MATH 4330/5330, Fourier Analysis
Section 4, The Heat Equation on the Circle
Fourier's Theorem

We imagine that we have a wire of length 1 that has been bent into a circle (with the ends 0 and 1 attached). We suppose that a function u of two variables t and x is such that the value $u(t, x)$ is the temperature at the point x on the wire at time t. Notice that the fact that the interval $[0, 1]$ is bent into a circle imposes a condition on the function u. Namely, $u(t, 0)$ must equal $u(t, 1)$ for all times $t \geq 0$.

We would like to be able to predict how this temperature function changes with time. Again, we consider the following so-called "initial value problem."

We suppose that we know the values $u(0, x) \equiv f(x)$ for all points $0 \leq x < 1$. These are the initial values. Is that enough information for us to be able to figure out the values $u(t, x)$ for a later time t?

REMARK. As before, physicists think that the temperature at a point on the wire is proportional to the velocity of the molecule at that point in the wire. Since the square of the velocity V^2 is proportional to the kinetic energy $mV^2/2$, we presume that the function $|u(t, x)|^2$ is proportional to the "instantaneous" energy at the point x, and so it must satisfy $\int_0^1 |u(t, x)|^2\, dx < \infty$ for every time t. The total energy must be finite at any given time.

Again, physicists also believe that this temperature function u must satisfy the same heat equation as in Section 1.

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x).$$

EXERCISE 4.1. Can you think of any solutions to this partial differential equation? Just as before, $u(t, x) = 0$ or $u(t, x) = 1$, or $u(t, x) = 2t + x^2$? How about $u(t, x) = e^{-\omega^2 t} \times e^{i\omega x}$? This time, these functions do satisfy the finite energy (L^2) requirement. Do any satisfy the requirement $u(t, 0) = u(t, 1)$ for all time t?

The preceding exercise shows that there are many solutions to the heat equation on the circle. Are there enough, i.e., given an arbitrary initial condition $f(x)$, is there necessarily a solution to the initial value problem? Are there too many solutions, i.e., given initial data $f(x)$, can there be more than one solution to the initial value problem?

We know from Section 2 that $u(t, x) = e^{-\omega^2 t} e^{i\omega x}$ is a solution of the heat equation, and its initial value is the function $f(x) = e^{i\omega x}$. (What do you think of an initial temperature function of the form $f(x) = e^{i\omega x}$? Would our physicist friends buy this?) The condition that $u(t, 0) = u(t, 1)$ then requires that $e^{i\omega} = 1$, and this only happens if $\omega = 2n\pi$ for some integer n. So, the only solutions of the initial value problem that we have thus far are functions of the form $u(t, x) = e^{-4\pi^2 n^2 t} e^{2\pi i n x}$, and the only initial conditions $f(x)$ for which we can solve the initial value problem for the heat equation are the functions $f(x) = e^{2\pi i n x}$.

EXERCISE 4.2. (a) Show that the set of solutions of the heat equation on the circle form a vectors space. That is, show that the sum of any two solutions is again a solution, and any scalar multiple of a solution is also a solution.

(b) Conclude that there is a solution to the initial value problem for the heat equation on the circle whenever the initial condition f is of the form

$$f(x) = \sum_{n=-N}^{N} c_n e^{2\pi inx},$$

where the c_n's are any complex numbers. In fact, exhibit an explicit solution. That is, write one down.

(c) Find solutions to the initial value problem for the heat equation, where the initial condition is $f(x) = \cos(2\pi x)$ or $f(x) = \sin(6\pi x)$. What about other initial value functions?

EXERCISE 4.3. (a) Show that if two functions $h(x)$ and $g(x)$ are square-integrable, then so is the sum $h(x) + g(x)$. Make use of the following inequalities:

$$\begin{aligned} |h(x) + g(x)|^2 &\le (|h(x)| + |g(x)|)^2 \\ &\le (2\max(|(h(x)|, |g(x)|))^2 \\ &= 4(\max(|h(x)|, |g(x)|)^2 \\ &\le 4(|h(x)|^2 + |g(x)|^2). \end{aligned}$$

(b) Conclude that the set of all square-integrable solutions to the heat equation is a vector space.

(c) Is it true, or clear, that there is just one solution to an initial value problem? That is, if $u(t,x)$ and $v(t,x)$ are both solutions to the heat equation, and if $u(0,x) = v(0,x)$ for all x, does it follow that $u(t,x) = v(t,x)$ for all t and x?

Periodic Functions

If f is a function, defined on the interval $[0,1)$, we may define a function, still called f, on the entire real line by extending f periodically. For example, for $1 \le x < 2$, we set $f(x)$ to be the same as $f(x-1)$. The number $x-1$ is between 0 and 1, so we know the value of f on that point. Then, for $2 \le x < 3$, we set $f(x) = f(x-1) = f(x-2)$. In general, for any real number x, write $x = [x] + \langle x \rangle$, where $[x]$ denotes the greatest integer $\le x$ and $\langle x \rangle$ denotes the fractional part of x, i.e., $x - [x]$. This extended function satisfies the condition $f(x) = f(x+1)$ for all real x.

A function on the whole real line that satisfies the equation $f(x+1) = f(x)$ for all $x \in \mathbb{R}$ is called a *periodic* function with period 1, or simply a *periodic* function. Any such periodic function is completely determined by its values on any interval of length 1, i.e., any interval of the form $[a, a+1)$. This particular function f is called the *periodic extension* of the original function f.

REMARK. We will often not distinguish between a function f on the interval $[0,1)$ and its periodic extension. They are completely interchangeable.

DEFINITION. Let p be a positive number. A function f on $\mathbb{R}$ is called *periodic with period p* if $f(x+p) = f(x)$ for all x.

Clearly, a periodic function with period p is totally determined by its values on any interval of the form $[a, a+p)$.

EXERCISE 4.4. (a) Show that the functions $\sin(2\pi x), \cos(2\pi x)$, and $e^{2\pi i x}$ are periodic.

(b) Show that the functions $|\sin(2\pi x)|$ and $|\cos(2\pi x)|$ are periodic with period $1/2$.

(c) Let α be a positive number. Show that the function $e^{i\alpha x}$ is periodic with period $2\pi/\alpha$.

Fourier's Claim

Fourier asserted that essentially every initial condition determines a unique solution of the initial value problem for the heat equation on the circle. He asserted this, because he claimed the following remarkable result:

THEOREM 4.1. (Fourier's Theorem)

(1) Every square-integrable function f on $[0, 1)$ can be represented as a (possibly infinite) linear combination of the exponential functions $\{e^{2\pi i n x}\}$:

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{2\pi i n x} \\ &= \lim_{N\to\infty} S_N(x). \end{aligned}$$

(2) For such an f, the function

$$u(t, x) = \sum_{n=-\infty}^{\infty} c_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

satisfies the initial value problem for the heat equation, with initial values $f(x)$.

EXERCISE 4.5. Assuming that part (1) is true, just how hard is it to prove part

(2)? What are the mathematical subtleties in the following computation?

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\begin{aligned}
\frac{\partial}{\partial t}(u(t,x)) &= \frac{\partial}{\partial t}(\sum_{n=-\infty}^{\infty} c_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}) \\
&= \frac{\partial}{\partial t}(\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}) \\
&= \lim_{N\to\infty} \frac{\partial}{\partial t}(\sum_{n=-N}^{N} c_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}) \\
&= \lim_{N\to\infty} (\sum_{n=-N}^{N} \frac{\partial}{\partial t} c_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}) \\
&= \sum_{n=-\infty}^{\infty} -4\pi^2 n^2 c_n e^{-4\pi^2 n^2 t} e^{2\pi i n x} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{-4\pi^2 n^2 t} \frac{\partial^2}{\partial x^2}(e^{2\pi i n x}) \\
&= \cdots \\
&= \frac{\partial^2}{\partial x^2}(\sum_{n=-\infty}^{\infty} c_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}) \\
&= \frac{\partial^2}{\partial x^2}(u(t,x)),
\end{aligned}
$$

and

$$u(0,x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} = f(x).$$

Moreover, Fourier is even more precise about part (1) of his theorem. He tells us exactly what the coefficients c_n are.

THEOREM 4.2. *If f is a square-integrable function on $[0,1)$, then the coefficients $\{c_n\}$ for which*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

are uniquely determined, and are given explicitly by the formulas

$$c_n = \int_0^1 f(t) e^{-2\pi i n t}\, dt.$$

We will come back to the proofs of these theorems in later sections.

DEFINITION. The infinite series $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$ is called the *Fourier series* for f, and the partial sums $S_N(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$ are called the *partial sums* of the Fourier series for f.

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EXERCISE 4.6. Use Euler's Theorem to show that part (1) of Fourier's Theorem can be stated as follows: If f is a square-integrable function on $[0,1)$, then f can be written in the form

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi kx) + \sum_{k=1}^{\infty} b_k \sin(2\pi kx),$$

where

$$a_0 = c_0 = \int_0^1 f(t)\, dt,$$

and, for $k \geq 1$,

$$a_k = 2\int_0^1 f(t)\cos(2\pi kt)\, dt$$

and

$$b_k = 2\int_0^1 f(t)\sin(2\pi kt)\, dt.$$

HINT: Justify the following calculations:

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx} \\
&= \sum_{n=-\infty}^{-1} c_n e^{2\pi inx} + c_0 + \sum_{n=1}^{\infty} c_n e^{2\pi inx} \\
&= c_0 + \sum_{n=1}^{\infty} c_{-n} e^{-2\pi inx} + \sum_{n=1}^{\infty} c_n e^{2\pi inx} \\
&= c_0 + \sum_{n=1}^{\infty} c_{-n}(\cos(-2\pi nx) + i\sin(-2\pi nx)) + \sum_{n=1}^{\infty} c_n(\cos(2\pi nx) + i\sin(2\pi nx)) \\
&= c_0 + \sum_{n=1}^{\infty} (c_{-n} + c_n)\cos(2\pi nx) + \sum_{n=1}^{\infty} i(c_n - c_{-n})\sin(2\pi nx) \\
&= c_0 + \sum_{k=1}^{\infty} (c_{-k} + c_k)\cos(2\pi kx) + \sum_{k=1}^{\infty} i(c_k - c_{-k})\sin(2\pi kx).
\end{aligned}$$

Now check that the coefficients of $\cos(2\pi kx)$ and $\sin(2\pi kx)$ are as claimed.

EXERCISE 4.7. How can Fourier's Theorem be stated for periodic functions f on $\mathbb{R}$? That is, "If f is a periodic function, ..."

The Fourier Transform on the Circle

For each square-integrable function f on $[0,1)$, define a function $\widehat{f}$ on the set $\mathbb{Z}$ of integers by

$$\widehat{f}(n) = \int_0^1 f(t)e^{-2\pi int}\, dt.$$

The assignment that takes f to $\widehat{f}$ is called the *Fourier transform* on the circle. If $f \in L^2([0,1))$, we often write $T(f)$ for its Fourier transform. It assigns to each square-integrable function f a doubly-infinite sequence $\{c_n\} \equiv \{\widehat{f}(n)\}$ of numbers. According to Fourier's theorem, the transform has an inverse; i.e., we can recover the function f from its transform $\widehat{F}$:

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

EXERCISE 4.8. Compute the Fourier transforms of the following functions on $[0, 1)$.

$f(x) = x$, $f(x) = x^2$, $f(x) = \sin(2\pi x)$, $f(x) = \sin x$, and $f(x) = e^{8\pi i x}$.

REMARK. In some sense, since we can go back and forth between them, the same information is encoded in the function f as is encoded in the function (sequence) $\widehat{f}$. Since f and $\widehat{f}$ are very different objects, it could be, and in fact often is, very informative that these two different objects both encode the same information. For example, a question or problem about functions can be "transformed" into a corresponding question or problem about sequences. While one of these two questions may be difficult to answer, the other one may be easy. And, since they are somehow really the same question in two different forms, a solution of one can be transformed into a solution of the other. This is perhaps the true essence of Fourier analysis.

EXAMPLE 4.1. Let α be a positive irrational number. Suppose f is a periodic function that is also periodic with period α. That is, f is periodic with respect to two different periods. What can be said about such an f? What can be said about its Fourier transform? Let us calculate:

$$\begin{aligned}
\widehat{f}(n) &= \int_0^1 f(t) e^{-2\pi i n t}\, dt \\
&= \int_{-\alpha}^{1-\alpha} f(s+\alpha) e^{-2\pi i n (s+\alpha)}\, ds \\
&= \int_{-\alpha}^{-\alpha+1} f(s) e^{-2\pi i n s} e^{-2\pi n \alpha}\, ds \\
&= e^{-2\pi i n \alpha} \int_0^1 f(s) e^{-2\pi i n s}\, ds \\
&= e^{-2\pi i n \alpha} \widehat{f}(n).
\end{aligned}$$

Hence, for any integer n such that $\widehat{f}(n) \neq 0$, we must have $e^{-2\pi i n \alpha} = 1$. But, $e^{it} = 1$ only if $t = 2k\pi$ for some integer k.

So, if $\widehat{f}(n) \neq 0$, we must have $-2\pi i n \alpha = 2\pi k$ or $-n\alpha = k$. This can't happen if $n \neq 0$, for that would mean that $\alpha = -k/n$ which is rational.

So the only n for which $\widehat{f}(n) \neq 0$ is for $n = 0$.

We know exactly what the Fourier transform of f looks like. $\widehat{f}(n) = 0$ for all $n \neq 0$. What does this mean about f itself? Fourier's Theorem tells us that

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x} = \widehat{f}(0).$$

That is, f is a constant function.

We have shown then, using Fourier's Theorem, that any periodic function that is also periodic with an irrational period, must be a constant function.

One final and extremely useful fact about the Fourier transform is this:

THEOREM 4.3. *Let $L^2([0,1))$ denote the set of all square-integrable functions on the interval $[0,1)$, and let $\mathcal{C}$ denote the set of all doubly-infinite sequences $\{c_n\}$. Then:*

(1) *$L^2([0,1))$ is a vector space.*
(2) *$\mathcal{C}$ is a vector space.*
(3) *The Fourier transform is a linear transformation from the vector space $L^2([0,1))$ into the vectors space $\mathcal{C}$. That is, $\widehat{af+bg} = a\widehat{f} + b\widehat{g}$ for all $f, g \in L^2([0,1))$ and all complex numbers a and b.*

PROOF. To see that $L^2([0,1))$ is a vector space, we use Exercise 4.3. (Why is this enough?)

That $\mathcal{C}$ is a vector space is even easier. We just note that the sum of two sequences is again a sequence, and any complex number times a sequence is again a sequence.

To see part (3), we just compute. It is going to work because the integral is a linear transformation.

$$\begin{aligned}
\widehat{af+bg}(n) &= \int_0^1 (af(t) + bg(t))e^{-2\pi int}\,dt \\
&= \int_0^1 (af(t)e^{-2\pi int} + bg(t)e^{-2\pi int})\,dt \\
&= a\int_0^1 f(t)e^{-2\pi int}\,dt + b\int_0^1 g(t)e^{-2\pi int}\,dt \\
&= a\widehat{f}(n) + b\widehat{g}(n).
\end{aligned}$$

EXERCISE 4.9. Use the preceding theorem and your answers to Exercise 4.8 to compute the Fourier transforms of $f(x) = \frac{1}{2} - x$ and $f(x) = \frac{1}{6} - x - x^2$.