that g is continuous on $[c,d]$ and continuously differentiable on each open subinterval of a certain partition of $[c,d]$. Therefore, by part (d) of Exercise 5.22, g' is improperly-integrable on (c,d).

Let $\{x_0 < x_1 < \ldots < x_p\}$ be a partition of $[a,b]$ for which ϕ_1' is continuous and nonzero on the subintervals (x_{j-1}, x_j). To show that $|\phi_1'|$ is improperly-integrable on (a,b), it will suffice to show this integrability on each subinterval (x_{j-1}, x_j). Thus, fix a closed interval $[a', b'] \subset (x_{j-1}, x_j)$, and let $[c', d']$ be the closed subinterval of $[c,d]$ such that g maps $[c', d']$ 1-1 and onto $[a', b']$. Hence, by part (e) of Exercise 5.22, we have

$$\begin{aligned}
\int_{a'}^{b'} |\phi_1'(t)|\,dt &= \int_{c'}^{d'} |\phi_1'(g(s))| g'(s)\,ds \\
&= \int_{c'}^{d'} |\phi_1'(g(s))||g'()s)|\,ds \\
&= \int_{c'}^{d'} |\phi_1'(g(s))g'(s)|\,ds \\
&= \int_{c'}^{d'} |(\phi_1 \circ g)'(s)|\,ds \\
&= \int_{c'}^{d'} |\phi_2'(s)|\,ds \\
&\leq \int_c^d |\phi_2'(s)|\,ds,
\end{aligned}$$

which, by taking limits as a' goes to x_{j-1} and b' goes to x_j, shows that $|\phi_1'|$ is improperly-integrable over (x_{j-1}, x_j) for every j, and hence integrable over all of

(a, b). Using part (e) of Exercise 5.22 again, and a calculation similar to the one above, we deduce the equality

$$\int_a^b |\phi_1'| = \int_c^d |\phi_2'|,$$

and the theorem is proved.

Exercise 6.6. (A curve of infinite length) Let $\phi : [0,1] : \mathbb{R}^2$ be defined by $\phi(0) = (0,0)$, and for $t > 0$, $\phi(t) = (t, t\sin(1/t))$. Let C be the smooth curve that is the range of ϕ.

(a) Graph this curve.

(b) Show that

$$\begin{aligned} |\phi'(t)| &= \sqrt{1 + \sin^2(1/t) - \frac{\sin(2/t)}{t} + \frac{\cos^2(1/t)}{t^2}} \\ &= \frac{1}{t}\sqrt{t^2 + t^2\sin^2(1/t) - t\sin(2/t) + \cos^2(1/t)}. \end{aligned}$$

(c) Show that

$$\int_\delta^1 |\phi'(t)|\,dt = \int_1^{1/\delta} \frac{1}{t}\sqrt{\frac{1}{t^2} + \frac{\sin^2(t)}{t^2} - \frac{\sin(2t)}{t} + \cos^2(t)}\,dt.$$

(d) Show that there exists an $\epsilon > 0$ so that for each positive integer n we have $\cos^2(t) - \sin(2t)/t > 1/2$ for all t such that $|t - n\pi| < \epsilon$.

(e) Conclude that $|\phi'|$ is not improperly-integrable on $(0, 1)$. Deduce that, if Formula (6.1) is correct for the length of a curve, then this curve has infinite length.

Next we develop a definition of the length of a parameterized curve from a purely mathematical or geometric point of view. Happily, it will turn out to coincide with the physically intuitive definition discussed above.

Let C be a piecewise smooth curve joining the points z_1 and z_2, and let $\phi : [a,b] \to C$ be a parameterization of C. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of the interval $[a,b]$. For each $0 \le j \le n$ write $z_j = \phi(t_j)$, and think about the polygonal trajectory joining these points $\{z_j\}$ in order. The length L_P^ϕ of this polygonal trajectory is given by the formula

$$L_P^\phi = \sum_{j=1}^n |z_j - z_{j-1}|,$$

and this length is evidently an approximation to the length of the curve C. Indeed, since the straight line joining two points is the shortest curve joining those points, these polygonal trajectories all should have a length smaller than or equal to the length of the curve. These remarks motivate the following definition.

DEFINITION. Let $\phi : [a,b] \to C$ be a parameterization of a piecewise smooth curve $C \subset \mathbb{C}$. By the *length* L^ϕ of C, relative to the parameterization ϕ, we mean the number $L^\phi = \sup_P L_P^\phi$, where the supremum is taken over all partitions P of $[a,b]$.

REMARK. Of course, the supremum in the definition above could well equal infinity in some cases. Though it is possible for a curve to have an infinite length, the ones we will study here will have finite lengths. This is another subtlety of this subject. After all, every smooth curve is a compact subset of $\mathbb{R}^2$, since it is the continuous image of a closed and bounded interval, and we think of compact sets as being "finite" in various ways. However, this finiteness does not necessarily extend to the length of a curve.

Exercise 6.7. Let $\phi : [a,b] \to \mathbb{R}^2$ be a parameterization of a piecewise smooth curve C, and let P and Q be two partitions of $[a,b]$.

(a) If P is finer than Q, i.e., $Q \subseteq P$, show that $L_Q^\phi \le L_P^\phi$.

(b) If $\phi(t) = u(t) + iv(t)$, express L_P^ϕ in terms of the numbers $u(t_j)$ and $v(t_j)$.

Of course, we again face the annoying possibility that the definition of length of a curve will depend on the parameterization we are using. However, the next theorem, taken together with Theorem 6.3, will show that this is not the case.

THEOREM 6.4. *If C is a piecewise smooth curve parameterized by $\phi : [a,b] \to C$, then*

$$L^\phi = \int_a^b |\phi'(t)|\, dt,$$

specifically meaning that one of these quantities is infinite if and only if the other one is infinite.

PROOF. We prove this theorem for the case when C is a smooth curve, leaving the general argument for a piecewise smooth curve to the exercises. We also only treat here the case when L^ϕ is finite, also leaving the argument for the infinite case to the exercises. Hence, assume that $\phi = u + iv$ is a smooth function on $[a,b]$ and that $L^\phi < \infty$.

Let $\epsilon > 0$ be given. Choose a partition $P = \{t_0 < t_1 < \ldots < t_n\}$ of $[a,b]$ for which

$$L^\phi - L_P^\phi = L^\phi - \sum_{j=1}^n |\phi(t_j) - \phi(t_{j-1})| < \epsilon.$$

Because ϕ is continuous, we may assume by making a finer partition if necessary that the t_j's are such that $|\phi(t_1) - \phi(t_0)| < \epsilon$ and $|\phi(t_n) - \phi(t_{n-1})| < \epsilon$. This means that

$$L^\phi - \sum_{j=2}^{n-1} |\phi(t_j) - \phi(t_{j-1})| < 3\epsilon.$$

The point of this step (trick) is that we know that ϕ' is continuous on the open interval (a,b), but we will use that it is uniformly continuous on the compact set $[t_1, t_{n-1}]$. Of course that means that $|\phi'|$ is integrable on that closed interval, and in fact one of the things we need to prove is that $|\phi'|$ is improperly-integrable on the open interval (a,b).

Now, because ϕ' is uniformly continuous on the closed interval $[t_1, t_{n-1}]$, there exists a $\delta > 0$ such that $|\phi'(t) - \phi'(s)| < \epsilon$ if $|t - s| < \delta$ and t and s are in the interval $[t_1, t_{n-1}]$. We may assume, again by taking a finer partition if necessary, that the mesh size of $P$is less than this δ. Then, using part (f) of Exercise 5.9, we may also assume that the partition P is such that

$$|\int_{t_1}^{t_{n-1}} |\phi'(t)|\,dt - \sum_{j=2}^{n-1} |\phi'(s_j)|(t_j - t_{j-1})| < \epsilon$$

no matter what points s_j in the interval (t_{j-1}, t_j) are chosen. So, we have the following calculation, in the middle of which we use the Mean Value Theorem on the two functions u and v.

$$
\begin{aligned}
0 &\le |L^\phi - \int_{t_1}^{t_{n-1}} |\phi'(t)|\,dt| \\
&\le |L^\phi - \sum_{j=2}^{n-1} |\phi(t_j) - \phi(t_{j-1})| \\
&\quad + |\sum_{j=2}^{n-1} |\phi(t_j) - \phi(t_{j-1})| - \int_{t_1}^{t_{n-1}} |\phi'(t)|\,dt| \\
&\le 3\epsilon + |\sum_{j=2}^{n-1} |\phi(t_j) - \phi(t_{j-1})| - \int_{t_1}^{g_{n-1}} |\phi'(t)|\,dt| \\
&= 3\epsilon + |\sum_{j=2}^{n-1} |u(t_j) - u(t_{j-1}) + i(v(t_j) - v(t_{j-1})| - \int_{t_1}^{t_{n-1}} |\phi'(t)|\,dt| \\
&= 3\epsilon + |\sum_{j=2}^{n-1} \sqrt{(u(t_j) - u(t_{j-1}))^2 + (v(t_j) - v(t_{j-1}))^2} \\
&\quad - \int_{t_1}^{g_{n-1}} |\phi'(t)|\,dt| \\
&= 3\epsilon + |\sum_{j=2}^{n-1} \sqrt{(u'(s_j))^2 + (v'(r_j))^2}(t_j - t_{j-1}) \\
&\quad - \int_{t_1}^{t_{n-1}} |\phi'(t)|\,dt| \\
&\le 3\epsilon + |\sum_{j=2}^{n-1} \sqrt{(u'(s_j))^2 + (v'(s_j))^2}(t_j - t_{j-1}) \\
&\quad - \int_{t_1}^{t_{n-1}} |\phi'(t)|\,dt| \\
&\quad + \sum_{j=2}^{n-1} |\sqrt{(u(s_j))^2 + (v'(r_j))^2} - \sqrt{(u(s_j))^2 + (v'(s_j))^2}|(t_j - t_{j-1}) \\
&= 3\epsilon + |\sum_{j=2}^{n-1} |\phi'(s_j)|(t_j - t_{j-1}) - \int_{t_1}^{t_{n-1}} |\phi'(t)|\,dt| \\
&\quad + \sum_{j=2}^{n-1} |\sqrt{(u(s_j))^2 + (v'(r_j))^2} - \sqrt{(u(s_j))^2 + (v'(s_j))^2}|(t_j - t_{j-1}) \\
&\le 4\epsilon + \sum_{j=2}^{n-1} \frac{|(v'(r_j))^2 - (v'(s_j))^2|}{\sqrt{(u'(s_j))^2 + (v'(r_j))^2} + \sqrt{(u'(s_j))^2 + (v'(s_j))^2}}(t_j - t_{j-1}) \\
&\le 4\epsilon + \sum_{j=2}^{n-1} \frac{|v'(r_j) - v'(s_j)||v'(r_j) + v'(s_j)|}{|v'(r_j)| + |v'(s_j)|}(t_j - t_{j-1}) \\
&\le 4\epsilon + \sum_{j=2}^{n-1} |v'(r_j) - v'(s_j)|(t_j - t_{j-1}) \\
&\le 4\epsilon + \sum_{j=2}^{n-1} |\phi'(r_j) - \phi'(s_j)|(t_j - t_{j-1}) \\
&\le 4\epsilon + \sum_{j=2}^{n-1} \epsilon(t_j - t_{j-1}) \\
&= 4\epsilon + \epsilon(t_{n-1} - t_1) \\
&< \epsilon(4 + b - a).
\end{aligned}
$$

This implies that

$$L^\phi - \epsilon(4+b-a) \le \int_{t_1}^{t_{n-1}} |\phi'| \le L^\phi + \epsilon(4+b-a).$$

If we now let t_1 approach a and t_{n-1} approach b, we get

$$L^\phi - \epsilon(4+b-a) \le \int_a^b |\phi'| \le L^\phi + \epsilon(4+b-a),$$

which completes the proof, since ϵ is arbitrary.

Exercise 6.8. (a) Take care of the piecewise case in the preceding theorem.
(b) Take care of the case when L^ϕ is infinite in the preceding theorem.

We now have all the ingredients necessary to define the length of a smooth curve.

DEFINITION. Let C be a piecewise smooth curve in the plane. The *length* or *arc length* $L \equiv L(C)$ of C is defined by the formula

$$L(C) = L^\phi = \sup_P L_P^\phi,$$

where ϕ is any parameterization of C.

If z and w are two points on a piecewise smooth curve C, we will denote by $L(z, w)$ the arc length of the portion of the curve between z and w.

REMARK. According to Theorems 6.3 and 6.4, we have the following formula for the length of a piecewise smooth curve:

$$L = \int_a^b |\phi'(t)|\, dt,$$

where ϕ is any parameterization of C.

It should come as no surprise that the length of a curve C from z_1 to z_2 is the same as the length of that same curve C, but thought of as joining z_2 to z_1. Nevertheless, let us make the calculation to verify this. If $\phi : [a, b] \to C$ is a parameterization of this curve from z_1 to z_2, then we have seen in part (f) of exercise 6.1 that $\psi : [a, b] \to C$, defined by $\psi(t) = \phi(a+b-t)$, is a parameterization of C from z_2 to z_1. We just need to check that the two integrals giving the lengths are equal. Thus,

$$\int_a^b |\psi'(t)|\, dt = \int_a^b |\phi'(a+b-t)(-1)|\, dt = \int_a^b |\phi'(a+b-t)|\, dt = \int_a^b |\phi'(s)|\, ds,$$

where the last equality follows by changing variables, i.e., setting $t = a+b-s$.

We can now derive the formula for the circumference of a circle, which was one of our main goals. TRUMPETS?

THEOREM 6.5. *Let C be a circle of radius r in the plane. Then the length of C is $2\pi r$.*

PROOF. Let the center of the circle be denoted by (h,k). We can parameterize the top half of the circle by the function ϕ on the interval $[0,\pi]$ by $\phi(t) = h + r\cos(t) + i(k + r\sin(t))$. So, the length of this half circle is given by

$$L = \int_0^\pi |\phi'(t)|\,dt = \int_0^\pi |-r\sin(t) + ir\cos(t)|\,dt = \int_0^\pi r\,dt = \pi r.$$

The same kind of calculation would show that the lower half of the circle has length πr, and hence the total length is $2\pi r$.

The integral formula for the length of a curve is frequently not much help, especially if you really want to know how long a curve is. The integrals that show up are frequently not easy to work out.

Exercise 6.9. (a) Let C be the portion of the graph of the function $y = x^2$ between $x = 0$ and $x = 1$. Let $\phi : [0,1] \to C$ be the parameterization of this curve given by $\phi(t) = t + t^2 i$. Find the length of this curve.

(b) Define $\phi : [-0,\pi] \to \mathbb{C}$ by $\phi(t) = a\cos(t) + ib\sin(t)$. What curve does ϕ parameterize, and can you find its length?

INTEGRATION WITH RESPECT TO ARC LENGTH

We introduce next what would appear to be the best parameterization of a piecewise smooth curve, i.e., a parameterization by arc length. We will then use this parameterization to define the integral of a function whose domain is the curve.

THEOREM 6.6. *Let C be a piecewise smooth curve of finite length L joining two distinct points z_1 to z_2. Then there exists a parameterization $\gamma : [0,L] \to C$ for which the arc length of the curve joining $\gamma(t)$ to $\gamma(u)$ is equal to $|u-t|$ for all $t < u \in [0,L]$.*

PROOF. Let $\phi : [a,b] \to C$ be a parameterization of C. Define a function $F : [a,b] \to [0,L]$ by

$$F(t) = \int_a^t |\phi'(s)|\,ds.$$

In other words, $F(t)$ is the length of the portion of C that joins the points $z_1 = \phi(a)$ and $\phi(t)$. By the Fundamental Theorem of Calculus, we know that the function F is continuous on the entire interval $[a,b]$ and is continuously differentiable on every subinterval (t_{i-1},t_i) of the partition P determined by the piecewise smooth parameterization ϕ. Moreover, $F'(t) = |\phi'(t)| > 0$ for all $t \in (t_{i-1},t_i)$, implying that F is strictly increasing on these subintervals. Therefore, if we write $s_i = F(t_i)$, then the s_i's form a partition of the interval $[0,L]$, and the function $F : (t_{i-1},t_i) \to (s_{i-1},s_i)$ is invertible, and its inverse F^{-1} is continuously differentiable. It follows then that $\gamma = \phi \circ F^{-1} : [0,L] \to C$ is a parameterization of C. The arc length between the points $\gamma(t)$ and $\gamma(u)$ is the arc length between $\phi(F^{-1}(t))$ and $\phi(F^{-1}(u))$, and this is given by the formula

$$\begin{aligned}\int_{F^{-1}(t)}^{F^{-1}(u)} |\phi'(s)|\,ds &= \int_a^{F^{-1}(u)} |\phi'(s)|\,ds - \int_a^{F^{-1}(t)} |\phi'(s)|\,ds \\ &= F(F^{-1}(u)) - F(F^{-1}(t)) \\ &= u - t,\end{aligned}$$

which completes the proof.

COROLLARY. *If γ is the parameterization by arc length of the preceding theorem, then, for all $t \in (s_{i-1}, s_i)$, we have $|\gamma'(s)| = 1$.*

PROOF OF THE COROLLARY. We just compute

$$\begin{aligned}
|\gamma'(s)| &= |(\phi \circ F^{-1})'(s)| \\
&= |\phi'(F^{-1}(s))(F^{-1})'(s)| \\
&= |\phi'(F^{-1}(s)|| \frac{1}{F'(F^{-1}(s))}| \\
&= |\phi'(f^{-1}(s))| \frac{1}{|\phi'(f^{-1}(s))|} \\
&= 1,
\end{aligned}$$

as desired.

We are now ready to make the first of our three definitions of integral over a curve. This first one is pretty easy.

Suppose C is a piecewise smooth curve joining z_1 to z_2 of finite length L, parameterized by arc length. Recall that this means that there is a 1-1 function γ from the interval $[0, L]$ onto C that satisfies the condidition that the arc length betweenthe two points $\gamma(t)$ and $\gamma(s)$ is exactly the distance between the points t and s. We can just identify the curve C with the interval $[0, L]$, and relative distances will correspond perfectly. A partition of the curve C will correspond naturally to a partition of the interval $[0, L]$. A step function on the dcurve will correspond in an obvious way to a step function on the interval $[0, L]$, and the formula for the integral of a step function on the curve is analogous to what it is on the interval. Here are the formal definitions:

DEFINITION. Let C be a piecewise smooth curve of finite length L joining distinct points, and let $\gamma : [0, L] \to C$ be a parameterization of C by arc length. By a *partition* of C we mean a set $\{z_0, z_1, \ldots , z_n\}$ of points on C such that $z_j = \gamma(t_j)$ for all j, where the points $\{t_0 < t_1 < \ldots < t_n\}$ form a partition of the interval $[0, L]$. The portions of the curve between the points z_{j-1} and z_j, i.e., the set $\gamma(t_{j-1}, t_j)$, are called the *elements* of the partition.

A *step fucntion* on C is a real-valued function h on C for which there exists a partition $\{z_0, z_1, \ldots , z_n\}$ of C such that $h(z)$ is a constant a_j on the portion of the curve between z_{j-1} and z_j.

Before defining the integral of a step function on a curve, we need to establish the usual consistency result, encountered in the previous cases of integration on intervals and integration over geometric sets, the proof of which this time we put in an exercise.

Exercise 6.10. Suppose h is a function on a piecewise smooth curve of finite length L, and assume that there exist two partitions $\{z_0, z_1, \ldots , z_n\}$ and $\{w_0, w_1, \ldots , w_m\}$ of C such that $h(z)$ is a constant a_k on the portion of the curve between z_{k-1} and z_k, and $h(z)$ is a constant b_j on the portion of the curve between w_{j-1} and w_j. Show that

$$\sum_{k=1}^{n} a_k L(z_{k-1}, z_k) = \sum_{j=1}^{m} b_j L(w_{j-1}, w_j).$$

HINT: Make use of the fact that $h \circ \gamma$ is a step function on the interval $[0, L]$.

Now we can make the definition of the integral of a step function on a curve.

DEFINITION. Let h be a step function on a piecewise smooth curve C of finite length L. The *integral, with respect to arc length* of h over C is denoted by $\int_C h(s)\,ds$, and is defined by

$$\int_C h(s)\,ds = \sum_{j=1}^n a_j L(z_{j-1}, z_j),$$

where $\{z_0, z_1, \dots, z_n\}$ is a partition of C for which $h(z)$ is the constant a_j on the portion of C between z_{j-1} and z_j.

Of course, integrable functions on C with respect to arc length will be defined to be functions that are uniform limits of step functions. Again, there is the consistency issue in the definition of the integral of an integrable function.

Exercise 6.11. (a) Suppose $\{h_n\}$ is a sequence of step functiohns on a piecewise smooth curve C of finite length, and assume that the sequence $\{h_n\}$ converges uniformly to a function f. Prove that the sequence $\{\int_C h_n(s)\,ds\}$ is a convergent sequence of real numbers.

(b) Suppose $\{h_n\}$ and $\{k_n\}$ are two sequences of step functions on a piecewise smooth curve C of finite length l, and that both sequences converge uniformly to the same function f. Prove that

$$\lim \int_C h_n(s)\,ds = \lim \int_C k_n(s)\,ds.$$

DEFINITION. Let C be a piecewise smooth curve of finite length L. A function f with domain C is called *integrable with respect to arc length* on C if it is the uniform limit of step functions on C.

The *integral with respect to arc length* of an integrable function f on C is again denoted by $\int_C f(s)\,ds$, and is defined by

$$\int_C f(s)\,ds = \lim \int_C h_n(s)\,ds,$$

where $\{h_n\}$ is a sequence of step functions that converges uniformly to f on C.

In a sense, we are simply identifying the curve C with the interval $[0, L]$ by means of the 1-1 parameterizing function γ. The next theorem makes this quite plain.

THEOREM 6.7. *Let C be a piecewise smooth curve of finite length L, and let γ be a parameterization of C by arc length. If f is an integrable function on C, then*

$$\int_C f(s)\,ds = \int_0^L f(\gamma(t))\,dt.$$

PROOF. First, if h is a step function on C, let $\{z_j\}$ be a partition of C for which $h(z)$ is a constant a_j on the portion of the curve between z_{j-1} and z_j. Let $\{t_j\}$ be the partition of $[0, L]$ for which $z_j = \gamma(t_j)$ for every j. Note that $h \circ \gamma$ is a step function on $[0, L]$, and that $h \circ \gamma(t) = a_j$ for all $t \in (t_{j-1}, t_j)$. Then,

$$\begin{aligned}\int_C h(s)\, ds &= \sum_{j=1}^{N} a_j L(z_{j-1}, z_j) \\ &= \sum_{j=1}^{n} a_j L(\gamma(t_{j-1}), \gamma(t_j)) \\ &= \sum_{j=1}^{n} a_j (t_j - t_{j-1}) \\ &= \int_0^L h \circ \gamma(t)\, dt,\end{aligned}$$

which proves the theorem for step functions.

Finally, if $f = \lim h_n$ is an integrable function on C, then the sequence $\{h_n \circ \gamma\}$ converges uniformly to $f \circ \gamma$ on $[0, L]$, and so

$$\begin{aligned}\int_C f(s)\, ds &= \lim \int_C h_n(s)\, ds \\ &= \lim \int_0^L h_n(\gamma(t))\, dt \\ &= \int_0^l f(\gamma(t))\, dt,\end{aligned}$$

where the final equality follows from Theorem 5.6. Hence, Theorem 6.7 is proved.

Although the basic definitions of integrable and integral, with respect to arc length, are made in terms of the particular parameterization γ of the curve, for computational purposes we need to know how to evaluate these integrals using different parameterizations. Here is the result:

THEOREM 6.8. *Let C be a piecewise smooth curve of finite length L, and let $\phi : [a, b] \to C$ be a parameterization of C. If f is an integrable function on C. Then*

$$\int_C f(s)\, ds = \int_a^b f(\phi(t))|\phi'(t)|\, dt.$$

PROOF. Write $\gamma : [0, L] \to C$ for a parameterization of C by arc length. As in the proof to Theorem 6.3, we write $g : [a, b] \to [0, L]$ for $\gamma^{-1} \circ \phi$. Just as in that proof, we know that g is a piecewise smooth function on the interval $[a, b]$. Hence, recalling that $|\gamma'(t)| = 1$ and $g'(t) > 0$ for all but a finite number of points, the following calculation is justified:

$$\begin{aligned}
\int_C f(s)\,ds &= \int_0^L f(\gamma(t))\,dt \\
&= \int_0^L f(\gamma(t))|\gamma'(t)|\,dt \\
&= \int_a^b f(\gamma(g(u)))|\gamma'(g(u))|g'(u)\,du \\
&= \int_a^b f(\gamma(g(u)))|\gamma'(g(u))||g'(u)|\,du \\
&= \int_a^b f(\phi(u))|\gamma'(g(u))g'(u)|\,du \\
&= \int_a^b f(\phi(u))|('gamma \circ g)'(u)|\,du \\
&= \int_a^b f(\phi(u))|\phi'(u)|\,du,
\end{aligned}$$

as desired.

Exercise 6.12. Let C be the straight line joining the points $(0,1)$ and $(1,2)$.

(a) Find the arc length parameterization $\gamma : [0,\sqrt{2}] \to C$.

(b) Let f be the function on this curve given by $f(x,y) = x^2y$. Compute $\int_C f(s)\,ds$.

(c) Let f be the function on this curve that is defined by $f(x,y)$ is the distance from (x,y) to the point $(0,3)$. Compute $\int_c f(s)\,ds$.

The final theorem of this section sums up the properties of integrals with respect to arc length. There are no surprises here.

THEOREM 6.9. *Let C be a piecewise smooth curve of finite length L, and write $I(C)$ for the set of all functions that are integrable with respect to arc length on C. Then:*

(1) *$I(C)$ is a vector space ovr the real numbers, and*

$$\int_C (af(s) + bg(s))\,ds = a\int_C f(s)\,ds + b\int_C g(s)\,ds$$

for all $f, g \in I(C)$ and all $a, b \in \mathbb{R}$.

(2) (Positivity) If $f(z) \geq 0$ for all $z \in C$, then $\int_C f(s)\,ds \geq 0$.

(3) If $f \in I(C)$, then so is $|f|$, and $|\int_C f(s)\,ds| \leq \int_C |f(s)|\,ds$.

(4) If f is the uniform limit of functions f_n, each of which is in $I(C)$, then $f \in I(C)$ and $\int_C f(s)\,ds = \lim \int_C f_n(s)\,ds$.

(5) Let $\{u_n\}$ be a sequence of functions in $I(C)$, and suppose that for each n there is a number m_n, for which $|u_n(z)| \leq m_n$ for all $z \in C$, and such that the infinite series $\sum m_n$ converges. Then the infinite series $\sum u_n$ converges uniformly to an integrable function, and $\int_C \sum u_n(s)\,ds = \sum \int_C u_n(s)\,ds$.

Exercise 6.13. (a) Prove the preceding theorem. Everything is easy if we compose all functions on C with the parameterization γ, obtaining functions on $[0, L]$, and then use Theorem 5.6.

(b) Suppose C is a piecewise smooth curve of finite length joining z_1 and z_2. Show that the integral with respect to arc length of a function f over C is the same whether we think of C as being a curve from z_1 to z_2 or, the other way around, a curve from z_2 to z_1.

REMARK. Because of the result in part (b) of the preceding exercise, we speak of "integrating over C" when we are integrating with respect to arc length. We do not speak of "integrating from z_1 to z_2," since the direction doesn't matter. This is in marked contrast to the next two kinds of integrals over curves that we will discuss.

here is one final bit of notation. Often, the curves of interest to us are graphs of real-valued functions. If $g : [a, b] \to \mathbb{R}$ is a piecewise smooth function, then its graph C is a piecewise smooth curve, and we write $\int_{\mathrm{graph}(g)} f(s)\, ds$ for the integral with respect to arc length of f over $C = \mathrm{graph}(g)$.

CONTOUR INTEGRALS

We discuss next what appears to be a simpler notion of integral over a curve. In this one, we really do regard the curve C as a subset of the complex plane as opposed to two-dimensional real space; we will be integrating complex-valued functions; and we explicitly think of the parameterizations of the curve as complex-valued functions on an interval $[a, b]$. Also, in this definition, a curve C from z_1 to z_2 will be distinguished from its reverse, i.e., the same set C thought of as a curve from z_2 to z_1.

DEFINITION. Let C be a piecewise smooth curve from z_1 to z_2 in the plane $\mathbb{C}$, parameterized by a (complex-valued) function $\phi : [a, b] \to C$. If f is a continuous, complex-valued function on C, The *contour integral of f from z_1 to z_2 along C* will be denoted by $\int_C f(\zeta)\, d\zeta$ or more precisely by $\int_{C\,z_1}^{z_2} f(\zeta)\, d\zeta$, and is defindd by

$$\int_{C\,z_1}^{z_2} f(\zeta)\, d\zeta = \int_a^b f(\phi(t))\phi'(t)\, dt.$$

REMARK. There is, as usual, the question about whether this definition depends on the parameterization. Again, it does not. See the next exercise.

The definition of a contour integral looks very like a change of variables formula for integrals. See Theorem 5.11 and part (e) of Exercise 5.22. This is an example of how mathematicians often use a true formula from one context to make a new definition in another context.

Notice that the only difference between the computation of a contour integral and an integral with respect to arc length on the curve is the absence of the absolute value bars around the factor $\phi'(t)$. This will make contour integrals more subtle than integrals with respect to arc length, just as conditionally convergent infinite series are more subtle than absolutely convergent ones.

Note also that there is no question about the integrability of $f(\phi(t))\phi'(t)$, because of Exercise 5.22. f is bounded, ϕ' is improperly-integrable on (a, b), and therefore so is their product.

Exercise 6.14. (a) State and prove the "independence of parameterization" result for contour integrals.

(b) Prove that

$$\int_{C\,z_1}^{z_2} f(\zeta)\,d\zeta = -\int_{C\,z_2}^{z_1} f(\zeta)\,d\zeta.$$

Just remember how to parameterize the curve in the opposite direction.

(c) Establish the following relation between the absolute value of a contour integral and a corresponding integral with respect to arc length.

$$|\int_C f(\zeta)\,d\zeta| \le \int_C |f(s)|\,ds.$$

Not all the usual properties hold for contour integrals, e.g., like those in Theorem 6.9 above. The functions here, and the values of their contour integrals, are complex numbers, so all the properties of integrals having to do with positivity and inequalities, except for the one in part (c) of Exercise 6.14, no longer make any sense. However, we do have the following results for contour integrals, the verification of which is just as it was for Theorem 6.9.

THEOREM 6.10. *Let C be a piecewise smooth curve of finite length joining z_1 to z_2. Then the contour integrals of continuous functions on C have the following properties.*

(1) *If f and g are any two continuous functions on C, and a and b are any two complex numbers, then*

$$\int_C (af(\zeta) + bg(\zeta))\,d\zeta = a\int_C f(\zeta)\,d\zeta + b\int_C g(\zeta)\,d\zeta.$$

(2) *If f is the uniform limit on C of a sequence $\{f_n\}$ of continuous functions, then $\int_C f(\zeta)\,d\zeta = \lim \int_C f_n(\zeta)\,d\zeta$.*

(3) *Let $\{u_n\}$ be a sequence of continuous functions on C, and suppose that for each n there is a number m_n, for which $|u_n(z)| \le m_n$ for all $z \in C$, and such that the infinite series $\sum m_n$ converges. Then the infinite series $\sum u_n$ converges uniformly to a continuous function, and $\int_C \sum u_n(\zeta)\,d\zeta = \sum \int_C u_n(\zeta)\,d\zeta$.*

In the next exercise, we give some important contour integrals, which will be referred to several times in the sequel. Make sure you understand them.

Exercise 6.15. Let c be a point in the complex plane, and let r be a positive number. Let C be the curve parameterized by $\phi : [-\pi, \pi - \epsilon] : C$ defined by $\phi(t) = c + re^{it} = c + r\cos(t) + ir\sin(t)$. For each integer $n \in \mathbb{Z}$, define $f_n(z) = (z-c)^n$.

(a) What two points z_1 and z_2 does C join, and what happens to z_2 as ϵ approaches 0?

(b) Compute $\int_C f_n(\zeta)\,d\zeta$ for all integers n, positive and negative.

(c) What happens to the integrals computed in part (b) when ϵ approaches 0?

(d) Set $\epsilon = \pi$, and compute $\int_C f_n(\zeta)\,d\zeta$ for all integers n.

(e) Again, set $\epsilon = \pi$. Evaluate

$$\int_C \frac{\cos(\zeta - c)}{\zeta - c}\,d\zeta \text{ and } \int_C \frac{\sin(\zeta - c)}{\zeta - c}\,d\zeta.$$

HINT: Make use of the infinite series representations of the trigonometric functions.

VECTOR FIELDS, DIFFERENTIAL FORMS, AND LINE INTEGRALS

We motivate our third definition of an integral over a curve by returning to physics. This definition is very much a real variable one, so that we think of the plane as $\mathbb{R}^2$ instead of $\mathbb{C}$. A connection between this real variable definition and the complex variable definition of a contour integral will emerge later.

DEFINITION. By a *vector field* on an open subset U of $\mathbb{R}^2$, we mean nothing more than a continuous function $\vec{V}(x,y) \equiv (P(x,y), Q(x,y))$ from U into $\mathbb{R}^2$. The functions P and Q are called the *components* of the vector field $\vec{V}$.

We will also speak of *smooth* vector fields, by which we will mean vector fields $\vec{V}$ both of whose component functions P and Q have continuous partial derivatives

$$\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x} textand \frac{\partial Q}{\partial y}$$

on U.

REMARK. The idea from physics is to think of a vector field as a force field, i.e., something that exerts a force at the point (x,y) with magnitude $|\vec{V}(x,y)|$ and acting in the direction of the vector $\vec{V}(x,y)$. For a particle to move within a force field, "work" must be done, that is energy must be provided to move the particle against the force, or energy is given to the particle as it moves under the influence of the force field. In either case, the basic definition of work is the product of force and distance traveled. More precisely, if a particle is moving in a direction $\vec{u}$ within a force field, then the work done on the particle is the product of the component of the force field in the direction of $\vec{u}$ and the distance traveled by the particle in that direction. That is, we must compute dot products of the vectors $\vec{V}(x,y)$ and $\vec{u}(x,y)$. Therefore, if a particle is moving along a curve C, parameterized with respect to arc length by $\gamma : [0,L] \to C$, and we write $\gamma(t) = (x(t), y(t))$, then the work $W(z_1, z_2)$ done on the particle as it moves from $z_1 = \gamma(0)$ to $z_2 = \gamma(L)$ within the force field $\vec{V}$, should intuitively be given by the formula

$$\begin{aligned} W(z_1, z_2) &= \int_0^L \langle \vec{V}(\gamma(t)) \mid \gamma'(t) \rangle \, dt \\ &= \int_0^L P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t) \, dt \\ &\equiv \int_C P \, dx + Q \, dy, \end{aligned}$$

where the last expression is explicitly defining the shorthand notation we will be using.

The preceding discussion leads us to a new notion of what kind of object should be "integrated" over a curve.

DEFINITION. A *differential form* on a subset U of $\mathbb{R}^2$ is denoted by $\omega = Pdx + Qdy$, and is determined by two continuous real-valued functions P and Q on U. We say that ω is *bounded* or *uniformly continuous* if the functions P and Q are bounded or uniformly continuous functions on U. We say that the differential form ω is *smooth of order* k if the set U is open, and the functions P and Q have continuous mixed partial derivatives of order k.

If $\omega = Pdx + Qdy$ is a differential form on a set U, and if C is any piecewise smooth curve of finite length contained in U, then we define the *line integral* $\int_C \omega$ of ω over C by

$$\int_C \omega = \int_C P\,dx + Q\,dy = \int_0^L P(\gamma(t))x'(t) + Q(\gamma(t))y'(t)\,dt,$$

where $\gamma(t) = (x(t), y(t))$ is a parameterization of C by arc length.

REMARK. There is no doubt that the integral in this definition exists, because P and Q are continuous functions on the compact set C, hence bounded, and γ' is integrable, implying that both x' and y' are integrable. Therefore $P(\gamma(t))x'(t) + Q(\gamma(t))y'(t)$ is integrable on $(0, L)$.

These differential forms ω really should be called "differential 1-forms." For instance, an example of a differential 2-form would look like $R\,dxdy$, and in higher dimensions, we could introduce notions of differential forms of higher and higher orders, e.g., in 3 dimension things like $P\,dxdy + Q\,dzdy + R\,dxdz$. Because we will always be dealing with $\mathbb{R}^2$, we will have no need for higher order differential forms, but the study of such things is wonderful. Take a course in Differential Geometry!

Again, we must see how this quantity $\int_C \omega$ depends, if it does, on different parameterizations. As usual, it does not.

Exercise 6.16. Suppose $\omega = Pdx + Qdy$ is a differential form on a subset U of $\mathbb{R}^2$.

(a) Let C be a piecewise smooth curve of finite length contained in U that joins z_1 to z_2. Prove that

$$\int_C \omega = \int_C P\,dx + Q\,dy = \int_a^b P(\phi(t))x'(t) + Q(\phi(t))y'(t)\,dt$$

for any parameterization $\phi : [a, b] \to C$ having components $x(t)$ and $y(t)$.

(b) Let C be as in part (a), and let $\widehat{C}$ denote the reverse of C, i.e., the same set C but thought of as a curve joining z_2 to z_1. Show that $\int_{\widehat{C}} \omega = -\int_C \omega$.

(c) Let C be as in part (a). Prove that

$$|\int_C P\,dx + Q\,dy| \le (M_P + M_Q)L,$$

where M_P and M_Q are bounds for the continuous functions $|P|$ and $|Q|$ on the compact set C, and where L is the length of C.

EXAMPLE. The simplest interesting example of a differential form is constructed as follows. Suppose U is an open subset of $\mathbb{R}^2$, and let $f : U \to \mathbb{R}$ be a differentiable real-valued function of two real variables; i.e., both of its partial derivatives exist at every point $(x, y) \in U$. (See the last section of Chapter IV.) Define a differential form $\omega = df$, called the *differential* of f, by

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy,$$

i.e., $P = \partial f/\partial x$ and $Q = \partial f/\partial y$. These differential forms df are called *exact differential forms.*

REMARK. Not every differential form ω is exact, i.e., of the form df. Indeed, determining which ω's are df's boils down to what may be the simplest possible partial differential equation problem. If ω is given by two functions P and Q, then saying that $\omega = df$ amounts to saying that f is a solution of the pair of simultaneous partial differential equations

$$\frac{\partial f}{\partial x} = P \text{ and } \frac{\partial f}{\partial y} = Q.$$

See part (b) of the exercise below for an example of a nonexact differential form.

Of course if a real-valued function f has continuous partial derivatives of the second order, then Theorem 4.22 tells us that the mixed partials f_{xy} and f_{yx} must be equal. So, if $\omega = Pdx + Qdy = df$ for some such f, Then P and Q would have to satisfy $\partial P/\partial y = \partial Q/\partial x$. Certainly not every P and Q would satisfy this equation, so it is in fact trivial to find examples of differential forms that are not differentials of functions. A good bit more subtle is the question of whether every differential form $Pdx + Qdy$, for which $\partial P/\partial y = \partial Q/\partial x$, is equal to some df. Even this is not true in general, as part (c) of the exercise below shows. The open subset U on which the differential form is defined plays a significant role, and, in fact, differential forms provide a way of studying topologically different kinds of open sets.

In fact, although it may seem as if a differential form is really nothing more than a pair of functions, the concept of a differential form is in part a way of organizing our thoughts about partial differential equation problems into an abstract mathematical context. This abstraction is a good bit more enlightening in higher dimensional spaces, i.e., in connection with functions of more than two variables. Take a course in Multivariable Analysis!

Exercise 6.17. (a) Solve the pair of simultaneous partial differential equations

$$\frac{\partial f}{\partial x} = x + y \text{ and } \frac{\partial f}{\partial y} = x - y.$$

(b) Show that it is impossible to solve the pair of simultaneous partial differential equations

$$\frac{\partial f}{\partial x} = x + y \text{ and } \frac{\partial f}{\partial y} = y^3.$$

Hence, conclude that the differential form $\omega = (x+y)dx+y^3dy$ is not the differential df of any real-valued function f.

(c) Let U be the open subset of $\mathbb{R}^2$ that is the complement of the single point $(0,0)$. Let $P(x,y) = -y/(x^2+y^2)$ and $Q(x,y) = x/(x^2+y^2)$. Show that $\partial P/\partial y = \partial Q/\partial x$ at every point of U, but that $\omega = Pdx + Qdy$ is not the differential df of any smooth function f on U.

HINT: If P were f_x, then f would have to be of the form $f(x,y) = -\tan^{-1}(x/y) + g(y)$, where g is some differentiable function of y. Show that if $Q = f_y$ then $g(y)$ is a constant c. Hence, $f(x,y)$ must be $-\tan^{-1}(x/y) + c$. But this function f is not continuous, let alone differentiable, at the point $(1,0)$. Consider $\lim f(1,1/n)$ and $\lim f(1,-1/n)$.

The next thing we wish to investigate is the continuity of $\int_C \omega$ as a function of the curve C. This brings out a significant difference in the concepts of line integrals versis integrals with respect to arc length. For the latter, we typically think of a fixed curve and varying functions, whereas with line integrals, we typically think of a fixed differential form and variable curves. This is not universally true, but should be kept in mind.

THEOREM 6.11. *Let $\omega = Pdx + Qdy$ be a fixed, bounded, uniformly continuous differential form on a set U in $\mathbb{R}^2$, and let C be a fixed piecewise smooth curve of finite length L, parameterized by $\phi : [a,b] \to C$, that is contained in U. Then, given an $\epsilon > 0$ there exists a $\delta > 0$ such that, for any curve $\widehat{C}$ contained in U, $|\int_C \omega - \int_{\widehat{C}} \omega| < \epsilon$ whenever the following conditions on the curve $\widehat{C}$ hold:*

(1) *$\widehat{C}$ is a piecewise smooth curve of finite length $\widehat{L}$ contained in U, parameterized by $\widehat{\phi} : [a,b] \to \widehat{C}$.*
(2) *$|\phi(t) - \widehat{\phi}(t)| < \delta$ for all $t \in [a,b]$.*
(3) *$\int_a^b |\phi'(t) - \widehat{\phi}'(t)|\, dt < \delta$.*

PROOF. Let $\epsilon > 0$ be given. Because both P and Q are bounded on U, let M_P and M_Q be upper bounds for the functions $|P|$ and $|Q|$ respectively. Also, since both P and Q are uniformly continuous on U, there exists a $\delta > 0$ such that if $|(c,d) - (c',d')| < \delta$, then $|P(c,d) - P(c',d')| < \epsilon/4L$ and $|Q(c,d) - Q(c',d')| < \epsilon/4L$. We may also choose this δ to be less than both $\epsilon/4M_P$ and $\epsilon/4M_Q$. Now, suppose $\widehat{C}$ is a curve of finite length $\widehat{L}$, parameterized by $\widehat{\phi} : [a,b] \to \widehat{C}$, and that $|\phi(t) - \widehat{\phi}(t)| < \delta$ for all $t \in [a,b]$, and that $\int_a^b |\phi'(t) - \widehat{\phi}'(t)| < \delta$. Writing

$\phi(t) = (x(t), y(t))$ and $\widehat{\phi}(t) = (\widehat{x}(t), \widehat{y}(t))$, we have

$$\begin{aligned}
0 &\leq |\int_C P\,dx + Q\,dy - \int_{\widehat{C}} P\,dx + Q\,dy| \\
&= |\int_a^b P(\phi(t))x'(t) - P(\widehat{\phi}(t))\widehat{x}'(t) + Q(\phi(t))y'(t) - Q(\widehat{\phi}(t))\widehat{y}'(t)\,dt| \\
&\leq \int_a^b |P(\phi(t))x'(t) - P(\widehat{\phi}(t))\widehat{x}'(t)|\,dt + \int_a^b |Q(\phi(t))y'(t) - Q(\widehat{\phi}(t))\widehat{y}'(t)|\,dt \\
&\leq \int_a^b |P(\phi(t)) - P(\widehat{\phi}(t))||x'(t)|\,dt + \int_a^b |P(\widehat{\phi}(t))||x'(t) - \widehat{x}'(t)|\,dt \\
&\quad + \int_a^b |Q(\phi(t)) - Q(\widehat{\phi}(t))||y'(t)|\,dt + \int_a^b |Q(\widehat{\phi}(t))||y'(t) - \widehat{y}'(t)|\,dt \\
&\leq \frac{\epsilon}{4L}\int_a^b |x'(t)|\,dt + M_P\int_a^b |x'(t) - \widehat{x}'(t)|\,dt \\
&\quad + \frac{\epsilon}{4L}\int_a^b |y'(t)|\,dt + M_Q\int_a^b |y'(t) - \widehat{y}'(t)|\,dt \\
&\leq \frac{\epsilon}{4L}\int_a^b |\phi'(t)|\,dt + M_P\int_a^b |\phi'(t) - \widehat{\phi}'(t)|\,dt \\
&\quad + \frac{\epsilon}{4L}\int_a^b |\phi'(t)|\,dt + M_Q\int_a^b |\phi'(t) - \widehat{\phi}'(t)|\,dt \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + M_P\delta + M_Q\delta \\
&< \epsilon,
\end{aligned}$$

as desired.

Again, we have a special notation when the curve C is a graph. If $g : [a, b] \to \mathbb{R}$ is a piecewise smooth function, then its graph C is a piecewise smooth curve, and we write $\int_{\mathrm{graph}(g)} P\,dx + Q\,dy$ for the line integral of the differential form $Pdx + Qdy$ over the curve $C = \mathrm{graph}(g)$.

As alluded to earlier, there is a connection between contour integrals and line integrals. It is that a single contour integral can often be expressed in terms of two line integrals. Here is the precise statement.

THEOREM 6.12. *Suppose C is a piecewise curve of finite length, and that $f = u + iv$ is a complex-valued, continuous function on C. Let $\phi : [a, b] \to C$ be a parameterization of C, and write $\phi(t) = x(t) + iy(t)$. Then*

$$\int_C f(\zeta)\,d\zeta = \int_C (U\,dx - v\,dy) + \int_C (v\,dx + u\,dy).$$

PROOF. We just compute:

$$\begin{aligned}
\int_C f(\zeta)\,d\zeta &= \int_a^b f(\phi(t))\phi'(t)\,dt \\
&= \int_a^b (u(\phi(t)) + iv(\phi(t)))(x'(t) + iy'(t))\,dt \\
&= \int_a^b (u(\phi(t))x'(t) - v(\phi(t))y'(t)) \\
&\quad + i(v(\phi(t))x'(t) + u(\phi(t))y'(t))\,dt \\
&= \int_a^b (u(\phi(t))x'(t) - v(\phi(t))y'(t))\,dt \\
&\quad + i\int_a^b (v(\phi(t))x'(t) + u(\phi(t))y'(t))\,dt \\
&= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy,
\end{aligned}$$

as asserted.

INTEGRATION AROUND CLOSED CURVES, AND GREEN'S THEOREM

Thus far, we have discussed integration over curves joining two distinct points z_1 and z_2. Very important in analysis is the concept of integrating around a closed curve, i.e., one that starts and ends at the same point. There is nothing really new here; the formulas for all three kinds of integrals we have defined will look the same, in the sense that they all are described interms of some parameterization ϕ. A parameterization $\phi : [a, b] \to C$ of a closed curve C is just like the parameterization for a curve joining two points, except that the two points $\phi(a)$ and $\phi(b)$ are equal.

Two problems are immediately apparent concerning integrating around a closed curve. First, where do we start on the curve, which point is the initial point? And second, which way to we go around the curve? Recall tha if $\phi : [a, b] \to C$ is a parameterization of C, then $\psi : [a, b] \to C$, defined by $\psi(t) = \phi(a + b - t)$, is a parameterization of C that is the reverse of ϕ, i.e., it goes around the curve in the other direction. If we are integrating with respect to arc length, this reverse direction won't make a difference, but, for contour integrals and line integrals, integrating in the reverse direction will introduce a minus sign.

The first question mentioned above is not so difficult to handle. It doesn't really matter where we start on a closed curve; the parameterization can easily be shifted.

Exercise 6.18. Let $\phi[a, b] \to \mathbb{R}^2$ be a piecewise smooth function that is 1-1 except that $\phi(a) = \phi(b)$. For each $0 < c < b - a$, define $\widehat{\phi} : [a + c, b + c] : \mathbb{R}^2$ by $\widehat{\phi}(t) = \phi(t)$ for $a + c \le t \le b$, and $\widehat{\phi}(t) = \phi(t - b + a$ for $b \le t \le b + c$.

(a) Show that $\widehat{\phi}$ is a piecewise smooth function, and that the range C of ϕ coincides with the range of $\widehat{\phi}$.

(b) Let f be an integrable (with respect to arc length) function on C. Show that

$$\int_a^b f(\phi(t))|\phi'(t)|\,dt = \int_{a+c}^{b+c} f(\widehat{\phi}(t))|\widehat{\phi}'(t)|\,dt.$$

That is, the integral $\int_C f(s)\,ds$ of f with respect to arc length around the closed curve C is independent of where we start.

(c) Let f be a continuous complex-valued function on C. Show that

$$\int_a^b f(\phi(t))\phi'(t)\,dt = \int_{a+c}^{b+c} f(\widehat{\phi}(t))\widehat{\phi}'(t)\,dt.$$

That is, the contour integral $\int_C f(\zeta)\,d\zeta$ of f around the closed curve C is independent of where we start.

(d) Let $\omega = Pdx + Qdy$ be a differential form on C. Prove that

$$\int_a^b P(\phi(t))x'(t) + Q(\phi(t))y'(t)\,dt = \int_{a+c}^{b+c} P(\widehat{\phi}(t))\widehat{x}'(t) + Q(\widehat{\phi}(t))\widehat{y}'(t)\,dt.$$

That is, the line integral $\int_C \omega$ of ω around C is independent of where we start.

The question of which way we proceed around a closed curve is one that leads to quite intricate and difficult mathematics, at least when we consider totaly general smooth curves. For our purposes it wil, suffice to study a special kind of closed curve, i.e., curves that are the boundaries of piecewise smooth geometric sets. Indeed, the intricate part of the general situation has a lot to do with determining which is the "inside" of the closed curve and which is the "outside," a question that is easily settled in the case of a geometric set. Simple pictures make this general question seem silly, but precise proofs that there is a definite inside and a definite outside are difficult, and eluded mathematicians for centuries, culminating in the famous Jordan Curve Theorem, which asserts exactly what our intuition predicts:

JORDAN CURVE THEOREM. *The complement of a closed curve is the union of two disjoint components, one bounded and one unbounded.*

We define the bounded component to be the inside of the curve and the unbounded component to be the outside.

We adopt the following convention for how we integrate around the boundary of a piecewise smooth geometric set S. That is, the curve C_S will consist of four parts: the lower boundary (graph of the lower bounding function l), the righthand boundary (a portion of the vertical line $x = b$), the upper boundary (the graph of the upper bounding function u), and finally the lefthand side (a portion of the vertical line $x = a$). By *integrating around* such a curve C_S, we will always mean proceeding counterclockwise around the curves. Specifically, we move from left to right along the lower boundary, from bottom to top along the righthand boundary, from right to left across the upper boundary, and from top to bottom along the lefthand boundary. Of course, as shown in the exercise above, it doesn't matter where we start.

Exercise 6.19. Let S be the closed piecewise smooth geometric set that is determined by the interval $[a, b]$ and the two piecewise smooth bounding functions u and l. Assume that the boundary C_S of S has finite length. Suppose the graph of u intersects the lines $x = a$ and $x = b$ at the points (a, c) and (b, d), and suppose that the graph of l intersects those lines at the points (a, e) and (b, f). Find a parameterization $\phi : [a', b'] \to C_S$ of the curve C_S.
HINT: Try using the interval $[a, b + d - f + b - a + c - e]$ as the domain $[a', b']$ of ϕ.

The next theorem, though simple to state and use, contains in its proof a combinatorial idea that is truly central to all that follows in this chapter. In its simplest form, it is just the realization that the line integral in one direction along a curve is the negative of the line integral in the opposite direction.

THEOREM 6.13. *Let $S_1, \dots, S_n$ be a collection of closed geometric sets that constitute a partition of a geometric set S, and assume that the boundaries of all the S_i's, as well as the boundary of S, have finite length. Suppose ω is a continuous differential form on all the boundaries $\{C_{S_k}\}$. Then*

$$\int_{C_S} \omega = \sum_{k=1}^{n} \int_{C_{S_k}} \omega.$$

PROOF. We give a careful proof for a special case, and then outline the general argument. Suppose then that S is a piecewise smooth geometric set, determined by the interval $[a, b]$ and the two bounding functions u and l, and assume that the boundary C_S has finite length. Suppose $m(x)$ is a piecewise smooth function on $[a, b]$, satisfying $\int_a^b |m'| < \infty$, and assume that $l(x) < m(x) < u(x)$ for all $x \in (a, b)$. Let S_1 be the geometric set determined by the interval $[a, b]$ and the two bounding functions m and l, and let S_2 be the geometric set determined by the interval $[a, b]$ and the two bounding functions u and m. We note first that the two geometric sets S_1 and S_2 comprise a partition of the geometric set S, so that this is indeed a pspecial case of the theorem.

Next, consider the following eight line integrals: First, integrate from left to write along the graph of m, second, up the line $x = b$ from $(b, m(b))$ to $(b, u(b))$, third, integrate from right to left across the graph of u, fourth, integrate down the line $x = a$ from $(a, u(a))$ to $(a, m(a))$, fifth, continue down the line $x = a$ from $(a, m(a))$ to $(a, l(a))$, sixth, integrate from left to right across the graph of l, seventh, integrate up the line $x = b$ from $(b, l(b))$ to $(b, m(b))$, and finally, integfrate from right to left across the graph of m.

The first four line integrals comprise the line integral around the geometric set S_2, and the last four comprise the line integral around the geometric set S_1. On the other hand, the first and eighth line integrals here cancel out, for one is just the reverse of the other. Hence, the sum total of these eight line integrals, integrals 2–7, is just the line integral around the boundary C_S of S. Therefore

$$\int_{C_S} \omega = \int_{C_{S_1}} \omega + \int_{C_{S_2}} \omega$$

as desired.

We give next an outline of the proof for a general partition $S_1, \dots, S_n$ of S. Let S_k be determined by the interval $[a_k, b_k]$ and the two bounding functions u_k and l_k. Observe that, if the boundary C_{S_k} of S_k intersects the boundary C_{S_j} of S_j in a curve C, then the line integral of ω along C, when it is computed as part of integrating counterclockwise around S_k, is the negative of the line integral along C, when it is computed as part of the line integral counterclockwise around S_j. Indeed, the first line integral is the reverse of the second one. (A picture could be helpful.) Consequently, when we compute the sum of the line integrals of ω around the C_{S_k}'s, All terms cancel out except those line integrals that ar computed along parts of the boundaries of the S_k's that intersect no other S_j. But such parts of the boundaries of the S_k's must coincide with parts of the boundary of S. Therefore, the sum of the line integrals of ω around the boundaries of the S_k's equals the line integral of ω around the boundary of S, and this is precisely what the theorem asserts.

Exercise 6.20. Prove the analog of Theorem 6.13 for contour integrals: Let $S_1, \ldots, S_n$ be a collection of closed geometric sets that constitute a partition of a geometric set S, and assume that the boundaries of all the S_i's, as well as the boundary of S, have finite length. Suppose f is a continuous complex-valued function on all the boundaries $\{C_{S_k}\}$ as well as on the boundary C_S. Then

$$\int_{C_S} f(\zeta)\, d\zeta = \sum_{k=1}^{n} \int_{C_{S_k}} f(\zeta)\, d\zeta.$$

We come now to the most remarkable theorem in the subject of integration over curves, Green's Theorem. Another fanfare, please!

THEOREM 6.14. (Green) Let S be a piecewise smooth, closed, geometric set, let C_S denote the closed curve that is the boundary of S, and assume that C_S is of finite length. Suppose $\omega = Pdx + Qdy$ is a continuous differential form on S that is smooth on the interior S^0 of S. Then

$$\int_{C_S} \omega = \int_{C_S} P\, dx + Q\, dy = \int_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

REMARK. The first thing to notice about this theorem is that it connects an integral around a (1-dimensional) curve with an integral over a (2-dimensional) set, suggesting a kind of connection between a 1-dimensional process and a 2-dimensional one. Such a connection seems to be unexpected, and it should therefore have some important implications, as indeed Green's Theorem does.

The second thing to think about is the case when ω is an exact differential df of a smooth function f of two real variables. In that case, Green's Theorem says

$$\int_{C_S} \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = \int_S (f_{yx} - f_{xy}),$$

which would be equal to 0 if $f \in C^2(S)$, by Theorem 4.22. Hence, the integral of df around any such curve would be 0. If U is an open subset of $\mathbb{R}^2$, there may or may not be some other ω's, called *closed differential forms,* having the property that their integral around every piecewise smooth curve of finite length in U is 0, and the study of these closed differential forms ω that are not exact differential forms df has led to much interesting mathematics. It turns out that the structure of the open set U, e.g., how many "holes" there are in it, is what's important. Take a course in Algebraic Topology!

The proof of Green's Theorem is tough, and we break it into several steps.

LEMMA 1. *Suppose S is the rectangle $[a,b] \times [c,d]$. Then Green's Theorem is true.*

PROOF OF LEMMA 1. We think of the closed curve C_S bounding the rectangle as the union of four straight lines, C_1, C_2, C_3 and C_4, and we parameterize them as follows: Let $\phi : [a,b] \to C_1$ be defined by $\phi(t) = (t,c)$; let $\phi : [b, b+d-c] \to C_2$ be defined by $\phi(t) = (b, t-b+c)$; let $\phi : [b+d-c, b+d-c+b-a] \to C_3$ be defined by $\phi(t) = (b+d-c+b-t, d)$; and let $\phi : [b+d-c+b-a, b+d-c+b-a+d-c] \to C_4$ be defined by $\phi(t) = (a, b+d-c+b-a+d-t)$. One can check directly to see that this ϕ parameterizes the boundary of the rectangle $S = [a,b] \times [c,d]$.

As usual, we write $\phi(t) = (x(t), y(t))$. Now, we just compute, use the Fundamental Theorem of Calculus in the middle, and use part (d) of Exercise 5.30 at the end.

$$\begin{aligned}
\int_{C_S} \omega &= \int_{C_1} \omega + \int_{C_2} \omega \\
&\quad + \int_{C_3} \omega + \int_{C_4} \omega \\
&= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy \\
&\quad + \int_{C_3} P\,dx + Q\,dy + \int_{C_4} P\,dx + Q\,dy \\
&= \int_a^b P(\phi(t))x'(t) + Q(\phi(t))y'(t)\,dt \\
&\quad + \int_b^{b+d-c} P(\phi(t))x'(t) + Q(\phi(t))y'(t)\,dt \\
&\quad + \int_{b+d-c}^{b+d-c+b-a} P(\phi(t))x'(t) + Q(\phi(t))y'(t)\,dt \\
&\quad + \int_{b+d-c+b-a}^{b+d-c+b-a+d-c} P(\phi(t))x'(t) + Q(\phi(t))y'(t)\,dt \\
&= \int_a^b P(t,c)\,dt + \int_b^{b+d-c} Q(b,t-b+c)\,dt \\
&\quad + \int_{b+d-c}^{b+d-c+b-a} P(b+d-c+b-t,d)(-1)\,dt \\
&\quad + \int_{b+d-c+b-a}^{b+d-c+b-a+d-c} Q(a,b+d-c+b-a+d-t)(-1)\,dt \\
&= \int_a^b P(t,c)\,dt + \int_c^d Q(b,t)\,dt \\
&\quad - \int_a^b P(t,d)\,dt - \int_c^d Q(a,t)\,dt \\
&= \int_c^d (Q(b,t) - Q(a,t))\,dt - \int_a^b (P(t,d) - P(t,c))\,dt \\
&= \int_c^d \int_a^b \frac{\partial Q}{\partial x}(s,t)\,dsdt \\
&\quad - \int_a^b \int_c^d \frac{\partial P}{\partial y}(t,s)\,dsdt \\
&= \int_S (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},
\end{aligned}$$

proving the lemma.

LEMMA 2. *Suppose S is a right triangle whose vertices are of the form (a,c), (b,c) and (b,d). Then Green's Theorem is true.*

PROOF OF LEMMA 2. We parameterize the boundary C_S of this triangle as follows: For $t \in [a,b]$, set $\phi(t) = (t,c)$; for $t \in [b, b+d-c]$, set $\phi(t) = (b, t+c-b)$;

and for $t \in [b+d-c, b+d-c+b-a]$, set $\phi(t) = (b+d-c+b-t, b+d-c+d-t)$. Again, one can check that this ϕ parameterizes the boundary of the triangle S.

Write $\phi(t) = (x(t), y(t))$. Again, using the Fundamental Theorem and Exercise 5.30, we have

$$\begin{aligned}
\int_{C_S} \omega &= \int_{C_S} P\,dx + Q\,dy \\
&= \int_a^b P(\phi(t))x'(t) + Q(\phi(t))y'(t)\,dt \\
&\quad + \int_b^{b+d-c} P(\phi(t))x'(t) + Q(\phi(t))y'(t)\,dt \\
&\quad + \int_{b+d-c}^{b+d-c+b-a} P(\phi(t))x'(t) + Q(\phi(t))y'(t)\,dt \\
&= \int_a^b P(t,c)\,dt + \int_b^{b+d-c} Q(b, t+c-b)\,dt \\
&\quad + \int_{b+d-c}^{b+d-c+b-a} P(b+d-c+b-t, b+d-c+d-t)(-1)\,dt \\
&\quad + \int_{b+d-c}^{b+d-c+b-a} Q(b+d-c+b-t, b+d-c+d-t)(-1)\,dt \\
&= \int_a^b P(t,c)\,dt + \int_c^d Q(b,t)\,dt \\
&\quad - \int_a^b P(s, (d + \frac{s-b}{a-b}(c-d)))\,ds \\
&\quad - \int_c^d Q(b + \frac{s-d}{c-d}(a-b)), s)\,ds \\
&= \int_c^d (Q(b,s) - Q((b + \frac{s-d}{c-d}(a-b)), s))\,ds \\
&\quad - \int_a^b (P(s, (d + \frac{s-b}{a-b}(c-d))) - P(s,c))\,ds \\
&= \int_c^d \int_{b+\frac{s-d}{c-d}(a-b)}^b \frac{\partial Q}{\partial x}(t,s)\,dtds \\
&\quad - \int_a^b \int_c^{d+\frac{s-b}{a-b}(c-d)} \frac{\partial P}{\partial y}(s,t)\,dtds \\
&= \int_S (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},
\end{aligned}$$

which proves Lemma 2.

LEMMA 3. *Suppose $S_1, \dots, S_n$ is a partition of the geometric set S, and that the boundary C_{S_k} has finite length for all $1 \le k \le n$. If Green's Theorem holds for each geometric set S_k, then it holds for S.*

PROOF OF LEMMA 3. From Theorem 6.13 we have

$$\int_{C_S} \omega = \sum_{k=1}^n \int_{C_{S_k}} \omega,$$

and from Theorem 5.24 we have

$$\int_S Q_x - P_y = \sum_{k=1}^{n} \int_{S_k} Q_x - P_y.$$

Since Green's Theorem holds for each k, we have that

$$\int_{C_{S_k}} \omega = \int_{S_k} Q_x - P_y,$$

and therefore

$$\int_{C_S} \omega = \int_S Q_x - P_y,$$

as desired.

Exercise 6.21. (a) Prove Green's Theorem for a right triangle with vertices of the form $(a, c), (b, c)$, and (a, d).

(b) Prove Green's Theorem for a trapezoid having vertices of the form $(a, c), (b, c)$, (b, d), and (a, e), where both d and e are greater than c.
HINT: Represent this trapezoid as the union of a rectangle and a right triangle that share a border. Then use Lemma 3.

(c) Prove Green's Theorem for S any quadrilateral that has two vertical sides.

(d) Prove Green's Theorem for any geometric set S whose upper and lower bounding functions are piecewise linear functions.
HINT: Show that S can be thought of as a finite union of quadrilaterals, like those in part (c), each one sharing a vertical boundary with the next. Then, using induction and the previous exercise finish the argument.

We need one final lemma before we can complete the general proof of Green's Theorem. This one is where the analysis shows up; there are carefully chosen ϵ's and δ's.

LEMMA 4. *Suppose S is contained in an open set U and that ω is smooth on all of U. Then Green's Theorem is true.*

PROOF OF LEMMA 4. Let the piecewise smooth geometric set S be determined by the interval $[a, b]$ and the two bounding functions u and l. Using Theorem 2.11, choose an $r > 0$ such that the neighborhood $N_r(S) \subseteq U$. Now let $\epsilon > 0$ be given, and choose *delta* to satisfy the following conditions:

(1) (a) $\delta < r/2$, from which it follows that the open neighborhood $N_\delta(S)$ is a subset of the compact set $\overline{N}_{r/2}(S)$. (See part (f) of Exercise 2.24.)
(2) (b) $\delta < \epsilon/4M$, where M is a common bound for all four continuous functions $|P|, |Q|, |P_y|$, and $|Q_x|$ on the compact set $\overline{N}_{r/2}(S)$.
(3) (c) $\delta < \epsilon/4M(b - a)$.
(4) (d) δ satisfies the conditions of Theorem 6.11.

Next, using Theorem 6.1, choose two piecewise linear functions p_u and p_l so that

(1) $|u(x) - p_u(x)| < \delta/2$ for all $x \in [a, b]$.
(2) $|l(x) - p_l(x)| < \delta/2$ for all $x \in [a, b]$.
(3) $\int_a^b |u'(x) - p_u'(x)|\, dx < \delta$.
(4) $\int_a^b |l'(x) - p_l'(x)|\, dx < \delta$.

Let $\widehat{S}$ be the geometric set determined by the interval $[a, b]$ and the two bounding functions $\widehat{u}$ and $\widehat{l}$, where $\widehat{u} = p_u + \delta/2$ and $\widehat{l} = p_l - \delta/2$. We know that both $\widehat{u}$ and $\widehat{l}$ are piecewise linear functions. We have to be a bit careful here, since for some x's it could be that $p_u(x) < p_l(x)$. Hence, we could not simply use p_u and p_l themselves as bounding functions for $\widehat{S}$. We do know from (1) and (2) that $u(x) < \widehat{u}(x)$ and $l(x) > \widehat{l}(x)$, which implies that the geometric set S is contained in the geometric set $\widehat{S}$. Also $\widehat{S}$ is a subset of the neighborhood $N_\delta(s)$, which in turn is a subset of the compact set $\overline{N}_{r/2}(S)$.

Now, by part (d) of the preceding exercise, we know that Green's Theorem holds for $\widehat{S}$. That is

$$\int_{C_{\widehat{S}}} \omega = \int_{\widehat{S}} (Q_x - P_y).$$

We will show that Green's Theorem holds for S by showing two things: (i) $|\int_{C_S} \omega - \int_{C_{\widehat{S}}} \omega| < 4\epsilon$, and (ii) $|\int_S (Q_x) - P_y) - \int_{\widehat{S}} (Q_x - P_y)| < \epsilon$. We would then have, by the usual adding and subtracting business, that

$$|\int_{C_S} \omega - \int_S (Q_x - P_y)| < 5\epsilon,$$

and, since ϵ is an arbitrary positive number, we would obtain

$$\int_{C_S} \omega = \int_S (Q_x - P_y).$$

Let us estabish (i) first. We have from (1) above that $|u(x) - \widehat{u}(x)| < \delta$ for all $x \in [a, b]$, and from (3) that

$$\int_a^b |u'(x) - \widehat{u}'(x)|\, dx = \int_a^b |u'(x) - p_u'(x)|\, dx < \delta.$$

Hence, by Theorem 6.11,

$$\int_{\mathrm{graph}(u)} \omega - \int_{\mathrm{graph}(\widehat{u})} \omega| < \epsilon.$$

Similarly, using (2) and (4) above, we have that

$$|\int_{\mathrm{graph}(l)} \omega - \int_{\mathrm{graph}(\widehat{l})} \omega| < \epsilon.$$

Also, the difference of the line integrals of ω along the righthand boundaries of S and $\widehat{S}$ is less than ϵ. Thus

$$\begin{aligned}
|\int_{C\,(b,l(b))}^{(b,u(b))} \omega - \int_{C\,(b,\widehat{l}(b))}^{(b,\widehat{u}(b))} \omega| &= |\int_{l(b)}^{u(b)} Q(b,t)\,dt - \int_{\widehat{l}(b)}^{\widehat{u}(b)} Q(b,t)\,dt| \\
&\le |\int_{u(b)}^{\widehat{u}(b)} Q(b,t)\,dt| + |\int_{\widehat{l}(b)}^{l(b)} Q(b,t)\,dt| \\
&\le M(|l(b) - \widehat{l}(b)| + |u(b) - \widehat{u}(b)|) \\
&\le M(\delta + \delta) \\
&= 2M\delta \\
&< \epsilon.
\end{aligned}$$

Of course, a similar calculation shows that

$$|\int_{C(a,u(a))}^{(a,l(a))} \omega - \int_{C(a,\widehat{u}(a))}^{(a,\widehat{l}(a))} \omega| < \epsilon.$$

These four line integral inequalities combine to give us that

$$|\int_{C_S} \omega - \int_{C_{\widehat{S}}} \omega| < 4\epsilon,$$

establishing (i).

Finally, to see (ii), we just compute

$$\begin{aligned}
0 &\leq |\int_{\widehat{S}} (Q_y - P_x) - \int_S (Q_y - P_x)| \\
&= |\int_a^b \int_{\widehat{l}(t)}^{\widehat{u}(t)} (Q_x((t,s) - P_y(t,s))\, dsdt - \int_a^b \int_{l(t)}^{u(t)} (Q_x(t,s) - P_y(t,s))\, dsdt| \\
&= |\int_a^b \int_{\widehat{l}(t)}^{l(t)} (Q_x(t,s) - P_y(t,s))\, dsdt + \int_a^b \int_{u(t)}^{\widehat{u}(t)} (Q_x(t,s) - P_y(t,s))\, dsdt| \\
&\leq 2M(\int_a^b |l(t) - \widehat{l}(t)| + |\widehat{u}(t) - u(t)|\, dt \\
&\leq 4M\delta(b-a) \\
&< \epsilon.
\end{aligned}$$

This establishes (ii), and the proof is complete.

At last, we can finish the proof of this remarkable result.

PROOF OF GREEN'S THEOREM. As usual, let S be determined by the interval $[a,b]$ and the two bounding functions u and l. Recall that $u(x) - l(x) > 0$ for all $x \in (a,b)$. For each natural number $n > 2$, let S_n be the geometric set that is determined by the interval $[a+1/n, b-1/n]$ and the two bounding functions u_n and l_n, where $u_n = u - (u-l)/n$ restricted to the interval $[a+1/n, b-1/n]$, and $l_n = l + (u-l)/n$ restricted to $[a+1/n, b-1/n]$. Then each S_n is a piecewise smooth geometric set, whose boundary has finite length, and each S_n is contained in the open set S^0 where by hypothesis ω is smooth. Hence, by Lemma 4, Green's Theorem holds for each S_n. Now it should follow directly, by taking limits, that Green's Theorem holds for S. In fact, this is the case, and we leave the details to the exercise that follows.

Exercise 6.22. Let S, ω, and the S_n's be as in the preceding proof.

(a) Using Theorem 6.11, show that

$$\int_{C_S} \omega = \lim \int_{C_{S_n}} \omega.$$

(b) Let f be a bounded integrable function on the geometric set S. Prove that

$$\int_S f = \lim \int_{S_n} f.$$

(c) Complete the proof to Green's Theorem; i.e., take limits.

REMARK. Green's Theorem is primarily a theoretical result. It is rarely used to "compute" a line integral around a curve or an integral of a function over a geometric set. However, there is one amusing exception to this, and that is when the differential form $\omega = x\,dy$. For that kind of ω, Green's Theorem says that the area of the geometric set S can be computed as follows:

$$A(S) = \int_S 1 = \int_S \frac{\partial Q}{\partial x} = \int_{C_S} x\,dy.$$

This is certainly a different way of computing areas of sets from the methods we developed earlier. Try this way out on circles, ellipses, and the like.