MATH 4330/5330, Fourier Analysis
Section 12, Abstract Fourier Theory

DEFINITION. Let V be a vector space over the field of complex numbers. By an *inner product* on V we mean a complex-valued function on $V \times V$, denoted by $\langle x \mid y \rangle$, that satisfies the following conditions:

(1) For each fixed vector $y \in V$, we have

$$\langle x + x' \mid y \rangle = \langle x \mid y \rangle + \langle x' \mid y \rangle$$

and

$$\langle cx \mid y \rangle = c\langle x \mid y \rangle;$$

i.e., The function that sends x to $\langle x \mid y \rangle$ is a linear functional on V.

(2) $\langle x \mid y \rangle = \overline{\langle y \mid x \rangle}$. That is, interchanging the order of the two variables in the inner product changes the value into its complex conjugate.

(3) If $x \neq 0$, then $\langle x \mid x \rangle > 0$.

If a vector space V has an inner product defined on it, then we call the vectors space V an *inner product space.*

EXAMPLE 1. Let V be the vector space of all continuous, periodic, complex-valued functions on the real line. For f and g in V, define

$$\langle f \mid g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt.$$

EXAMPLE 2. Let $V = l^2$ be the vector space of all square-summable sequences $\{c_n\}_{-\infty}^{\infty}$. If $x = \{c_n\}$ and $y = \{d_n\}$ are two elements of l^2, define

$$\langle x \mid y \rangle = \sum_{n=-\infty}^{\infty} c_n \overline{d_n}.$$

EXERCISE 12.1. (a) Verify that the definition of the inner product in Example 1 satisfies the three required conditions.

(b) Verify that the inner product defined in Example 2 satisfies the required three conditions. By the way, is it clear that the infinite series defining this inner product actually converges?

(c) Suppose V is an inner product space. Prove that $\langle 0 \mid y \rangle = 0$ for all y. Show also that

$$\langle x \mid y + y' \rangle = \langle x \mid y \rangle + \langle x \mid y' \rangle$$

What about $\langle x \mid cy \rangle$? Is it equal to $c\langle x \mid y \rangle$? Is the assignment $y \to \langle x \mid y \rangle$ a linear functional?

DEFINITION. Let V be an inner product space. We say that two vectors x and y in V are *orthogonal* or *perpendicular* if their inner product $\langle x \mid y \rangle$ is 0. A collection $x_1, x_2, \ldots$ is called an *orthogonal set* if any two distinct elements of this set are orthogonal. That is, if x_i and x_j are two different elements of this set, then $\langle x_i \mid x_j \rangle = 0$.

EXERCISE 12.2. (a) Prove that the functions $\{\phi_n(x)\} \equiv \{e^{2\pi inx}\}$ form an orthogonal set in the inner product space of example 1 above.

(b) Let $\{c_n\}$ be the element of the inner product space of Example 2 above defined by $c_0 = 0$, and, for $n \neq 0$, $c_n = 1/n$. Let $\{d_n\}$ be the element defined by $d_0 = 2\pi$, and for $n \neq 0$, $d_n = 1/|n|$. Prove that these are orthogonal vectors.

(c) Let f be the element of the inner product space of Example 1 defined by $f(x) = 1/2 - x$. Show that f is orthogonal to the constant function 1. Next, find a nonzero quadratic function $ax^2 + bx + c$ that is orthogonal to both f and the constant function 1.

DEFINITION. Let V be an inner product space. If $v \in V$, define the *norm* $\|v\|$ of v by

$$\|v\| = \sqrt{\langle v \mid v\rangle}.$$

We will show that this definition of the norm of a vector makes sense, and we will use it to have a notion of convergence in the inner product space. What is necessary to make this definition of the norm intuitive is to show that it satisfies the triangle inequality. This is not easy, and we will only be able to prove it after another important inequality is established.

THEOREM 12.1. (Cauchy-Schwarz Inequality) Let V be an inner product space, and let v and w be two elements of V. Then

$$|\langle v \mid w\rangle| \leq \|v\|\|w\|.$$

PROOF. Note first that this inequality is definitely satisfied if $w = 0$. In fact, both sides of the inequality are 0 in that case. Hence, assume that w is not 0.

Next, note that, for any complex number c, we have the following calculation.

$$\begin{aligned}
0 &\leq \|v + cw\|^2 \\
&= \langle v + cw \mid v + cw\rangle \\
&= \langle v \mid v + cw\rangle + \langle cw \mid v + cw\rangle \\
&= \langle v \mid v\rangle + \langle v \mid cw\rangle + \langle cw \mid v\rangle + \langle cw \mid cw\rangle \\
&= \|v\|^2 + \overline{c}\langle v \mid w\rangle + c\langle w \mid v\rangle + |c|^2\|w\|w^2.
\end{aligned}$$

Here comes the trick. Since this inequality is true for any choice of complex number c, let's plug in $c = -\langle v \mid w\rangle/\|w\|^2$. We get

$$\begin{aligned}
0 &\leq \|v\|^2 + \frac{-\overline{\langle v \mid w\rangle}}{\|w\|^2}\langle v \mid w\rangle + \frac{-\langle v \mid w\rangle}{\|w\|^2}\langle w \mid v\rangle + \frac{|\langle v \mid w\rangle|^2}{\|w\|^4}\|w\|^2 \\
&= \|v\|^2 - \frac{|\langle v \mid w\rangle|^2}{\|w\|^2} - \frac{|\langle v \mid w\rangle|^2}{\|w\|^2} + \frac{|\langle v \mid w\rangle|^2}{\|w\|^2} \\
&= \|v\|^2 - \frac{|\langle v \mid w\rangle|^2}{\|w\|^2},
\end{aligned}$$

from which it follows that

$$|\langle v \mid w\rangle| \leq \|v\|\|w\|,$$

as desired.

REMARK. Notice that the only place in the calculations above where an actual inequality could occur is in the very first step $0 \leq \|v + cw\|^2$. Hence, the Cauchy-Schwarz Inequality is actually a strict inequality unless there is some complex number c such that $v + cw = 0$, i.e., v is a multiple of w. Another way of putting this is as follows: If the Cauchy-Schwarz Inequality is actually an equality, $|\langle v \mid w\rangle| = \|v\|\|w\|$, then one of the two vectors must be a multiple of the other one.

EXERCISE 12.3. (a) Write out explicitly what the norm is for Example 1. Then write out explicitly what the Cauchy-Schwarz Inequality looks like.
(b) Repeat part (a) for the inner product space of Example 2.

THEOREM 12.2. (Triangle Inequality for the Norm) Suppose V is an inner product space, and let v and w be two elements of V. Then

$$\|v + w\| \leq \|v\| + \|w\|.$$

PROOF. We will use the Cauchy-Schwarz Inequality in the middle of this computation, and we will also use the triangle inequality for complex numbers $|z + z'| \leq |z| + |z'|$.

$$\begin{aligned} \|v + w\|^2 &= \langle v + w \mid v + w\rangle \\ &= \langle v \mid v\rangle + \langle v \mid w\rangle + \langle w \mid v\rangle + \langle w \mid w\rangle \\ &= \|v\|^2 + \langle v \mid w\rangle + \langle w \mid v\rangle + \|w\|^2. \end{aligned}$$

So,

$$\begin{aligned} \|v + w\|^2 &= |\|v + w\|^2| \\ &= |\|v\|^2 + \langle v \mid w\rangle + \langle w \mid v\rangle + \|w\|^2| \\ &\leq |\|v\|^2| + |\langle v \mid w\rangle| + |\langle w \mid v\rangle| + |\|w\|^2| \\ &\leq \|v\|^2 + \|v\|\|w\| + \|w\|\|v\| + \|w\|^2 \\ &= (\|v\| + \|w\|)^2, \end{aligned}$$

and the Triangle Inequality follows by taking square roots of this inequality.

DEFINITION. A collection $\{x_1, x_2, \dots\}$ of vectors in an inner product space is called an *orthonormal* set if it is an orthogonal set, and each vector x_i has norm equal to 1.

REMARK. The functions $\{\phi_n(x)\} \equiv \{e^{2\pi inx}\}$ form what is probably the most famous orthonormal set. Make sure you can prove this (well, not the most famous part). In fact, as we will see, every orthonormal set behaves pretty much exactly like the ϕ_n's.

EXERCISE 12.4. Let $\{v_1, v_2, \dots, v_N\}$ be an orthonormal set in an inner product space V. Suppose x is a linear combination of the v_i's: $x = \sum_{n=1}^{N} c_n v_n$. Show that

$$\|x\|^2 = \sum_{n=1}^{N} |c_n|^2.$$

Compare with part (4) of Proposition 7.2.

THEOREM 12.3. (Bessel's Inequality) Let $v_1, \ldots, v_N$ be an orthonormal set in an inner product space V, and let x be an element of V. Then

$$\sum_{n=1}^{N} |\langle x \mid v_n\rangle|^2 \leq \|x\|^2.$$

EXERCISE 12.5. Prove the preceding theorem. Remember that it begins with the obviously correct inequality

$$0 \leq \|x - \sum_{n=1}^{N} \langle x \mid v_n\rangle v_n\|^2.$$

Compare with Theorem 7.3.

REMARK. Given an (possibly infinite) orthonormal set $\{v_n\}$ in an inner product space V, we can define a kind of generalized Fourier transform on V. Namely, if $x \in V$, let $\widehat{x}$ be the function given by

$$\widehat{x}(n) = \langle x \mid v_n\rangle.$$

According to Bessel's Inequality above, the numbers $\{\widehat{x}(n)\}$ are square-summable, and more over

$$\sum_n |\widehat{x}(n)|^2 = \sum_n |\langle x \mid v_n\rangle|^2 \leq \|x\|^2.$$

The set of ϕ_n's in the classical Fourier series case were sufficiently rich, so that the transform in that case was 1-1 and had a nice expression for its inverse. In a general inner product space, what does "rich" mean?

DEFINITION. A set $\{v_n\}$ of orthonormal vectors in an inner product space V is called *maximal* if there is no nonzero vector $v \in V$ for which $\langle v \mid v_n\rangle = 0$ for all n. The idea is that, if there were such a nonzero vector v, then a larger collection of orthonormal vectors could be made by using the v_n's together with the additional vector $(1/\|v\|)v$.

THEOREM 12.4. *Suppose $\{v_n\}$ is a maximal collection of orthonormal vectors in an inner product space V. Then the generalized Fourier transform on V that is determined by this set of orthonormal vectors is 1-1. That is, if x and y are two elements of V for which $\widehat{x} = \widehat{y}$, then x must equal y.*

PROOF. The generalized Fourier transform on V determined by the orthonormal set $\{v_N\}$ is given by

$$\widehat{x}(n) = \langle x \mid v_n\rangle.$$

Suppose $\widehat{x} = \widehat{y}$. We must show that $x = y$.

Let $z = x - y$. Then

$$\begin{aligned}\langle z \mid v_n\rangle &= \langle x - y \mid v_n\rangle \\ &= \langle x \mid v_n\rangle - \langle y \mid v_n\rangle \\ &= \widehat{x}(n) - \widehat{y}(n) \\ &= 0.\end{aligned}$$

Because the set $\{v_n\}$ is a maximal set of orthonormal vectors, it must be that z is 0. But that means that $x = y$.

REMARK. So, a maximal set of orthonormal vectors gives us a generalized Fourier transform that is 1-1. What about the nice expression for its inverse? For that, we need something else.

DEFINITION. An inner product space H is called a *Hilbert space* if there exists a maximal orthonormal set $\{v_n\}$ in H that has the following property: If $\{c_n\}$ is any square-summable sequence of complex numbers, then the infinite series $\sum_n c_n v_n$ is summable in the norm on H. This means that there exists an element $y \in H$ such that

$$\lim_{N\to\infty} \|y - S_N\| = 0,$$

where

$$S_N = \sum_{n=1}^{N} c_n v_n.$$

Any such maximal set of orthonormal vectors is called an *orthonormal basis* for H.

EXERCISE 12.6. (a) Prove that the set of periodic, square-integrable functions is a Hilbert space. In fact, show that the collection $\{\phi_n\}$ is an orthonormal basis. (This just amounts to quoting the appropriate theorems to show that these functions satisfy the requirements.)

(b) In the inner product space of Example 2, let v_n be the sequence that is 1 at the nth coordinate and 0 at all other coordinates. Prove that these vectors $\{v_n\}$ form an orthonormal basis.

THEOREM 12.5. *Let H be a Hilbert space and let $\{v_n\}$ be an orthonormal basis for H. Then the generalized Fourier transform on H determined by the v_n's has an inverse given by*

$$x = \sum_n \widehat{x}(n) v_n.$$

That is, there is an explicit formula for recovering x from its transform $\widehat{x}$.

EXERCISE 12.7. (a) Prove Theorem 12.5. Adapt the argument given in the proof of Theorem 7.1.

(b) Prove the general Parseval Equality. That is, if $\{v_n\}$ is an orthonormal basis in a Hilbert space H, and if x is any vector in H, then

$$\|x\|^2 = \sum_n |\widehat{x}(n)|^2 = \sum_n |\langle x \mid v_n \rangle|^2.$$